Large scale properties in turbulent spherically symmetric accretion

Arnab K. Ray

Harish–Chandra Research Institute,
Chhatnag Road, Jhunsi, Allahabad 211 019, INDIA

arnab@mri.ernet.in

and

J. K. Bhattacharjee

Department of Theoretical Physics,
Indian Association for the Cultivation of Science,
Jadavpur, Calcutta (Kolkata) 700 032, INDIA

tpjkb@mahendra.iacs.res.in

ABSTRACT

The role of turbulence in a spherically symmetric accreting system has been studied on very large spatial scales of the system. This is also a highly subsonic flow region and here the accreting fluid has been treated as nearly incompressible. It has been shown here that the coupling of the mean flow and the turbulent fluctuations, gives rise to a scaling relation for an effective "turbulent viscosity". This in turn leads to a dynamic scaling for sound propagation in the accretion process. As a consequence of this scaling, the sonic horizon of the transonic inflow solution is shifted inwards, in comparison with the inviscid flow.

Subject headings: accretion, accretion disks — hydrodynamics — methods: analytical — turbulence

1. Introduction

The purpose of this work has been to study the dynamic scaling behaviour of the coefficients of viscosity arising out of turbulence in a spherically symmetric accreting system, and how such scaling behaviour leads to a scale dependence for the speed of sound as well. In its turn this will be shown to have an important bearing on the sonic point of the transonic inflow solution, since it is with the speed of sound that both the bulk velocity of the flow and the sonic point are scaled. A study of this kind should be useful in addressing recent observational discrepancies, for which the classical (and inviscid) Bondi theory has proved somewhat inadequate.

In accretion studies, turbulence is of great relevance, since in almost all cases of physical interest, the accreting astrophysical fluid is in a turbulent state (Choudhuri 1999). It is presently a well established fact that in the two dimensional case of a thin accretion disc, viscous shearing between two differentially rotating adjacent layers, accomplishes the outward transport of angular momentum and effectively facilitates the infall of matter (Pringle 1981; Frank et al. 1992). However, in this situation, ordinary molecular viscosity has been known to be quite an inadequate mechanism to explain the rate of the transport process. On the other hand, it is to a very high value of the Reynold's number that the flow corresponds, and as such the flow is widely acknowledged to be turbulent (Frank et al. 1992; Balbus & Hawley 1998). In such a situation, turbulence — as quantitatively characterized by a "turbulent viscosity" in the Navier-Stokes' equation — becomes a prime candidate for a physical mechanism that brings about an enhanced outward transport of angular momentum. The very well known α prescription of Shakura & Sunyaev (1973) is based on this principle.

As opposed to the facilitating role that it plays in a rotationally accreting flow, viscosity — even presumably "turbulent viscosity" — affects the more paradigmatic spherically symmetric accreting flow, somewhat differently. In the latter case, it has been seen that the role of viscosity is actually directed towards inhibiting the process of gravity driven infall, and in doing so, viscosity also sets up a limiting length scale for the effectiveness of gravity — the "viscous shielding radius" (Ray 2003). However, as in the thin disc system, even in the spherically symmetric case, molecular viscosity would be far too weak a mechanism to bring about a significant quantitative impact. It would then be well worth investigating into the question of how significantly would spherically symmetric accretion be affected by a large and scale dependent turbulent viscosity. Following the qualitative insights obtained with the introduction of molecular viscosity in the governing hydrodynamical equations, it is possible as a matter of standard practice to study both the qualitative and the quantitative extent of the influence of an *effective* turbulent viscosity on the hydrodynamical processes. The main purpose here would be to show that the turbulent fluctuations of the interstellar medium are capable of renormalizing on large length scales, the small molecular viscosity given in the Navier-Stokes equation. This renormalized effective viscosity, as pictured by Heisenberg (1948) in his theory of turbulence, can very well be instrumental in setting a noticeable limiting length scale on the effectiveness of gravity to drive the accretion process. Indeed, the renormalizing of the viscosity would be robust enough to make viscosity be comparable with pressure, on the same scale of length.

Related to this contention, one example that may be cited is that of the resistive role against gravity that turbulence plays in another spherically symmetric system of astrophysical interest — that of the self-gravity driven Jeans collapse of a gas cloud without angular momentum. Studies carried out by Bonazzola et al. (1987, 1992) have shown that turbulence acts as a stabilizing agent against a self-gravity driven collapse. A renormalization approach has shown that a renormalized turbulent pressure acts against gravity. In addition to this, what is being contended in this work is that a scaled-up renormalized viscosity also enfeebles the influence of gravity. Both the pressure term and the viscous term derive from the stress tensor in the Navier-Stokes equation (Tennekes & Lumley 1972). For the spherically symmetric case, the contribution comes only from one diagonal (hence, isotropic) element of the stress tensor. In such a situation, the renormalized pressure and the viscosity terms would both manifest themselves through the same physical effect, and the physical contents of the arguments presented by Bonazzola et al. (1987, 1992) for the self-gravity driven spherical collapse of a gas cloud, would match those in this study of turbulent spherically symmetric accretion. Having noted this point, it would also be instructive to have an understanding of the difference between the two physical cases being compared here. Whereas Bonazzola et al. (1992) have studied the response to large scale density perturbations on a stationary turbulent solution in a self-gravity driven system, what is being studied in this work, is the influence of spontaneous fluctuations on the mean stationary solution of a system, in which gravity comes into play through an external accretor. An analysis of the latter nature is all the more contextual with regard to accretion studies, because spherically symmetric accretion is exemplified by the infall of interstellar matter on to an isolated accretor, and it has been well recognized that the interstellar medium displays turbulent behaviour (Jokipii & Lerche 1969; Jokipii et al. 1969; Lee & Jokipii 1976).

While dwelling on this matter, it would be important to mention that certain previous studies in spherically symmetric accretion on to a black hole, have in fact quantitatively accounted for the physical role of turbulence in very efficiently converting gravitational energy to radiation. In the works of Mészáros (1975) and Mészáros & Silk (1977), it has been argued that for spherical accretion on to a massive black hole, turbulent dissipation would be one of the factors which would result in the luminosity of the system being enhanced by quite a few orders of magnitude — indeed to such an extent that the spherically symmetric system could be compared with disc models as an X-ray source.

2. The equations of turbulent spherical accretion

The effect of turbulent fluctuations has been studied here on very large spatial (and therefore highly subsonic scales) of the spherically symmetric accreting system. The physical effects of turbulence are appreciably manifested on these scales. Since all physically feasible flow solutions have to pass through the subsonic flow region, the turbulent fluctuations here must have a significant influence on the flow. And more to the point, on these subsonic scales, the flow could be studied in the nearly incompressible regime.

Turbulence is an attribute of the fluid flow (Tennekes & Lumley 1972), while molecular viscosity is an intrinsic physical property of the fluid. And yet the two can be very closely related to each other through the Navier-Stokes equation (Frisch 1999), which, as one of the governing equations of the flow, is given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\nabla P}{\rho} + \frac{GM}{r^2}\hat{\mathbf{r}} = \nu\nabla^2\mathbf{v} + \mu\nabla(\nabla\cdot\mathbf{v}) \tag{1}$$

where ν and μ are the two kinematic coefficients of viscosity. The pressure P is related to the density through a general polytropic equation of state $P = k\rho^\gamma$. Here γ is the polytropic exponent with an admissible range given by $1 < \gamma < 5/3$ — these restrictions having been imposed by the isothermal limit and the adiabatic limit respectively. The flow is also governed by the continuity equation, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{2}$$

The total velocity and density fields are written as $\mathbf{v} = \mathbf{v_0} + \mathbf{u}$ and $\rho = \rho_0 + \delta\rho$, in which $\mathbf{v_0}$ and ρ_0, which are functions of the radial coordinate only, are the mean velocity and density profiles for the spherically symmetric transonic flow, while $\mathbf{u}$ and $\delta\rho$ generally are time-dependent and three-dimensional random fluctuations about the transonic solution. The implicit understanding here is that in an accreting system naturally evolving in real time, the transonic solution is accorded primacy over all possible other stationary solutions in both the inviscid (Bondi 1952; Garlick 1979; Ray & Bhattacharjee 2002) and viscous regimes (Axford & Newman 1967). Under the assumption that cross-correlations of the density and the velocity fluctuations would be negligible, i.e. $\langle\nabla\cdot(\mathbf{u}\delta\rho)\rangle = 0$, the average (and steady) solutions would be obtained as

$$\nabla\cdot(\rho_0\mathbf{v_0}) = 0 \tag{3}$$

and

$$(\mathbf{v_0}\cdot\nabla)\mathbf{v_0} + \langle(\mathbf{u}\cdot\nabla)\mathbf{u}\rangle + \frac{\gamma k}{\gamma - 1}\nabla\rho_0^{\gamma-1} + \frac{GM}{r^2}\hat{\mathbf{r}} = \nu\nabla^2\mathbf{v_0} + \mu\nabla(\nabla\cdot\mathbf{v_0}) \tag{4}$$

In the nearly incompressible regime, only the first order term in the expansion of the density fluctuations about the mean density need be retained. The fluctuating density and velocity fields are therefore seen to satisfy,

$$\frac{\partial}{\partial t}\delta\rho + \nabla\cdot(\rho_0 \mathbf{u}) + \nabla\cdot(\mathbf{v_0}\delta\rho) + \nabla\cdot(\mathbf{u}\delta\rho) = 0 \tag{5}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\left[c_s^2\frac{\delta\rho}{\rho_0}\right] = \nu\nabla^2\mathbf{u} + \mu\nabla(\nabla\cdot\mathbf{u}) - \Big[(\mathbf{u}\cdot\nabla)\mathbf{v_0} + (\mathbf{v_0}\cdot\nabla)\mathbf{u} - \langle(\mathbf{u}\cdot\nabla)\mathbf{u}\rangle\Big] \tag{6}$$

respectively, with c_s being the steady value of the speed of sound, which is related to the mean density by $c_s^2 = \gamma k \rho_0^{\gamma-1}$.

In the very much subsonic region of the flow, the variation of the mean density ρ_0 may be neglected, since in this region the mean density very closely assumes an ambient value, which is a constant. Under this approximation, equation (3) gives the relation $\nabla\cdot\mathbf{v_0} \cong 0$. Furthermore, on these scales the continuity equation also governs the asymptotic behaviour of the mean velocity, which implies that the variation of the mean velocity, at its most rapid, is given by $v_0 \sim r^{-2}$ (Petterson et al. 1980; Chakrabarti 1990). On the other hand (under these asymptotic conditions) the turbulent velocity fluctuations are much greater than the mean velocity itself, and in fact are of the order of the speed of sound. Hence on these large length scales, ignoring all terms involving the mean velocity and the gradient of the mean density and its fluctuations, it would be meaningful to retain only the primary signature of a compressible flow, namely $\nabla\cdot\mathbf{u} \neq 0$, whence equation (5) simplifies to

$$\frac{1}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \nabla\cdot\mathbf{u} = 0 \tag{7}$$

which is an expression that has found quite regular mention in the study of a nearly incompressible fluid flow with random fluctuations (Staroselsky et al. 1990; Bhattacharjee 1993).

At this stage it should be important to be assured of the consistency in neglecting the higher powers of $\delta\rho/\rho_0$. Under the chosen working approximations, the terms in the left hand side of equation (6) can be written as $\dot{u}_\alpha$, $u_\beta \partial_\beta u_\alpha$ and $c_s^2\partial_\alpha(\delta\rho/\rho_0)$ respectively. If u_α and c_s are to scale as L^ϵ, then the time t scales as $L^{1-\epsilon}$, while $(\delta\rho/\rho_0)$, of course, remains independent of any scaling. Here ϵ is arbitrary, but anticipating that a one-loop calculation will yield a positive value for ϵ, all nonlinearities involving $\delta\rho/\rho_0$ (with ρ_0 being asymptotically a constant) may be ignored in favour of $u_\beta \partial_\beta u_\alpha$. This, arguably, should suffice for a study of the scaling dependence in the flow. If the resulting calculations lead to a positive ϵ, the adopted procedure would be justified and would be consistent with itself. That this is precisely what happens, will be demonstrated in the following sections.

Equation (6) is likewise simplified, and closed with the help of equation (7), to give

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = c_s^2 \nabla \Big(\frac{\partial}{\partial t}\Big)^{-1} (\nabla\cdot\mathbf{u}) + \nu\nabla^2\mathbf{u} + \mu\nabla(\nabla\cdot\mathbf{u}) + \mathbf{f} \tag{8}$$

in which $\mathbf{f} = -\left[(\mathbf{u}\cdot\nabla)\mathbf{v_0} + (\mathbf{v_0}\cdot\nabla)\mathbf{u} - \langle(\mathbf{u}\cdot\nabla)\mathbf{u}\rangle\right]$. The primary complication at this stage is in this term $\mathbf{f}$, which couples the fluctuating flow to the mean flow. Gravity of the central accretor maintains the mean flow, from which energy is transferred to the fluctuating flow, through its coupling with the mean flow. So effectively what happens is that the turbulent fluctuations are sustained by gravitation, via the nonlinear coupling in the term $\mathbf{f}$. Various approximations in the theory of turbulence have involved a modelling of this nature of energy input to the turbulent flow. Prandtl's mixing length theory is one of the most well known (Faber 1995; Choudhuri 1999). A more recent point of view treats this force as an as yet unspecified force external to the turbulent flow (Forster et al. 1977; De Dominicis & Martin 1979). Its dependence on the random field $\mathbf{u}$, makes it random and hence the modelling endows $\mathbf{f}$ with random properties. Even for this accretion problem it would therefore be quite possible to conceive of a randomly forced turbulent flow described by (for the nearly incompressible flow that is being studied here)

$$\partial_t u_i + (u_j\partial_j)u_i = c_s^2\partial_i(\partial_t^{-1}\partial_j u_j) + \nu\partial_j\partial_j u_i + \mu\partial_i(\partial_j u_j) + f_i \tag{9}$$

in which, for the Gaussian forcing, the correlation function is specified as

$$\langle f_i(\mathbf{r},t) f_j(\mathbf{r},t)\rangle = \delta_{ij} C_0(|\mathbf{r}-\mathbf{r}'|)\delta(t-t') \tag{10}$$

These two equations (9) and (10) will be necessary to develop a dynamic scaling theory for the turbulent spherically symmetric flow.

3. Dynamic scaling for turbulent spherical accretion

To carry out a dynamic scaling analysis with the help of equations (9) and (10), it would be convenient to work in Fourier transform space. This would necessitate writing

$$u_i(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int u_i(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \mathrm{d}^3\mathbf{k}\,\mathrm{d}\omega \tag{11}$$

in terms of which equation (9) becomes

$$\Big[(-i\omega + \nu k^2)\delta_{ij} + \mu k_i k_j\Big]u_j - c_s^2\frac{k_i k_j}{i\omega}u_j = f_i - i\sum_{\mathbf{p},\omega'} p_j u_j(\mathbf{k}-\mathbf{p},\omega')u_i(\mathbf{p},\omega-\omega') \tag{12}$$

The technique that has been adopted here is to expand the velocity field as $u_i = u_i^{(0)} + u_i^{(1)} + u_i^{(2)} + \ldots$, in which $u_i^{(0)}$ is the solution in the absence of the nonlinear term. The subsequent terms are the effect of the nonlinear term in equation (12). The lowest order solution can then be written as

$$u_i^{(0)} = G_{ij}^{(0)} f_j \tag{13}$$

in which

$$\left[G_{ij}^{(0)}\right]^{-1} = \Big(-i\omega + \nu k^2\Big)\delta_{ij} - \Big(\frac{c_s^2}{i\omega} - \mu\Big)k_i k_j \tag{14}$$

The first-order correction, $u_i^{(1)}$, satisfies

$$\Big[\Big(-i\omega + \nu k^2\Big)\delta_{ij} - \Big(\frac{c_s^2}{i\omega} - \mu\Big)k_i k_j\Big]u_i^{(1)} = -i\sum_{\mathbf{p},\omega'} p_j u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(0)}(\mathbf{p},\omega-\omega') \tag{15}$$

and its solution is given by

$$u_i^{(1)} = -iG_{ij}^{(0)}(\mathbf{k},\omega)\sum_{\mathbf{p},\omega'} p_k u_k^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_j^{(0)}(\mathbf{p},\omega-\omega') \tag{16}$$

As has been stressed by Heisenberg (1948), the momentum transfer term, given in the right hand side of equation (12), gives rise to an effective turbulent shear viscosity (the eddy viscosity) for an incompressible flow. This is the physical content of all subsequent theories — the different kinds of renormalized perturbation expansion (McComb 1990), the renormalization group (McComb 1990), the self-consistent mode coupling (McComb 1990) and the very recent Lagrangian picture approach (L'vov & Procaccia 1995a,b). In this compressible case, it will be easy to see that the right hand side of equation (12) will give rise to the effective shear viscosity, bulk viscosity and speed of sound. The simplest way of arriving at this result is to examine the average value of the right hand side of equation (12), averaged over the distribution of the random force f_i. This is done perturbatively. In what follows in this section, the salient results of the perturbative analysis have been brought forth. The details of the calculations have been presented in the Appendix.

The averaging of the nonlinear term in equation (12) will lead to its equivalent linearized representation (see the Appendix), given by

$$\langle -i\sum_{\mathbf{p},\omega'} p_j u_j(\mathbf{k}-\mathbf{p},\omega')u_i(\mathbf{p},\omega-\omega')\rangle \equiv -\sigma_{il}^{(0)}(\mathbf{k},\omega)u_l^{(0)}(\mathbf{k},\omega) \tag{17}$$

where

$$\sigma_{il}^{(0)}(\mathbf{k},\omega) = 2\sum_{\mathbf{p},\omega'} p_j k_k\Big[G_{jl}^{(0)}(\mathbf{k}-\mathbf{p},\omega')\tilde{C}_{ki}^{(0)}(\mathbf{p},\omega-\omega') + G_{il}^{(0)}(\mathbf{p},\omega-\omega')\tilde{C}_{kj}^{(0)}(\mathbf{k}-\mathbf{p},\omega')\Big] \tag{18}$$

and in which

$$\tilde{C}^{(0)}_{kj}(\mathbf{p},\omega') = G^{(0)}_{kl}(\mathbf{p},\omega')C_0(\mathbf{p})G^{(0)}_{lj}(-\mathbf{p},-\omega') \tag{19}$$

Clearly, $\tilde{C}^{(0)}_{ij}$ is the zeroth-order correlation function for the velocity field. The nonlinear term in the equation of motion now has the structure $\sigma^{(0)}_{il}(\mathbf{k},\omega)u^{(0)}_l(\mathbf{k},\omega)$ in this lowest order of the perturbation theory. The coefficient $\sigma^{(0)}_{il}$ can clearly be identified as making a contribution to the two coefficients of viscosity and the speed of sound, by comparing with the linear term in the equation of motion shown in equation (12). The $\sigma^{(0)}_{il}$ that has been obtained is called the self energy and constitutes a dressing of the bare coefficients. This is exactly in conformity with all the different ways of doing perturbation theory (McComb 1990). The step beyond perturbation theory goes to say that as the higher order terms are considered, $\sigma^{(0)}_{il}$ will be converted to the full self energy σ_{il}. It must be emphasized here that conversion to the full self energy will not differently affect the scaling arguments that will be developed here on the basis of the lowest order in the perturbation theory.

To make any further progress, it would be instructive to examine the structure of $\sigma^{(0)}_{ij}(\mathbf{k},\omega)u^{(0)}_l(\mathbf{k},\omega)$. By comparing with the form of $[G^{(0)}_{ij}]^{-1}$ in equation (14), it is possible to write

$$\begin{aligned}\sigma^{(0)}_{ij}(\mathbf{k},\omega) &= 2\sum_{\mathbf{p},\omega'} p_m k_n \Big[G^{(0)}_{mj}(\mathbf{k}-\mathbf{p},\omega')\tilde{C}^{(0)}_{ni}(\mathbf{p},\omega-\omega') \\ &\qquad + G^{(0)}_{im}(\mathbf{p},\omega-\omega')\tilde{C}^{(0)}_{jn}(\mathbf{k}-\mathbf{p},\omega')\Big] \\ &= k^2\left[\sigma^{(0)}_1(\mathbf{k},\omega)\delta_{ij} + \sigma^{(0)}_2(\mathbf{k},\omega)\frac{k_ik_j}{k^2}\right]\end{aligned} \tag{20}$$

where $\sigma^{(0)}_1$ and $\sigma^{(0)}_2$ are the frequency and momentum dependent components of the self energy tensor which must have the structure shown in equation (20) from the isotropy of space. Evidently, $\sigma^{(0)}_1(\mathbf{k},\omega=0)$ dresses the shear viscosity, while $\sigma^{(0)}_2(\mathbf{k},\omega)$ dresses the bulk viscosity and the speed of sound. To have any information about the dressing of the speed of sound, the $(i\omega)^{-1}$ part would have to be extracted from $\sigma^{(0)}_2(\mathbf{k},\omega)$ and the rest of the integral would have to be evaluated at $\omega = 0$, to yield the dressed bulk viscosity.

The renormalization of ν, μ and c_s converts them into the renormalized quantitites $\tilde{\nu}$, $\tilde{\mu}$ and $\tilde{c}_s$ respectively. To have any idea of how the two coefficients of viscosity and the speed of sound get renormalized, the Green's function would have to be written out by inversion of the matrix implied by $[G^{(0)}_{ij}]^{-1}$ in equation (14). Substitution of the unrenormalized quantities in the Green's function by the renormalized ones (see the Appendix), will then give the fully dressed Green's function as

$$G_{ij}(\mathbf{k},\omega) = \frac{1}{-i\omega+\tilde{\nu}k^2}\left[\delta_{ij} - k_ik_j\frac{(\tilde{\mu}-\tilde{c}^2_s/i\omega)}{-i\omega+\tilde{\nu}k^2+k^2(\tilde{\mu}-\tilde{c}^2_s/i\omega)}\right] \tag{21}$$

The poles of the Green's function, occurring at $\omega = -i\tilde{\nu}k^2$ and the roots of $\omega^2 = \tilde{c}_s^2 k^2 - i\omega k^2(\tilde{\nu}+\tilde{\mu})$, would deliver a dispersion relation for ω. It is satisfying to note here that the second relation (the quadratic in ω) is identical to the one obtained by Bonazzola et al. (1992), barring a term arising from the self-gravity of the system that they were studying. In the long wavelength limit (k small), which contains the interesting features about the scaling behaviour, the quadratic in ω can be approximated as

$$\omega \cong \pm \tilde{c}_s k - \frac{i}{2} k^2 (\tilde{\nu} + \tilde{\mu}) \tag{22}$$

The renormalized quantities will have a power law dependence on k. If dynamic scaling is to be invoked, then the frequency will be proportional to some definite power of k, which means that all the terms in the right hand side of equation (22) must scale in the same way. In physical terms this would mean that both the propagating term and the dissipative term in equation (22), would be comparably effective on the same scale. If a power law were to be written in the form $\tilde{\nu} \propto k^{-y}$, then it will clearly also indicate that $\tilde{\mu} \propto k^{-y}$ and $\tilde{c}_s \propto k^{1-y}$. The main concern will be to set a value for y. To make any progress in that direction, the forcing term would have to be specified, which actually would imply specifying the correlation function $C_0(|\mathbf{r}-\mathbf{r}'|)$ in equation (10). Since this term dominates at large distance, a scaling form $C_0(|\mathbf{r}-\mathbf{r}'|) \propto |\mathbf{r}-\mathbf{r}'|^{\alpha}$ may be assumed. This would then transform in the momentum space as $C_0(k) \sim k^{-(D+\alpha)}$, where D is the dimensionality of the space. At this point a significant departure from Bonazzola et al. (1992) is being made, by suggesting that both the renormalized $\sigma_1^{(0)}$ and $\sigma_2^{(0)}$, as given by equation (20), would have to be treated as comparable with each other. Consequently, information on the scaling of the sound velocity (arising from the pressure term) can be had from the scaling of the shear viscosity, which is clearly a dissipative effect.

The fully dressed self-consistent form of equation (20) can now be written down as an integral (see the Appendix), given by

$$\sigma_{ij}(\mathbf{k},\omega) = 4 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}\omega'}{2\pi} p_m k_n G_{mj}(\mathbf{k}-\mathbf{p}, \omega - \omega') \tilde{C}_{ni}(\mathbf{p}, \omega') \tag{23}$$

where $\tilde{C}_{ni} = G_{nr} C_0 G^*_{ri}$.

For an incompressible flow, the Kolmogorov spectrum requires that $\alpha = 0$, to characterize the nature of the transfer of energy between the mean flow and the fluctuating flow. Since this transfer characteristic should be independent of the speed of sound, it would be possible to write $\alpha = 0$ for the case of near incompressibility being discussed here. The left hand side of equation (23) then scales as k^{2-y}, while the right hand side scales as k^{2y-2}. For the scaling properties of both sides of equation (23) to agree, it should be necessary to set

$2 - y = 2y - 2$, which will yield $y = 4/3$. This will then establish the result

$$\tilde{\nu} \sim k^{-4/3}, \qquad \tilde{\mu} \sim k^{-4/3}, \qquad \tilde{c}_s \sim k^{-1/3} \tag{24}$$

which, it may be mentioned at this point, is identical to the scaling relation obtained by Staroselsky et al. (1990) for the case of a randomly stirred compressible fluid. It must also be emphasized here that simple dimensional arguments would not entirely suffice. Indeed, Staroselsky et al. (1990) make this point amply clear by saying that the renormalization of the speed of sound is essential to understanding the physics of compressible flows, since the appearance of the speed of sound as a dimensional parameter, makes simple dimensional considerations invalid.

Of immediate interest would be the scaling behaviour of the speed of sound, which in terms of the radial distance may be written as $\tilde{c}_s \sim r^{1/3}$. The steady state solution of the continuity equation (with ρ_0 written in terms of c_s), gives a dependence for the steady velocity of the flow v_0, which goes as (Chakrabarti 1990)

$$v_0 \sim r^{-2} {c_s}^{-2n} \tag{25}$$

where $n = (\gamma - 1)^{-1}$ is the polytropic index, whose admissible range of values for inflow solutions is given by $3/2 < n < \infty$ (Chakrabarti 1990). Using the renormalized speed of sound and its associated scaling relation in equation (25), will give a scaling behaviour for the steady flow velocity as

$$v_0 \sim r^{-2(1+n/3)} \tag{26}$$

from which it is quite evident that regardless of the value of n, on large length scales, the steady flow velocity would die out — a fact that is in conformity with the boundary condition of the flow. The result given by equation (26) highlights another very interesting issue. It has been discussed earlier that on large length scales, the mean flow is limited by the equation of continuity, and therefore its variation is given by $v_0 \sim r^{-2}$. This is a result that is easily derived from the classical and inviscid Bondi flow (Petterson et al. 1980; Chakrabarti 1990). What equation (26) indicates is that turbulent fluctuations, sustaining themselves at the expense of the mean flow, detracts even further from the r^{-2} scaling law for the mean flow velocity — something that, from considerations of energy dissipation, can be qualitatively intuited about the influence of turbulence on the mean flow.

The results in equations (24) and (26) also lead to the conclusion that in the renormalized situation, there would be a scale dependence for the position of the sonic horizon as well. For large length scales, i.e. concomitantly for a large effective turbulent viscosity, the sonic horizon would be shifted inwards. This happens because, seen on a large length scale, an enhanced scale dependent speed of sound, could only be matched by the steady flow velocity deeper within the gravitational potential well. Since the flow has to pass through the subsonic region in any case, this effect of subsonic turbulence in shifting the sonic point inwards, is also seen to have a bearing on the transonicity of the inflow solution. This observation would be entirely compatible with the role of a weak molecular viscosity in inwardly shifting the position of the critical point of the inflow solution (Ray 2003).

4. Concluding remarks

It has been seen so far, how on the large length scales of a spherically symmetric accreting system, turbulence is capable of setting a scaling behaviour for both viscosity and the speed of sound. However, it need not be supposed that given the scaling relation $\tilde{c}_s \sim r^{1/3}$, there would be an arbitrarily large scaling for the speed of sound on large length scales. This is because in spherical symmetry, turbulence itself will also play a role in limiting the accretion process. The physical quantity $\dot{m}/\nu\rho$ (with $\dot{m}$ being the accretion flow rate) has the dimension of length, and this has been understood to be a viscous shielding radius, $r_{\rm visc}$ (Ray 2003). If the value of ν is enhanced by the introduction of a large and scale dependent kinematic viscosity, then $r_{\rm visc}$ will define a noticeable spatial limit for the accretion process.

The $r^{1/3}$ scaling behaviour for sound propagation is also apparently surprising, with its physical implication being that the flow is heated up more at larger radial distances. On the other hand, the classical Bondi theory shows that the speed of sound increases as the flow moves inward, i.e. the flow gets heated up more at smaller radii. The point to remember is that this property of classical spherical accretion is not violated by the mean flow, and the dressing of sound propagation that the turbulent fluctuations bring about, is manifest over and above the standard features that the mean flow is expected to show. The extent of energy dissipation that turbulence brings about is not accounted for by the Bondi theory. This energy dissipation shows itself as an enhanced scaling for the speed of sound (larger scales are more energetic in this sense), and had temperature been chosen as a dynamical variable, this would have shown no contradiction. A cautionary reminder that is to be sounded here is that all the scaling relations have been derived under the assumption of near incompressibility on large length scales, which is a condition that cannot be applied too far into the inner region of the flow, and in consequence, the $r^{1/3}$ scaling for sound propagation is not to be extended too much to small length scales either.

It would also be instructive here to have an understanding of the dynamic scaling of both the speed of sound and the steady flow velocity, which could be derived on using the prescription for an effective viscosity forwarded by Mészáros & Silk (1977). They proposed a scaling behaviour for the kinematic coefficient of turbulent viscosity $\nu_{\rm t}$, which could be conceived of as a product of a characteristic length scale $l_{\rm t}$ of the turbulent cells, and the magnitude of their associated turbulent velocity fluctuations $v_{\rm t}$. It was assumed by Mészáros & Silk (1977) that for all r, the length $l_{\rm t}$ would be some fraction of the radial distance r, while $v_{\rm t}$ would be a fraction of the free fall velocity $v_{\rm ff}$, which varies as $r^{-1/2}$. For large length scales, in which the density of the accreting fluid approaches its constant ambient value, this prescription would lead to a scaling behaviour given by $\nu_{\rm t} \sim v_{\rm t} l_{\rm t} \sim r^{1/2}$.

In this case, it would then be easy to see from the dispersion relation given by equation (22), that the speed of sound would be scaled by the relation $c_s \sim r^{-1/2}$, while scaling for the steady flow velocity, from equation (25), would be given by $v_0 \sim r^{n-2}$. The difficulty arises for $n > 2$, since for large length scales, v_0 would actually increase, contrary to a common understanding of the boundary condition that v_0 should decrease over large radial distances. This discrepancy arises because of considering the characteristic eddy velocity to be a fraction of the free fall velocity. Even though this looks well founded on dimensional principles alone, this scaling behaviour breaks down on large length scales, because on these scales free fall conditions do not hold. Rather, this is the region of the *ambient* conditions, where the mean flow velocity, even under inviscid conditions, varies at the most as r^{-2}, and therefore the velocity fluctuations would have to have a different scaling behaviour. Indeed, by being coupled to the mean flow, the velocity fluctuations alter the scaling behaviour for the mean velocity as well, as equation (26) indicates. Thus it would probably be more correct to suggest that within the sonic radius and close to the accretor, for a highly supersonic mean flow, free fall conditions can have a bearing on the velocity fluctuations. However, the extent of the influence of turbulence on such small scales would be a somewhat contentious issue, and is not within the scope of this work.

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A. Appendix

It has been seen that equation (12) is in the form

$$\Big[(-i\omega + \nu k^2)\delta_{ij} + \mu k_i k_j\Big] u_j - c_s^2 \frac{k_i k_j}{i\omega} u_j = f_i - i \sum_{\mathbf{p},\omega'} p_j u_j(\mathbf{k}-\mathbf{p},\omega') u_i(\mathbf{p},\omega-\omega') \qquad \text{(A1)}$$

of which, the right hand side is averaged over the distribution of the random force f_i. For the nonlinear term, the perturbative expansion of u_i can be written for its first two terms as

$$\begin{aligned}\langle -i\sum_{\mathbf{p},\omega'} p_j u_j(\mathbf{k}-\mathbf{p},\omega')u_i(\mathbf{p},\omega-\omega')\rangle &= \langle -i\sum_{\mathbf{p},\omega'} p_j u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(0)}(\mathbf{p},\omega-\omega')\rangle \\ &\quad +\langle -i\sum_{\mathbf{p},\omega'} p_j [u_j^{(1)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(0)}(\mathbf{p},\omega-\omega') \\ &\qquad +u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(1)}(\mathbf{p},\omega-\omega')]\rangle\end{aligned} \tag{A2}$$

In the above equation, the first term on the right hand, with $u_i^{(0)}$ substituted from equation (13), can be written as

$$\begin{aligned}\langle -i\sum_{\mathbf{p},\omega'} p_j u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(0)}(\mathbf{p},\omega-\omega')\rangle &= -i\sum_{\mathbf{p},\omega'} p_j G_{jm}^{(0)}(\mathbf{k}-\mathbf{p},\omega')G_{in}^{(0)}(\mathbf{p},\omega-\omega') \\ &\qquad \times\langle f_m(\mathbf{k}-\mathbf{p},\omega')f_n(\mathbf{p},\omega-\omega')\rangle \\ &= -i\sum_{\mathbf{p},\omega'} p_j G_{jn}^{(0)}(-\mathbf{p},\omega')C_0(-\mathbf{p})G_{in}^{(0)}(\mathbf{p},-\omega')\end{aligned}$$

As can easily be seen, the expression above does not produce any momentum ($\mathbf{k}$) or frequency (ω) dependent term and hence is not responsible for momentum transfer. The second term in equation (A2), with $u_i^{(1)}$ substituted from equation (16), can be written down as

$$\begin{aligned}&\langle -i\sum_{\mathbf{p},\omega'} p_j [u_j^{(1)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(0)}(\mathbf{p},\omega-\omega') + u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')u_i^{(1)}(\mathbf{p},\omega-\omega')]\rangle \\ =\ & -\langle\sum_{\mathbf{p},\omega'} p_j \Big[G_{jl}^{(0)}(\mathbf{k}-\mathbf{p},\omega')\sum_{\mathbf{q},\omega''} q_k u_k^{(0)}(\mathbf{k}-\mathbf{p}-\mathbf{q},\omega'')u_l^{(0)}(\mathbf{q},\omega'-\omega'')u_i^{(0)}(\mathbf{p},\omega-\omega') \\ & +u_j^{(0)}(\mathbf{k}-\mathbf{p},\omega')G_{il}^{(0)}(\mathbf{p},\omega-\omega')\sum_{\mathbf{q},\omega''} q_k u_k^{(0)}(\mathbf{p}-\mathbf{q},\omega'')u_l^{(0)}(\mathbf{q},\omega-\omega'-\omega'')\Big]\rangle \\ =\ & -2\sum_{\mathbf{p},\mathbf{q},\omega',\omega''} p_j\Big[G_{jl}^{(0)}(\mathbf{k}-\mathbf{p},\omega')q_k G_{km}^{(0)}(\mathbf{k}-\mathbf{p}-\mathbf{q},\omega'')G_{in}^{(0)}(\mathbf{p},\omega-\omega') \\ &\qquad \times\langle f_m(\mathbf{k}-\mathbf{p}-\mathbf{q},\omega'')f_n(\mathbf{p},\omega-\omega')\rangle u_l^{(0)}(\mathbf{q},\omega'-\omega'') \\ & +G_{il}^{(0)}(\mathbf{p},\omega-\omega')q_k G_{km}^{(0)}(\mathbf{p}-\mathbf{q},\omega'')G_{jn}^{(0)}(\mathbf{k}-\mathbf{p},\omega') \\ &\qquad \times\langle f_m(\mathbf{p}-\mathbf{q},\omega'')f_n(\mathbf{k}-\mathbf{p},\omega')\rangle u_l^{(0)}(\mathbf{q},\omega-\omega'-\omega'')\Big]\end{aligned}$$

The factor of 2 appears in the expression above because $u_i^{(0)}$ could be expressed in two ways with the help of equation (13). Using the correlation function implied by equation (10), will now give from the result above

$$
\begin{aligned}
& -2\sum_{\mathbf{p},\omega'} p_j k_k \Big[G^{(0)}_{jl}(\mathbf{k}-\mathbf{p},\omega')G^{(0)}_{kn}(-\mathbf{p},-\omega+\omega')G^{(0)}_{in}(\mathbf{p},\omega-\omega')C_0(\mathbf{p}) \\
& +G^{(0)}_{il}(\mathbf{p},\omega-\omega')G^{(0)}_{kn}(\mathbf{p}-\mathbf{k},-\omega')G^{(0)}_{jn}(\mathbf{k}-\mathbf{p},\omega')C_0(\mathbf{k}-\mathbf{p})\Big]u^{(0)}_l(\mathbf{k},\omega) \\
= & -2\sum_{\mathbf{p},\omega'} p_j k_k \Big[G^{(0)}_{jl}(\mathbf{k}-\mathbf{p},\omega')\tilde{C}^{(0)}_{ki}(\mathbf{p},\omega-\omega') \\
& \qquad +G^{(0)}_{il}(\mathbf{p},\omega-\omega')\tilde{C}^{(0)}_{kj}(\mathbf{k}-\mathbf{p},\omega')\Big]u^{(0)}_l(\mathbf{k},\omega) \\
= & -\sigma^{(0)}_{il}(\mathbf{k},\omega)u^{(0)}_l(\mathbf{k},\omega)
\end{aligned}
$$

where

$$
\sigma^{(0)}_{il}(\mathbf{k},\omega) = 2\sum_{\mathbf{p},\omega'} p_j k_k \Big[G^{(0)}_{jl}(\mathbf{k}-\mathbf{p},\omega')\tilde{C}^{(0)}_{ki}(\mathbf{p},\omega-\omega') + G^{(0)}_{il}(\mathbf{p},\omega-\omega')\tilde{C}^{(0)}_{kj}(\mathbf{k}-\mathbf{p},\omega')\Big]
$$

and

$$
\tilde{C}^{(0)}_{kj}(\mathbf{p},\omega') = G^{(0)}_{kl}(\mathbf{p},\omega')C_0(\mathbf{p})G^{(0)}_{lj}(-\mathbf{p},-\omega')
$$

In the lowest order of the perturbation theory, the nonlinear term now has an equivalent linearized representation given by $\sigma^{(0)}_{il}(\mathbf{k},\omega)u^{(0)}_l(\mathbf{k},\omega)$. The $\sigma^{(0)}_{il}$ that has been obtained is called the self energy and it serves the purpose of dressing the bare coefficients in the equation of motion. Considering all the higher order terms, $\sigma^{(0)}_{il}$ will be converted to the full self energy σ_{il}. The self energy can be compared with equation (14) and can be seen to make a contribution to the two coefficients of viscosity and the speed of sound. Seen in this way it can be written as

$$
\begin{aligned}
\sigma^{(0)}_{ij}(\mathbf{k},\omega) &= 2\sum_{\mathbf{p},\omega'} p_m k_n \Big[G^{(0)}_{mj}(\mathbf{k}-\mathbf{p},\omega')\tilde{C}^{(0)}_{ni}(\mathbf{p},\omega-\omega') \\
& \qquad +G^{(0)}_{im}(\mathbf{p},\omega-\omega')\tilde{C}^{(0)}_{jn}(\mathbf{k}-\mathbf{p},\omega')\Big] \\
&= k^2\Big[\sigma^{(0)}_1(\mathbf{k},\omega)\delta_{ij} + \sigma^{(0)}_2(\mathbf{k},\omega)\frac{k_i k_j}{k^2}\Big]
\end{aligned}
\tag{A3}
$$

in which $\sigma^{(0)}_1(\mathbf{k},\omega=0)$ dresses the shear viscosity, and $\sigma^{(0)}_2(\mathbf{k},\omega)$ dresses the bulk viscosity and the speed of sound.

To understand the effect of renormalization, it would be necessary first to obtain the Green's function by inversion of the matrix implied by equation (14). In this way the Green's function is given as

$$
G^{(0)}_{ij}(\mathbf{k},\omega) = \frac{1}{-i\omega+\nu k^2}\left[\delta_{ij} - k_i k_j \frac{(\mu - c_s^2/i\omega)}{-i\omega+\nu k^2 + k^2(\mu - c_s^2/i\omega)}\right] \tag{A4}
$$

It is important to check if the incompressible limit is to be correctly obtained. The incompressible limit implies $c_s \longrightarrow \infty$ and in that limit it is seen that $G_{ij}^{(0)}(\mathbf{k},\omega) = P_{ij}(\mathbf{k})[-i\omega + \nu k^2]^{-1}$, where $P_{ij}(\mathbf{k}) = \delta_{ij} - (k_i k_j)/k^2$, is the projection operator, which is as it should be.

The renormalization of ν, μ and c_s converts them into the renormalized quantities $\tilde{\nu}$, $\tilde{\mu}$ and $\tilde{c}_s$ respectively. In the event of the two coefficients of viscosity and the speed of sound getting renormalized, the fully dressed Green's function is given by

$$G_{ij}(\mathbf{k},\omega) = \frac{1}{-i\omega + \tilde{\nu}k^2}\left[\delta_{ij} - k_i k_j \frac{(\tilde{\mu} - \tilde{c}_s^2/i\omega)}{-i\omega + \tilde{\nu}k^2 + k^2(\tilde{\mu} - \tilde{c}_s^2/i\omega)}\right] \tag{A5}$$

The poles of the Green's function occur at $\omega = -i\tilde{\nu}k^2$ and the roots of $\omega^2 = \tilde{c}_s^2 k^2 - i\omega k^2(\tilde{\nu} + \tilde{\mu})$, and on solving the quadratic in ω, the dispersion relation is given by

$$2\omega = -ik^2(\tilde{\nu} + \tilde{\mu}) \pm \sqrt{4\tilde{c}_s^2 k^2 - k^4(\tilde{\nu} + \tilde{\mu})^2} \tag{A6}$$

The long wavelength limit (k small) yields

$$\omega \cong \pm \tilde{c}_s k - \frac{i}{2}k^2(\tilde{\nu} + \tilde{\mu}) \tag{A7}$$

Dynamic scaling would imply that the frequency would be proportional to some power of k, and this in its turn would mean that each term in the right hand side of equation (A7) must scale in the same way. Assuming a power law of the form $\tilde{\nu} \propto k^{-y}$, will also clearly lead to having $\tilde{\mu} \propto k^{-y}$ and $\tilde{c}_s \propto k^{1-y}$. To know the value of y, the function $C_0(|\mathbf{r}-\mathbf{r}'|)$ in equation (10) would have to be specified first. Assuming a scaling form given by $C_0(|\mathbf{r}-\mathbf{r}'|) \propto |\mathbf{r}-\mathbf{r}'|^{\alpha}$, yields the corresponding transformation in the momentum space as $C_0(k) \sim k^{-(D+\alpha)}$, with D being the dimensionality of the space.

The fully dressed self-consistent form of equation (A3) can be set in an integral form given by (on noting that the two summations in the right hand side of equation (A3) are quite identical)

$$\sigma_{ij}(\mathbf{k},\omega) = 4\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{d}\omega'}{2\pi} p_m k_n G_{mj}(\mathbf{k}-\mathbf{p},\omega-\omega')\tilde{C}_{ni}(\mathbf{p},\omega') \tag{A8}$$

where $\tilde{C}_{ni} = G_{nr}C_0 G_{ri}^*$. This is the generalization of the self-consistent mode coupling of the incompressible turbulent flow, to the compressible turbulent flow and is the mode coupling version of the renormalization group arguments of Staroselsky et al. (1990). As in all such problems, the mode coupling integral is valid over a larger momentum scale and hence, in principle, allows more than the asymptotic analysis of the renormalization group. In this work, the focus is only on the exponent α, which is an asymptotic result. In the incompressible limit, it is to be noted that $G_{ij} = P_{ij}(-i\omega + \tilde{\nu}k^2)^{-1}$, which forces $\tilde{C}_{ij} = P_{ij}\tilde{C}_0(\omega^2 + \tilde{\nu}^2 k^4)^{-1} k^{-(D+\alpha)}$, where $\tilde{C}_0$ is a constant. This reduces equation (A8) to

$$\tilde{\nu}(\mathbf{k}, \omega) = \frac{4}{k^2} \int \frac{\mathrm{d}^D \mathbf{p}}{(2\pi)^D} \int \frac{\mathrm{d}\omega'}{2\pi} \frac{k_m k_n P_{mj}(\mathbf{k}-\mathbf{p}) P_{ni}(\mathbf{p}) P_{jl}(\mathbf{k}) P_{li}(\mathbf{k})}{[-i(\omega - \omega') + \tilde{\nu}(\mathbf{k}-\mathbf{p})^2][\omega^2 + \tilde{\nu}^2 p^4] p^{D+\alpha}} \qquad \text{(A9)}$$

which is very close to the expression obtained by Bhattacharjee (1991). The Kolmogorov spectrum for incompressible turbulence requires that $\alpha = 0$, which characterizes the nature of the energy transfer between the mean and the random flow. This transfer characteristic should be independent of the speed of sound, and so what holds for $c_s \longrightarrow \infty$, should also hold at finite c_s. Consequently, the forcing function is characterized by $\alpha = 0$ in the nearly incompressible regime that is being studied here.

A scaling analysis of equation (A8) can now be carried out. The left hand side scales as k^{2-y}. The right hand side clearly scales as $k^{D+2-y+2} k^{y-2} k^{-D} k^{2(y-2)} = k^{2y-2}$. For the scaling properties of the right and the left hand sides to agree, it is necessary to have the condition $2 - y = 2y - 2$, which gives $y = 4/3$. This leads to the result $\tilde{\nu} \sim k^{-4/3}$, $\tilde{\mu} \sim k^{-4/3}$ and $\tilde{c}_s \sim k^{-1/3}$.

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