Poincaré and Celestial Mechanics

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Introduction

Most students of physics are familiar with Newton's great triumph – explaining the motion of the moon around the Earth, and planets around the Sun. The basic laws of motion and the inverse square law of gravitation were sufficient to derive the three laws formulated by Kepler. These state (i) that planets move in ellipses, (ii) the speed along the orbit is such that equal areas are swept out by the planet – sun line in equal times and (iii) the square of the orbital period is proportional to the cube of the semi-major axis. Most text books of mechanics stop at this point, more than three hundred years ago! This "Classroom" piece carries the story of celestial mechanics upto Poincaré's work, for students and teachers who want more.

The Problem of Small Perturbations

Consider the moon's motion around the Earth. Surely the Sun, inspite of being much further away, must have some influence on it. The extra force on the moon due to the Sun is about half a per cent of that exerted by the Earth [1]. In many branches of science, an error of half a per cent would be quite tolerable. But imagine what would happen if the half per cent effects on each successive orbit added up! This is what celestial mechanics call a 'secular' effect – one that builds up with time. In fact, there *is* such an effect, because the plane of the moon's orbit turns once in about eighteen years. (This is well known to all astrologers as the movement of Rahu and Ketu, the directions in which the moon's orbital plane intersects the plane of the earth's orbit). But centuries of observation show that the distance from the Earth does not evolve significantly in fifty months nor does the eccentricity of the orbit. Clearly, the effects of such small extra forces, known as perturbations, have to be understood. Otherwise, even the stability of the solar system over the billions of years that it took life to develop on Earth is not explained. For example, the force which Jupiter exerts on the Earth is about one

[1] One warning. If the reader wants to check this, she should calculate the *difference* of the forces that the Sun exerts on Earth and moon. The *average* is responsible for pulling both in orbit around the Sun, and does not affect the *relative* motion of moon and earth.

part in twenty five thousand of that which the Sun exerts. How are we sure that this weak-looking force would not have produced some strong effects acting over four billion orbits, the age of the solar system?

A simple-minded first guess can be made, based on what we know about the harmonic oscillator (i.e simple pendulum with a small angle of swing). This has a natural frequency ω_{natural} related to the period by $T=2\pi/\omega$. If it is pushed periodically at the same frequency, the oscillations become larger and larger, and this is called resonance (that is why this journal pushes your interest in science once a month!). If the external push is at a slightly different frequency, then the amplitude builds up to a value which varies inversely with respect to the small difference in frequencies,i.e the amplitude is proportional to $1/(\omega_{\text{ext}} - \omega_{\text{natural}})$.

Resonances and their Consequences

Now consider the Sun-Jupiter-Saturn system. The pull from Jupiter has a frequency of once every twelve years, while the frequency of Saturn is one rotation per thirty years. Far from resonance, so nothing to worry about? But wait. The force of Jupiter on Saturn contains expressions like $1/|\mathbf{r}_S - \mathbf{r}_J|^2$. Mathematically, this is not *linear* in the co-ordinates of Jupiter and Saturn. This means that if we assume motions at ω_S and ω_J for the two planets as the first approximation, the next approximation produces forces proportional to higher powers of the two co-ordinates. For example, one might encounter terms like $\cos^5(\omega_J t)\cos^2(\omega_S t)$. It comes as no surprise to learn that the founders of celestial mechanics, Laplace, Lagrange and others (why were so many of them French?) had to worry about higher order resonances, i.e differences between *multiples* of the frequencies of different planets, occurring in the denominators of their mathematical expressions [2]. Dividing by zero is bad. Dividing by something small is allowed but casts doubt on whether one's successive corrections are really getting smaller.

[2] For example $5\omega_S - 2\omega_J$ is very close to zero!

The situation when Poincaré entered the scene was thus as follows. There was an elaborate machinery for calculating the positions of planets for all time as a series, but there was no guarantee that this series would converge, because of resonances. This was regarded as an outstanding problem, and a prize was instituted for the best solution [3]. Poincaré's work did not give a final solution to the problem as stated. But the level of additional insight obtained was so great that the Swedish Academy had no hesitation in awarding the prize to him. Later, Poincaré collected his contributions in the three volume treatise, '*Les Methodes Nouvelles de la Mecanique Celeste*' (New methods of Celestial Mechanics). In translation these extend to about a thousand pages [4]. It is said that Poincaré wrote rapidly, and did not believe in polishing his presentation repeatedly, since he would prefer to use that time to do more original work. But the book is full of new ideas which influenced the field for a long time thereafter.

[3] To celebrate the 60th birthday of King Oskar II of Sweden, a prize was instituted for progress on this problem and Poincaré won it in 1889.

[4] The translation with a detailed introduction was published by the AIP (American Institute of Physics) in 1993.

The Complexity of Two Dimensional Motion

The remarkable fact which Poincaré discovered is that when a particle moves in two (or more) space dimensions, the motion can be much more complicated than the planetary orbits we are familiar with. The reason is that even after coming back to the same point in space (i.e same value of the co-ordinates, x and y), the velocity can be in a completely different direction. An orbit of the three-body problem with this property is shown in *Figure* 1, and you can see that it is quite complicated. Of course, we can use the idea of energy conservation to fix the kinetic energy at the given point x and y, since we know the potential energy at that point. But kinetic energy only fixes the magnitude of the velocity, not its direction. Poincaré tells us that we should now really be asking the opposite question. Why is it that in some cases, the motion in two space dimensions can be simple? One example is Newton's solution for planetary motion. For the inverse square force it is just an ellipse. Even for a force which is not inverse square, it is a *precessing* ellipse. *Figure* 2 shows that for such an orbit, at a given point the velocity only has one of two possible

Figure 1. Computer generated orbit for a third light test particle moving around two heavy masses which are themselves in circular orbit around each other. The plot is made in a rotating frame of reference.

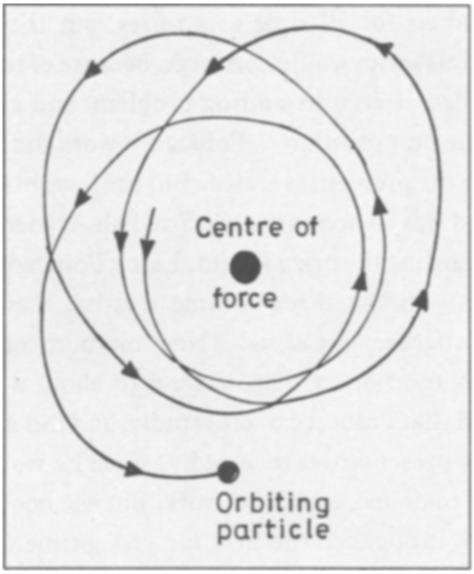

Figure 2. A precessing orbit for a particle moving under a general central force. Note that at any given point, the velocity vector takes only two values, with the same tangential component and radial components differing in sign.

directions, which makes the motion much simpler than the most general possibility allowed by energy conservation, of *Figure* 1. The reason why these or-bits are simple is that there is a *further* restriction when the force acts towards the centre. This is the law of conservation of angular momentum, or equivalently Kepler's second law. This fixes the magnitude of the tangential component of the velocity at a given point (because that determines the rate at which the radius vector sweeps out area). Once the tangential component is fixed, and we also know the magnitude of the velocity, the radia! component can only take one of two values, pointing inwards and outwards, and that is exactly what *Figure* 2 shows us.

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Poincaré's Surface of Section

Without giving details of all of Poincaré's innovations, we can state the basic idea behind one of them, using an analogy. Quite often, an engineer needs to study a rapidly rotating piece of machinery. One trick used is to illuminate it with a 'strobe light', a lamp that emits flashes at a frequency which can be controlled. The continuous motion is now seen as a series of still pictures. For example, if the frequency matches that of the rotating wheel, it appears to be at rest, an illusion so convincing that one should resist the temptation to put one's hand in! One sees this effect in a weaker form when a fan starts to speed up under a fluorescent (tube) light, or a car starts in a movie. The rotation can appear to be in a different direction, or stop briefly, etc. The famous 'Poincaré section', or 'surface of section' of celestial mechanics is a mathematical trick (cleverer than the stroboscope) reducing the study of continuous motion to the study of 'maps' in lower dimensions. The map is a discrete transformation which maps the position and velocity to their values at a later time, when the particle has come 'back' (say to the x-axis).

Box. Even Poincaré' Was Not Perfect.

Poincaré's prize winning essay on the n-body problem actually contained an error, pointed out by Phragmen, a Swedish mathematician! Correcting this error led Poincaré to one of the basic ideas of modern chaos theory. This story is documented in the preface to the translation of Poincaré's treatise, referred to earlier.

Conclusion

Using the properties of maps, Poincaré was able to give a criterion for finding periodic i.e. closed orbits even for systems as complicated as the three body problem. Further, he was able to set up the machinery to calculate when such an orbit would be stable. That is, if we started the particle with position and velocity very close to that which it has in the periodic orbit, would it stay close to that orbit? When such an orbit was unstable, he was able to show that the motion starting near it could be very complicated. In his own (translated) words... . "One will be struck by the complexity of this figure, which I shall not even attempt to draw. Nothing is more suitable for providing us with an idea of the complex nature of the three body problem, and of all the problems of dynamics in general". The potential of many of Poincaré's ideas was exploited only later, by Birkhoff, Kolmogorov, Arnold and others to build up the modern understanding of celestial mechanics.[5] And his qualitative picture of the motion was amply borne out when powerful computers were used to calculate orbits. He brought in new disciplines of geometry, algebra, topology, etc. into dynamics which had earlier been regarded purely as a study of differential equations [6]. Seeing connections between different kinds of mathematics was Poincaré's great strength. He pioneered the study of *qualitative* questions like the infinite time stability of the n-body problem.

[5] See Govindan Rangarajan, Kolmogorov-Arnold-Moser Theorem, *Resonance*, Vol.3, No.4, p.43, 1998.

[6] For example, Lagrange was proud that his 'Mecanique Analytique' did not contain a single figure!