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Atmospheric Noise on the Bispectrum in Optical Speckle Interferometry

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Abstract. Based on a simple picture of speckle phenomena in optical interferometry it is shown that the recent signal-to-noise ratio estimate for the so called bispectrum, due to Wirnitzer (1985), does not possess the right limit when photon statistics is unimportant. In this wave-limit, which is true for bright sources, his calculations over-estimate the signal-to-noise ratio for the bispectrum by a factor of the order of the square root of the number of speckles.

Key words: speckle interferometry—bispectrum—signal-to-noise ratio

1. Introduction

Though large ground-based optical telescopes have large light collecting capacity, the Earth's turbulent atmosphere degrades their resolution (smallest detectable angular separation) to a value far poorer than the Rayleigh limit corresponding to their diameter. Random density fluctuations associated with the atmospheric turbulence offer random refractive-index inhomogeneities to the incoming light waves. The telescope then behaves as if it were aberrated. These aberrations due to the atmosphere vary on both spatial (typically 10 cm) as well as temporal (about ten milliseconds) scales.

Let $I(x)$ be the focal plane distribution of intensity which is the convolution of the source structure $S(x)$ and the system (telescope + atmosphere) response $R(x)$. Equivalently, in the Fourier representation

$$I_u = R_u S_u \tag{1}$$

where u, v are spatial frequencies. As mentioned before, due to atmospheric turbulence, R_u is random in u and in time. If one integrates the image over times much longer than about ten milliseconds then the average response $\langle R_u \rangle$ is significantly different from zero only for spatial frequencies less than about one (arcsec) $^{-1}$, which corresponds to resolution of about one arcsec. A 10-m telescope, in the absence of the atmosphere, should have a resolution of about ten milliarcsec, *i.e.,* the diffraction-limited response should be significant for spatial frequencies upto 100 (arcsec) $^{-1}$.

The stochastic nature of the system response requires one to use statistical methods of image restoration. Bispectrum analysis is one such method. In this note we point out that when photon noise is unimportant, previous calculations due to Wirnitzer (1985) overestimate the signal-to-noise ratio for the bispectrum.

2. Statistical methods of image restoration

To get the information about the source structure for higher spatial frequencies it is necessary to measure those correlations of the I_u's for which corresponding correlations of the system responses, R_u' s, are nonzero right upto the diffraction limit of the telescope. Labeyrie (1970) proposed and successfully demonstrated (Gezari, Labeyrie & Stachnik 1972) the use of the second order correlation (power spectrum)

$$\langle I_u I_{-u} \rangle \tag{2}$$

in high-resolution astronomy. This form of speckle-interferometry has the limitation that it yields only the modulus of the Fourier coefficients and not their phases. It is known (Dainty & Greenaway 1979) that, for bright sources, the signal-to-noise ratio for the power spectrum measurements is of the order of unity, in one realization:

$$\mathrm{SNR}_{|I_u|} \sim 1 \tag{3}$$

To reconstruct the phase of the object Fourier transform is important and Weigelt (1977) has proposed the use of the so-called bispectrum

$$\langle I_u I_v I_{-u\,-v} \rangle \tag{4}$$

a third-order correlation, which is nonzero upto the diffraction limit of the telescope. This method, discussed in detail by Bartlet, Lohmann & Wirnitzer (1984), has been applied with some success to astronomical interferometry (Lohmann, Weigelt & Wirnitzer 1983).

Wirnitzer (1985) has calculated the SNR for the bispectrum as well as for phase restoration for general light levels. For bright sources and one frame of data his results can be summarized as follows:

$$\mathrm{SNR}_{\mathrm{Bispectrum}} \sim 1, \tag{5}$$

$$\mathrm{SNR}_{\mathrm{Phase}} \sim N^{1/2}, \tag{6}$$

where N is the average number of speckles per frame. Note (also from Fig. 3 of Wirnitzer 1985) that the SNR for the bispectrum is of the same order as that for the power spectrum. Paradoxically, it appears that the phase of a Fourier coefficient is better determined (Eq. 6) than the amplitude (Eq. 3)! This paradox is removed below.

3. Present calculations

In this section we use an extremely simplified picture of the speckle phenomenon to estimate the SNR for the bispectrum in the wave limit. The system response $R(x)$ can be considered to contain N speckles, each with roughly the diffraction-limited size, spread over an area about a square arcsec. The number of speckles is about 100 D^2 where D is the diameter of the telescope in metres. The intensity of each speckle is random, but in the following we shall regard it a constant, say I_0. The position x_j of each speckle is random and uncorrelated to every other. Thus for the system response we use

$$R(x) = I_0 \sum_{j=1}^{N} \delta(x - x_j)$$

or

$$R_u = R_0 \sum_{j=1}^{N} \exp(iux_j). \tag{7}$$

We see that the system response function in the Fourier representation, R_u, is just a sum of N uncorrelated complex numbers. Because of the assumption of randomness, the average of all these numbers is zero. In this simple picture the triple product $R_u R_v R_{-u-v}$ is given by

$$R_u R_v R_{-u-v}/R_0^3 = \sum_{j=1}^{N} \exp[iux_j] \sum_{k=1}^{N} \exp[ivx_k] \sum_{l=1}^{N} \exp[i(-u-v)x_l]$$

$$= \sum_{j=1}^{N} 1 \qquad N \text{ terms} \tag{8.1}$$

$$+ \sum_{\substack{j=k \\ j \neq l}} \exp[i(u+v)(x_j - x_l)] \qquad N(N-1)\text{terms} \tag{8.2}$$

$$+ \sum_{\substack{j=l \\ k \neq l}} \exp[iv(x_k - x_l)] \qquad N(N-1)\text{ terms} \tag{8.3}$$

$$+ \sum_{\substack{k=l \\ j \neq l}} \exp[iu(x_j - x_l)] \qquad N(N-1)\text{ terms} \tag{8.4}$$

$$+ \sum_{\substack{j,k,l \\ \text{all distinct}}} \exp[iu(x_j - x_l) + iv(x_k - x_l)] \qquad N(N-1)(N-2)\text{terms} \tag{8.5}$$

Note that the triple product $R_u\ R_u\ R_{-u-u}$ contains N deterministic terms and $N(N^2 - 1)$ random terms whose average is zero. It is then clear that for the average of the bispectrum it is the terms like (8.1) which contribute:

$$\langle R_u R_v R_{-u-v} \rangle \sim N R_0^3. \tag{9}$$

Now, consider the modulus of the bispectrum:

$(R_u\ R_v\ R_{-u-v})(R_u\ R_v\ R_{-u-v})^*$

$= (N + N(N^2 - 1)$ random terms$)$ $(N + N(N^2 - 1)$ random terms$)^*$

$= (N^2 + N(N^2 - 1)$ + cross-terms with nonzero phase factors.

The cross-terms, which have nonzero phase factors, vanish on averaging. Thus:

$$\langle |R_u R_v R_{-u-v}|^2 \rangle \sim N^2 + N(N^2 - 1) \sim N^3 \quad \text{as } N \gg 1. \tag{10}$$

This gives us the SNR for the bispectrum, apart from a factor which depends on the S_u's:

$$\text{SNR}_{\text{Bispectrum}} \sim N/N^{3/2} = N^{-1/2} \sim (10D)^{-1} \tag{11}$$

where D is the diameter of the telescope in metres. The approximation of a speckle pattern by a sum of delta functions may appear drastic and has been chosen only to illustrate the scaling with N in the simplest possible way. It is easy to see that allowance for the finite size of the speckles and the fact that their intensities and shapes vary cannot alter this scaling. Similar approximations, for a speckle image, have recently been used in high-resolution optical astronomy (see for example: Roddier 1986 and Christou *et al.* 1986) arid are known to reproduce, qualitatively, the results based on more refined models of the speckle image.

The number of independent points in the image, or number of spatial frequencies, is of the order of N. The number of points in the bispectrum, which is a function of two independent spatial frequencies, is proportional to N^2 (Wirnitzer 1985). Thus the phase information is contained redundantly in the bispectrum. This is expected (Wirnitzer 1985) to lead to a significant improvement, by a factor $N^{1/2}$ as seen from the equations 5 and 6, for recovering the phase of the object Fourier transform.

4. Conclusion

With our estimate (Eq. 11) of the SNR for the bispectrum one should then get the SNR for phase restoration as about unity which is similar to that for the amplitude, thus resolving the paradox. We emphasize that this result does not question the significance of the bispectrum for optical interferometry. A SNR of the order unity per frame in the wave limit, for phase restoration, would still make the triple correlation very useful in high-resolution astronomy. In this note, however, we do not consider further the question of phase retrieval from the bispectrum.

A brief version of these results was presented earlier (Karbelkar & Nityananda 1986).

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