MEAN MATRICES AND INFINITE DIVISIBILITY

Rajendra Bhatia[1]

and

Hideki Kosaki[2]

To Roger Horn on his 65th birthday

Abstract. We consider matrices M with entries $m_{ij} = m(\lambda_i, \lambda_j)$ where $\lambda_1, \ldots, \lambda_n$ are positive numbers and m is a binary mean dominated by the geometric mean, and matrices W with entries $w_{ij} = 1/m(\lambda_i, \lambda_j)$ where m is a binary mean that dominates the geometric mean. We show that these matrices are infinitely divisible for several much-studied classes of means.

AMS Subject Classifications: 15A45, 15A57, 42A82, 47A63.

Keywords: mean, positive definite matrix, infinitely divisible matrix, operator monotone function, Fourier transform.

[1]Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 7, S. J. S. Sansanwal Marg, New Delhi - 110 016, India. Email: rbh@isid.ac.in

[2]Graduate School of Mathematics, Kyushu University, Higashi-ku, Fukuoka 812-8581, Japan. Email: kosaki@math.kyushu-u.ac.jp

1. Introduction

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ positive semidefinite matrices. By the well-known theorem of Schur the *Hadamard product* $A \circ B = [a_{ij} b_{ij}]$ is positive semidefinite. Thus for each positive integer m, the mth *Hadamard power* $A^{\circ m} = [a_{ij}^m]$ is positive semidefinite.

Suppose A is positive semidefinite and all its entries a_{ij} are nonnegative. We say A is *infinitely divisible* if for every real number $r \geq 0$ the matrix $A^{\circ r} = [a_{ij}^r]$ is positive semidefinite. By Schur's theorem and continuity A is infinitely divisible if and only if every *fractional Hadamard power* $A^{\circ 1/m}$ is positive semidefinite.

It is easy to see that every 2×2 positive semidefinite matrix with nonnegative entries is infinitely divisible. This is not always the case for matrices of order $n > 2$. We refer the reader to some old papers [H2], [H3] on infinitely divisible matrices and the recent work [B2] where diverse examples of such matrices are given.

The motivation for this paper stems from the following observation. Let $\lambda_1, \ldots, \lambda_n$ be any given positive numbers. Consider the matrices A whose entries are given by one of the following rules:

$$\begin{aligned}
a_{ij} &= \min(\lambda_i, \lambda_j), \\
a_{ij} &= \frac{1}{\max(\lambda_i, \lambda_j)}, \\
a_{ij} &= H(\lambda_i, \lambda_j), \\
a_{ij} &= \frac{1}{A(\lambda_i, \lambda_j)}, \\
a_{ij} &= \sqrt{\lambda_i \lambda_j},
\end{aligned}$$

where $H(\lambda_i, \lambda_j)$ is the harmonic mean of λ_i and λ_j, and $A(\lambda_i, \lambda_j)$ their arithmetic mean. Then all these five matrices are infinitely divisible. How general is this phenomenon?

A binary operation m on positive numbers is called a *mean* if it satisfies the following conditions:

(i) $m(a,b) = m(b,a)$.

(ii) $\min(a,b) \leq m(a,b) \leq \max(a,b)$.

(iii) $m(\alpha a, \alpha b) = \alpha m(a,b)$ for all $\alpha > 0$.

(iv) $m(a,b)$ is an increasing function of a and b.

(v) $m(a,b)$ is a continuous function of a and b.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be positive numbers and let $m(a,b)$ be a mean. Suppose $m(a,b) \leq \sqrt{ab}$ for all a and b. Let M be the matrix with entries

$$m_{ij} = m(\lambda_i, \lambda_j).$$

On the other hand, suppose $\sqrt{ab} \leq m(a,b)$ for all a and b. Then let W be the matrix with entries

$$w_{ij} = \frac{1}{m(\lambda_i, \lambda_j)}.$$

Are the matrices M and W infinitely divisible? We will see that this is the case for several families of means. However, the domination criterion vis a vis the geometric mean is not sufficient to guarantee infinite divisibility of these matrices and we give an example to show that.

Some of the key ideas used here occur in our earlier work, especially in the papers of Bhatia and Parthasarathy [BP] and Hiai and Kosaki [HK2] . One of them is the use of "congruence transformations": if X is a diagonal matrix with positive diagonal entries then the two matrices C and XCX are positive definite (infinitely divisible, respectively) at the same time. Another is the use of positive definite functions. A (complex-valued) function f on $\mathbf{R}$ is said to be positive definite if for all choices of n real numbers $\lambda_1, \ldots, \lambda_n$ the $n \times n$ matrices $[f(\lambda_i - \lambda_j)]$ are positive semidefinite. We will say that f is *infinitely divisible* if for every $r \geq 0$ the function $(f(x))^r$ is positive definite.

We will use a theorem of Roger Horn [H1] on operator monotone functions. We refer the reader to [B1, Chapter V] for the theory of such functions. One of the key facts is that a (differentiable) function $f : [0,\infty) \to [0,\infty)$ is operator monotone if and only if for all choices of n positive numbers $\lambda_1, \dots, \lambda_n$, the $n \times n$ matrices

$$\left[\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\right] \tag{1}$$

are positive semidefinite. (If $\lambda_i = \lambda_j$, the difference quotient is taken to mean $f'(\lambda_i)$.) This was proved by C. Loewner and the matrices in (1) are called *Loewner matrices*. Another theorem of Loewner says that f is operator monotone if and only if it has an analytic continuation to a mapping of the upper half-plane into itself. Horn [H1] showed that this analytic continuation is a one-to-one (also called *univalent* or *schlicht*) map if and only if all Loewner matrices (1) are infinitely divisible.

The matrix E all whose entries are equal to one is called the *flat matrix*. This is clearly infinitely divisible. Hence, if $G(\lambda_i, \lambda_j)$ represents the geometric mean of λ_i and λ_j, then the matrices $[G(\lambda_i, \lambda_j)]$ and $[1/G(\lambda_i, \lambda_j)]$ both are infinitely divisible. As a consideration of 2×2 matrices shows, for no other mean can these two matrices be positive definite at the same time.

A matrix C whose entries are

$$c_{ij} = \frac{1}{\lambda_i + \lambda_j},$$

is called a *Cauchy matrix*. This is an infinitely divisible matrix. See [B2] for different proofs of this fact. From this it follows that the matrix W with entries

$$w_{ij} = \frac{1}{A(\lambda_i, \lambda_j)},$$

where $A(.,.)$ represents the arithmetic mean is infinitely divisible, as is the matrix M with entries

$$m_{ij} = H(\lambda_i, \lambda_j),$$

where H represents the harmonic mean. This fact about Cauchy matrices will be used again in the next section.

2. Examples

2.1. **The logarithmic mean.** The *logarithmic mean* $L(a,b)$ is defined as

$$L(a,b)=\begin{cases}\frac{a-b}{\log a-\log b}, & (a\neq b),\\ a & (a=b).\end{cases}$$

We have $\sqrt{ab}\leq L(a,b)\leq\frac{1}{2}(a+b)$, which is a refinement of the arithmetic-geometric mean inequality. The matrix W with entries

$$w_{ij}=\frac{1}{L(\lambda_i,\lambda_j)}=\frac{\log\lambda_i-\log\lambda_j}{\lambda_i-\lambda_j} \tag{2}$$

is the Loewner matrix of the schlicht function $\log z$ mapping the upper half-plane into itself. Hence, by the theorem of Horn [H1] this matrix is infinitely divisible. We will see other proofs of this fact later in this paper.

Another representation for the mean L is given by the integral formula

$$\frac{1}{L(a,b)}=\int_0^\infty\frac{dt}{(t+a)(t+b)}.$$

For each $t\geq 0$, the matrix with entries

$$\frac{1}{(t+\lambda_i)(t+\lambda_j)}$$

is congruent to the flat matrix, and is thus positive definite (and infinitely divisible). It follows immediately that the matrix (2) is positive definite.

It was observed in [BP] that the positive definiteness of all matrices (2) is equivalent to the function

$$f(x)=\frac{x}{\sinh\, x}$$

being positive definite. The same argument now shows that this function is infinitely divisible.

2.2. **The Heinz means.** For $0 \le \nu \le 1$, the *Heinz mean* is defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}.$$

For each pair (a,b) of positive numbers the function $H_\nu(a,b)$ of ν is symmetric about the point $\nu = 1/2$ and attains its minimum value there. The minimum value is $H_{1/2}(a,b) = \sqrt{ab}$. The maximum value is $H_0(a,b) = H_1(a,b) = \frac{1}{2}(a+b)$. For $0 \le \nu \le 1/2$ let W be the matrix with entries

$$\begin{aligned} w_{ij} &= \frac{1}{H_\nu(\lambda_i,\lambda_j)} = \frac{2}{\lambda_i^\nu \lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^\nu} \\ &= \frac{2}{\lambda_i^\nu \left(\lambda_i^{1-2\nu} + \lambda_j^{1-2\nu}\right)\lambda_j^\nu}. \end{aligned}$$

Then $W = XCX$, where X is a positive diagonal matrix and C is a Cauchy matrix. Hence W is infinitely divisible.

2.3. **The Binomial means.** The binomial means also called *power means,* are defined as

$$B_\alpha(a,b) = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{1/\alpha}, \quad -\infty \le \alpha \le \infty.$$

It is understood that

$$\begin{aligned} B_0(a,b) &= \lim_{\alpha\to 0} B_\alpha(a,b) = \sqrt{ab}, \\ B_\infty(a,b) &= \lim_{\alpha\to\infty} B_\alpha(a,b) = \max(a,b), \\ B_{-\infty}(a,b) &= \lim_{\alpha\to -\infty} B_\alpha(a,b) = \min(a,b). \end{aligned}$$

For fixed a and b the function $B_\alpha(a,b)$ is increasing in α. Further

$$B_{-\alpha}(a,b) = \frac{ab}{B_\alpha(a,b)}. \tag{3}$$

For $\alpha \ge 0$ let W be the matrix with entries

$$w_{ij} = \frac{1}{B_\alpha(\lambda_i,\lambda_j)} = \frac{2^{1/\alpha}}{\left(\lambda_i^\alpha + \lambda_j^\alpha\right)^{1/\alpha}}.$$

This matrix is infinitely divisible since every Cauchy matrix has that property. The relation (3) then shows that for each $\alpha \geq 0$ the matrix M with entries

$$m_{ij} = B_{-\alpha}(\lambda_i, \lambda_j)$$

is also infinitely divisible.

2.4. **The Lehmer means.** This family is defined as

$$L_p(a,b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}}, \quad -\infty \leq p \leq \infty.$$

The special values $p = 0, 1/2$, and 1 give the harmonic, geometric, and arithmetic means, respectively. For fixed a and b, the function $L_p(a,b)$ is an increasing function of p. We have

$$\begin{aligned} L_\infty(a,b) &= \lim_{p \to \infty} L_p(a,b) = \max(a,b), \\ L_{-\infty}(a,b) &= \lim_{p \to -\infty} L_p(a,b) = \min(a,b). \end{aligned}$$

A small calculation shows that

$$L_{1-p}(a,b) = \frac{ab}{L_p(a,b)}. \tag{4}$$

We will show that for each $p \geq 1/2$ the matrix W with entries

$$w_{ij} = \frac{1}{L_p(\lambda_i, \lambda_j)} = \frac{\lambda_i^{p-1} + \lambda_j^{p-1}}{\lambda_i^p + \lambda_j^p}, \tag{5}$$

is infinitely divisible.

First, observe that it is enough to prove this for $p \geq 1$, because that would say that every matrix of the form

$$\left[\frac{\lambda_i^\nu + \lambda_j^\nu}{\lambda_i + \lambda_j} \right], \quad 0 < \nu < 1, \tag{6}$$

is infinitely divisible. If $1/2 \leq p \leq 1$, we let $r = 1 - p$, and note that $0 \leq r \leq 1/2$. The expression (5) in this case can be written as

$$w_{ij} = \frac{\lambda_i^{-r} + \lambda_j^{-r}}{\lambda_i^p + \lambda_j^p} = \frac{1}{\lambda_i^r} \, \frac{\lambda_i^r + \lambda_j^r}{\lambda_i^p + \lambda_j^p} \, \frac{1}{\lambda_j^r}.$$

Since $r/p \leq 1$, the infinite divisibility of this last matrix W follows from that of (6).

Observe further that if the matrices in (5) have been proved to be infinitely divisible for $p \geq 1/2$, then the relation (4) can be used to show that for each $p \leq 1/2$, the matrix M with entries

$$m_{ij} = L_p(\lambda_i, \lambda_j)$$

is infinitely divisible. Thus we may restrict our attention to the matrices in (6).

Following the ideas in [BP] we make the substitution $\lambda_i = e^{x_i}$, and then write the entries of (6) as

$$\frac{e^{\nu x_i} + e^{\nu x_i}}{e^{x_i} + e^{x_j}} = \frac{e^{\nu x_i/2}}{e^{x_i/2}} \; \frac{e^{\nu(x_i - x_j)/2} + e^{\nu(x_j - x_i)/2}}{e^{(x_i - x_j)/2} + e^{(x_j - x_i)/2}} \; \frac{e^{\nu x_j/2}}{e^{x_j/2}}.$$

Thus the matrix in (6) is infinitely divisible if and only if the matrix

$$\left[\frac{\cosh\, \nu(x_i - x_j)}{\cosh\,(x_i - x_j)}\right], \quad 0 < \nu < 1,$$

is infinitely divisible. This is equivalent to the statement of the following theorem:

Theorem 1. For $0 < \nu < 1$ the function

$$f(x) = \frac{\cosh\, \nu x}{\cosh\, x}$$

is infinitely divisible.

Proof. We will show that for $a, b > 0$ the function

$$\frac{\cosh\, bx}{\cosh(a+b)x}$$

is infinitely divisible. Using the identity

$$\cosh\,(a+b)x = 2\, \cosh\, ax\, \cosh\, bx - \cosh\,(a-b)x$$

we obtain

$$\frac{\cosh\, bx}{\cosh\,(a+b)x} = \frac{1}{2\, \cosh\, ax} \; \frac{1}{1 - \frac{\cosh\,(a-b)x}{2\, \cosh\, ax\, \cosh\, bx}}. \tag{7}$$

Let r be any real number in $(0,1)$. Then for $|t|<1$ we have the power series expansion

$$\frac{1}{(1-t)^r} = \sum_{n=0}^{\infty} a_n t^n,$$

where the coefficients a_n are the nonnegative numbers given by $a_0 = 1$ and

$$a_n = \frac{r(r+1)(r+2)\cdots(r+m+1)}{m!}, \quad m > 1.$$

Hence we have from (7)

$$(8) \qquad \left(\frac{\cosh\, bx}{\cosh(a+b)x}\right)^r = \frac{1}{2^r(\cosh\, ax)^r}\sum_{n=0}^{\infty}\frac{a_n}{2^n}\frac{\cosh^n(a-b)x}{\cosh^n\, ax\, \cosh^n\, bx}.$$

We already know that the function $1/\cosh(x)$ is infinitely divisible. So the factor outside the summation in (8) is positive definite. We know also that for $0 \le \nu \le 1$, the function $\cosh(\nu x)/\cosh(x)$ is positive definite. Consider each of the summands in (8). Depending on whether $a \ge b$ or $a \le b$, one of

$$\frac{\cosh(a-b)x}{\cosh\, ax} \quad \text{and} \quad \frac{\cosh(a-b)x}{\cosh\, bx}$$

is positive definite. Hence, in either case

$$\frac{\cosh(a-b)x}{\cosh\, ax\, \cosh\, bx}$$

is positive definite, and so are all its nth powers. Thus the series in (8) represents a positive definite function for $0<r<1$. This is enough to show that the function in (7) is infinitely divisible. ■

2.5. **Power difference means.** This is not a standard terminology for the following family of means that are of interest and have been studied in detail in [HK2] and [HK3]. For any real number p let

$$K_p(a,b) = \frac{p-1}{p}\;\frac{a^p - b^p}{a^{p-1} - b^{p-1}}.$$

It is understood that

$$K_p(a,b) = a.$$

For fixed a and b, the quantity $K_p(a,b)$ is an increasing function of p. This family includes some of the most familiar means:

$$\begin{aligned}
K_{-\infty}(a,b) &= \min(a,b), \\
K_{-1}(a,b) &= \frac{2}{a^{-1}+b^{-1}}, \quad \text{the harmonic mean,} \\
K_{1/2}(a,b) &= \sqrt{ab}, \quad \text{the geometric mean,} \\
K_1(a,b) &= \lim_{p\to 1} \ K_p(a,b) = \frac{a-b}{\log\, a - \log\, b}, \\
& \qquad\qquad\qquad \text{the logarithmic mean,} \\
K_2(a,b) &= \frac{a+b}{2}, \quad \text{the arithmetic mean,} \\
K_{\infty}(a,b) &= \max(a,b).
\end{aligned}$$

The analysis of these means is very similar to that of Lehmer means.

A small calculation shows that

$$K_{1-p}(a,b) = \frac{ab}{K_p(a,b)}, \tag{9}$$

and as for Lehmer means it is enough to show that for $p > 1$, the matrix W with entries

$$w_{ij} = \frac{1}{K_p(\lambda_i,\lambda_j)} = \frac{p}{p-1} \ \frac{\lambda_i^{p-1}-\lambda_j^{p-1}}{\lambda_i^p - \lambda_j^p} \tag{10}$$

is infinitely divisible. (The reader can check that from this it follows that this matrix is infinitely divisible also for $1/2 \le p < 1$; and then using the relation (9) one can see that for $p \le 1/2$, the matrix M with entries $m_{ij} = K_p(\lambda_i,\lambda_j)$ is infinitely divisible.)

So consider the matrix (10) with $p > 1$. This is infinitely divisible if every matrix of the form

$$\left[\frac{\lambda_i^\nu - \lambda_j^\nu}{\lambda_i - \lambda_j}\right], \quad 0 < \nu < 1, \tag{11}$$

is infinitely divisible. We can prove this by appealing to Horn's theorem cited earlier. Alternately, we can follow our analysis in Section 2.5. Now the function cosh is replaced by sinh and we have the following theorem in place of Theorem 1. We note that this theorem can be deduced from Horn's theorem on schicht maps, but we give a direct proof akin to our proof of Theorem 1.

Theorem 2. For $0 < \nu < 1$ the function

$$g(x) = \frac{\sinh\ \nu x}{\sinh\ x} \tag{12}$$

is infinitely divisible.

Proof. Use the identity

$$\sinh(a+b)x = 2\sinh\ ax\ \cosh\ bx - \sinh(a-b)x$$

to obtain

$$\frac{\sinh\ ax}{\sinh(a+b)x} = \frac{1}{2\cosh\ bx}\ \frac{1}{1 - \frac{\sinh(a-b)x}{2\sinh\ ax\ \cosh\ bx}}.$$

Let $0 \le b \le a$ and $0 < r < 1$. We have the expansion

$$\left(\frac{\sinh\ ax}{\sinh(a+b)x}\right)^r = \frac{1}{2^r \cosh^r\ bx} \sum_{n=0}^{\infty} \frac{a_n}{2^n} \frac{\sinh^n(a-b)x}{\sinh^n\ ax\ \cosh^n\ bx}. \tag{13}$$

Compare this with (8). We know that the function $\sinh(\nu x)/\sinh(x)$ is positive definite for $0 < \nu < 1$. See [BP]. Thus the argument used in the proof of Theorem 1 shows that (13) represents a positive definite function. Since we assumed $0 \le b \le a$, this shows that the function (12) is infinitely divisible for $1/2 \le \nu \le 1$. But if ν is any number in $(0,1)$ we can choose a sequence

$$\nu = \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_m = 1$$

with $\nu_i/\nu_{i+1} \ge 1/2$. Then

$$\frac{\sinh \nu x}{\sinh x} = \prod_{i=0}^{m-1} \frac{\sinh \nu_i x}{\sinh \nu_{i+1} x}$$

is infinitely divisible since each factor in the product has that property.

■

Taking the limit $\nu \downarrow 0$ of the function $\dfrac{\sinh \nu x}{\nu \sinh x}$ we get from Theorem 2 another proof of the fact that the function $\dfrac{x}{\sinh x}$ is infinitely divisible.

2.6. **Stolarsky means.** Another favourite family of mean theorists is the class of *Stolarsky means* defined for $-\infty < \gamma < \infty$ as

$$S_\gamma(a,b) = \left(\frac{a^\gamma - b^\gamma}{\gamma(a-b)}\right)^{1/(\gamma-1)} = \left(\frac{1}{b-a}\int_a^b t^{\gamma-1}dt\right)^{1/(\gamma-1)}.$$

For fixed a and b, $S_\gamma(a,b)$ is an increasing function of γ. Some special values are

$$\begin{aligned} S_2(a,b) &= \frac{a+b}{2}, \quad \text{the arithmetic mean,} \\ S_0(a,b) &= \frac{a-b}{\log a - \log b}, \quad \text{the logarithmic mean,} \\ S_{-1}(a,b) &= \sqrt{ab}, \quad \text{the geometric mean.} \end{aligned}$$

It is understood that

$$S_1(a,b) = \lim_{\gamma\to 1} \; S_\gamma(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}.$$

This is called the *identric mean* of a and b.

This family too leads to infinitely divisible matrices. Consider first the case $\gamma > 1$, and the matrix W with entries

$$w_{ij} = \frac{1}{S_\gamma(\lambda_i,\lambda_j)} = \left(\frac{\gamma(\lambda_i - \lambda_j)}{\lambda_j^\gamma - \lambda_j^\gamma}\right)^{1/(\gamma-1)}. \tag{14}$$

From the result proved in Section 2.5 the matrix

$$\left[\frac{\lambda_i - \lambda_j}{\lambda_i^\gamma - \lambda_j^\gamma}\right]$$

is infinitely divisible, and therefore so is the matrix W in (14). Next let $0 < \gamma < 1$ and consider the matrix W whose entries are

$$w_{ij} = \frac{1}{S_\gamma(\lambda_i,\lambda_j)} = \left(\frac{\lambda_i^\gamma - \lambda_j^\gamma}{\gamma(\lambda_i - \lambda_j)}\right)^{1/(1-\gamma)}.$$

Again, by the infinite divisibility of (11) this matrix too has that property. Now consider the case $-1 < \gamma < 0$. Then $\gamma = -\delta$, where $0 < \delta < 1$. The matrix W with entries

$$w_{ij} = \frac{1}{S_\gamma(\lambda_i, \lambda_j)} = \left(\frac{\lambda_i^\delta - \lambda_j^\delta}{\delta(\lambda_i - \lambda_j)\lambda_i^\delta \lambda_j^\delta} \right)^{1/(\delta+1)}, \tag{15}$$

is a positive Hadamard power of a matrix of the form XLX, where X is a positive diagonal matrix and L is a Loewner matrix of the form

$$\left[\frac{\lambda_i^\delta - \lambda_j^\delta}{\lambda_i - \lambda_j} \right].$$

This matrix is infinitely divisible, and therefore so is the matrix W in (15).

Finally, let $\gamma < -1$. Then $\gamma = -\delta$ where $\delta > 1$. Let M be the matrix with entries

$$m_{ij} = S_\gamma(\lambda_i, \lambda_j) = \left(\frac{\delta \lambda_i^\delta \lambda_j^\delta (\lambda_i - \lambda_j)}{\lambda_i^\delta - \lambda_j^\delta} \right)^{1/(\delta+1)}.$$

The arguments in the earlier cases can be applied again to show that this matrix is infinitely divisible.

2.7. **Heron means.** The pattern established by our examples so far is broken by this family of means defined as

$$F_\alpha(a, b) = (1 - \alpha)\sqrt{ab} + \alpha \, \frac{a + b}{2}, \;\; 0 \le \alpha \le 1.$$

This is the linear interpolant between the geometric and the arithmetic means, and each member of this family dominates the geometric mean. Let W be the matrix with entries

$$w_{ij} = \frac{1}{F_\alpha(\lambda_i, \lambda_j)} = \frac{2}{\alpha(\lambda_i + \lambda_j) + 2(1 - \alpha)\sqrt{\lambda_i \lambda_j}}. \tag{16}$$

The question of positive definiteness of such matrices has been studied in [B3]. Changing variables, this reduces to the question: for what values of t is the matrix V with entries

$$v_{ij} = \frac{1}{\lambda_i^2 + \lambda_j^2 + t\lambda_i\lambda_j} \tag{17}$$

infinitely divisible? It has been observed in [B2] that V is infinitely divisible for $-2 < t \le 2$. When $n = 2$, the matrix V is known to be positive definite for all $t > -2$; hence it is infinitely divisible as well. In general, however, a matrix of the form V need not be positive definite for $t > 2$. See [BP].

Returning to (16), we can conclude from the discussion above that the matrix W is infinitely divisible for $1/2 \le \alpha \le 1$. However, when $0 < 1/2 < \alpha$ not all such matrices are positive definite, even though the mean F_α dominates the geometric mean.

As observed in [BP], the positive definiteness of all matrices V of the form (17) for $-2 < t \le 2$ is equivalent to the positive definiteness of the function

$$f(x) = \frac{1}{\cosh\, x + t}, \quad -1 < t \le 1. \tag{18}$$

The infinite divisibility of the matrices V shows that this function is, in fact, infinitely divisible. We discuss this again in Section 3.

3. Further results and remarks.

More theorems on positive definiteness and infinite divisibility can be obtained from the examples in Section 2. As in our earlier work, Schur's theorem, congruence, positive definite functions, and hyperbolic functions play an important role.

Theorem 3. The function

$$f(x) = \frac{x\, \cosh\, ax}{\sinh\, x} \tag{19}$$

is infinitely divisible for $-1/2 \le a \le 1/2$.

Proof. Making the substitution $\lambda_i = e^{x_i}$, the matrix W in (2) may be written as

$$w_{ij} = \frac{x_i - x_j}{e^{x_i} - e^{x_j}} = \frac{1}{e^{x_i/2}} \, \frac{(x_i - x_j)/2}{\sinh(x_i - x_j)/2} \, \frac{1}{e^{x_j/2}}.$$

So, the infinite divisibility of W implies that the function $x/\sinh x$ is infinitely divisible. The identity

$$\frac{x \ \cosh\, ax}{\sinh\, x} = \frac{x/2}{\sinh\, x/2} \, \frac{\cosh\, ax}{\cosh\, x/2}$$

displays $f(x)$ as the product of two functions, the first of which is infinitely divisible, and by Theorem 1 so is the second, provided $-1/2 \leq a \leq 1/2$. ■

In [BP], [K1], [HK1] and [HK2] the positive definiteness of functions like (19) was used to obtain inequalities for norms of operators. The next corollary of Theorem 3 is a refinement of some of these. Here $|||\cdot|||$ stands for a unitarily invariant norm (see [B1, Chap.IV] for instance).

Corollary. Let A and B be positive definite matrices and let X be any matrix. Then for $1/4 \leq \nu \leq 3/4$ we have

$$\frac{1}{2}|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leq |||\int_0^1 A^t X B^{1-t} dt|||. \tag{20}$$

Proof. As explained in [BP] and [HK1], this inequality is a consequence of the positive definiteness of the matrix V with entries

$$v_{ij} = \ \frac{\lambda_i^{\nu}\lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^{\nu}}{2} \, \frac{\log\lambda_i - \log\lambda_j}{\lambda_i - \lambda_j}$$

for $1/4 \leq \nu \leq 3/4$. Making the substitution $\lambda_i = e^{x_i}$, a small calculation shows

$$v_{ij} = \frac{(\lambda_i - \lambda_j)/2 \cdot \ \cosh((2\nu - 1)(\lambda_i - \lambda_j)/2)}{\sinh(\lambda_i - \lambda_j)/2}.$$

The positive definiteness of all such matrices is equivalent to the function (19) being positive definite. ■

To put the inequality (20) in perspective, let us recall the generalised Heinz inequality proved by Bhatia and Davis [BD1]:

$$|||A^{1/2}XB^{1/2}||| \leq \frac{1}{2}|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leq \frac{1}{2}|||AX + XB|||$$

for $0 \leq \nu \leq 1$; and the operator arithmetic-logarithmic-geometric mean inequality proved by Hiai and Kosaki [HK1]

$$|||A^{1/2}XB^{1/2}||| \leq |||\int_0^1 A^t XB^{1-t}dt||| \leq \frac{1}{2}|||AX + XB|||.$$

The inequality (20) is a refinement of these two.

The next two propositions are generalisations of Theorems 1 and 2, respectively.

Proposition 4. Let $\nu_1, \nu_2, \ldots, \nu_n$ be nonnegative real numbers and suppose $\sum_{i=1}^n \nu_i \leq 1$. Then the function

$$f(x) = \frac{\prod_{i=1}^n \cosh(\nu_i x)}{\cosh\, x}$$

is infinitely divisible. In particular, if n and m are positive integers with $n \geq m$, then the function $\cosh^m\, x/\cosh\,(nx)$ is infinitely divisible.

Proof. We use induction on n. The case $n = 1$ is covered by Theorem 1. The equation (8) can be written in another form as

$$\left(\frac{\cosh\,\nu_1\, x}{\cosh\, x}\right)^r = \frac{2^{-r}}{\cosh\,(1-\nu_1)x}\sum_{n=0}^{\infty}\frac{a_n}{2^n}\,\frac{\cosh^n\,(1-2\nu_1)x}{\cosh^n(1-\nu_1)x\,\cosh^n\,\nu_1\, x}.$$

Multiply both sides of this equation by $\left(\prod_{i=2}^n \cosh\,\nu_i\, x\right)^r$ to get

$$\left(\frac{\prod_{i=1}^n \cosh\,\nu_i x}{\cosh\, x}\right)^r = 2^{-r}\left(\frac{\prod_{i=2}^n \cosh\nu_i x}{\cosh\,(1-\nu_1)x}\right)^r \sum_{n=0}^{\infty}\frac{a_n}{2^n}\,\frac{\cosh^n\,(1-2\nu_1)x}{\cosh^n(1-\nu_1)x\,\cosh^n\,\nu_1 x}.$$

Since $\sum_{i=2}^n \nu_i \leq 1-\nu_1$, the induction hypothesis implies that

$$\left(\frac{\prod_{i=2}^n \cosh\nu_i x}{\cosh\,(1-\nu_1)x}\right)^r$$

is positive definite. The infinite divisibility of f can now be deduced by repeating the arguments in Theorem 1. ∎

Proposition 5. Let $\nu_0, \nu_1, \ldots, \nu_n$ be nonnegative real numbers. Suppose $\sum_{i=0}^n \nu_i \le 1$ and $\sum_{i=1}^n \nu_i \le 1/2$. Then the function

$$f(x) = \frac{\sinh\, \nu_0 x\, \prod_{i=1}^n \cosh\, \nu_i x}{\sinh\, x} \tag{21}$$

is infinitely divisible.

Proof. The function f can be expressed as

$$f(x) = \frac{\sinh\, \nu_0 x}{\sinh\,(1 - \sum_{i=1}^n \nu_i)\, x}\; \frac{\sinh\,(1 - \sum_{i=1}^n \nu_i)\, x\ \ \prod_{i=1}^n \cosh\, \nu_i x}{\sinh\ x}.$$

The given conditions imply that $\nu_0 \le 1 - \sum_{i=1}^n \nu_i$. So, by Theorem 2 the first factor in the product above is infinitely divisible. So to prove the infinite divisibility of the function (21) we may, and do, assume that $\nu_0 = 1 - \sum_{i=1}^n \nu_i$. Then, we have $\nu_0 \ge 1/2$ by the given conditions. As in the proof of Theorem 2, we have instead of (13) the equality

$$\left(\frac{\sinh\, \nu_0 x}{\sinh\, x}\right)^r = \frac{2^{-r}}{\cosh^r(1-\nu_0)x} \sum_{n=0}^{\infty} \frac{a_n}{2^n} \frac{\sinh^n(2\nu_0 - 1)x}{\sinh^n\, \nu_0 x\ \cosh^n\,(1-\nu_0)x}.$$

Hence

$$\begin{aligned}\left(\frac{\sinh\, \nu_0 x\, \prod_{i=1}^n \cosh\, \nu_i x}{\sinh\, x}\right)^r \\ = 2^{-r} \left(\frac{\prod_{i=1}^n \cosh\, \nu_i x}{\cosh\,(1-\nu_0)x}\right)^r \sum_{n=0}^{\infty} \frac{a_n}{2^n} \frac{\sinh^n(2\nu_0 - 1)x}{\sinh^n\, \nu_0 x\ \cosh^n(1-\nu_0)x}.\end{aligned}$$

The factor outside the summation is positive definite by Proposition 4. The function represented by the infinite sum above is positive definite by the argument used for the sum in (13). Hence f is infinitely divisible. ∎

Remark. The requirements in Proposition 5 are optimal: it is known that if $a, b \ge 0$ and $a + b \le 1$, then the function

$$\frac{\sinh\, ax\ \cosh\, bx}{\sinh\, x}$$

is positive definite if and only if $b \le 1/2$. See [K2].

We observed in Section 2 that the function (18) is infinitely divisible. This may be concluded also by a calculation of Fourier transforms that may have independent interest.

Proposition 6. The Fourier transform of the function

$$f(x) = \frac{1}{(\cosh\, x + t)^r}, \quad -1 < t < 1, \quad 0 < r < 1$$

is given by the formula

$$\hat{f}(\xi) = \frac{2\sin \pi r}{\sinh \pi\xi}\left[\int_0^{\arccos t} \frac{\sinh(\alpha\xi)\, d\alpha}{(\cos\alpha - t)^r} + \int_0^{\infty} \frac{\sin(\alpha\xi)\, d\alpha}{(\cosh\, \alpha - t)^r}\right]. \tag{22}$$

Proof. We use the well-known integral

$$x^r = \frac{\sin\, \pi r}{\pi}\int_0^\infty \frac{x}{x+\lambda}\frac{d\lambda}{\lambda^{1-r}}, \quad x \ge 0$$

to write f as

$$\begin{aligned} f(x) &= \frac{\sin\, \pi r}{\pi}\int_0^\infty \frac{(\cosh\, x+t)^{-1}}{(\cosh\, x+t)^{-1}+\lambda}\frac{d\lambda}{\lambda^{1-r}} \\ &= \frac{\sin\, \pi r}{\pi}\int_0^\infty \frac{1}{\lambda(\cosh\, x+t)+1}\frac{d\lambda}{\lambda^{1-r}} \\ &= \frac{\sin\, \pi r}{\pi}\int_0^\infty \frac{1}{\cosh\, x+t+\frac{1}{\lambda}}\frac{d\lambda}{\lambda^{2-r}} \\ &= \frac{\sin\, \pi r}{\pi}\Big[\int_0^{\frac{1}{1-t}} \frac{1}{\cosh\, x+t+\frac{1}{\lambda}}\frac{d\lambda}{\lambda^{2-r}} \\ &\qquad\qquad + \int_{\frac{1}{1-t}}^\infty \frac{1}{\cosh\, x+t+\frac{1}{\lambda}}\frac{d\lambda}{\lambda^{2-r}}\Big]. \end{aligned} \tag{23}$$

The quantity $t+1/\lambda$ appearing in the denominators decreases from ∞ to 1 as λ varies from 0 to $1/(1-t)$, and it decreases from 1 to t as λ varies from $1/(1-t)$ to ∞. Change variables by putting

$$u = t + \frac{1}{\lambda} \quad \text{(and hence } du = -\frac{d\lambda}{\lambda^2},\ \lambda = (u-t)^{-1}).$$

Then we obtain from (23)

$$f(x) = \frac{\sin \pi r}{\pi}\Big[\int_1^\infty \frac{1}{\cosh x + u}\,\frac{du}{(u-t)^r} + \int_t^1 \frac{1}{\cosh x + u}\,\frac{du}{(u-t)^r}\Big]. \tag{24}$$

Using Fubini's theorem we get from (24)

$$\hat{f}(\xi) = \frac{\sin \pi r}{\pi}\left[I_1 + I_2\right],$$

where

$$I_1 = \int_1^\infty \hat{g}(\xi)\frac{du}{(u-t)^r}, \quad I_2 = \int_t^1 \hat{g}(\xi)\frac{du}{(u-t)^r}. \tag{25}$$

The Fourier transform of g is known; see e.g. [BD2] Section 3. When $u > 1$ we have

$$\hat{g}(\xi) = \frac{2\pi}{\sqrt{u^2-1}}\frac{\sin(\xi \operatorname{arccosh} u)}{\sinh \pi\xi}.$$

Put this into (25) and then change the variable u to $\cosh\,\alpha$. This gives

$$I_1 = \frac{2\pi}{\sinh\,\pi\xi}\int_0^\infty \frac{\sin\,\alpha\xi}{(\cosh\,\alpha - t)^r}\,d\alpha.$$

When $-1 < u < 1$ we have

$$\hat{g}(\xi) = \frac{2\pi}{\sqrt{1-u^2}}\,\frac{\sinh(\xi\,\arccos\,u)}{\sinh\,\pi\xi}.$$

Put this expression into (25) and then change the variable u to $\cos\alpha$. This gives

$$I_2 = \frac{2\pi}{\sinh\,\pi\xi}\int_0^{\arccos\,t} \frac{\sinh\;\alpha\xi}{(\cos\alpha - t)^r}\,d\alpha.$$

Putting everything together we get the formula (22). ■

We claim that $\hat{f}(\xi) \geq 0$ for all ξ. Being the Fourier transform of the even function $f(x)$, $\hat{f}$ is even. Hence it suffices to show that $\hat{f}(\xi) \geq 0$ for all $\xi > 0$. Consider, one by one, the quantities occurring on the right hand side of (22). The factor outside the brackets is clearly positive. So is the first of the two integrals. For fixed ξ and t, the function $(\cosh\,\alpha - t)^{-r}$ decreases monotonically as α increases while $\sin\,\alpha\xi$ is oscillatory. Hence the second integral in (22) is also positive. Thus $\hat{f}(\xi) \geq 0$.

It follows from Bochner's theorem that the function f of Proposition 6 is positive definite. Hence the function (18) is infinitely divisible.

We end this section with a few remarks and questions.

In the earlier works [BP], [HK3], several ratios of means have been studied and many matrices arising from these have been proved to be positive definite. It seems most of them are also infinitely divisible. Several more examples using computations with Fourier transforms will appear in the paper by Kosaki [K2]. In a recent paper Drissi [D] has shown that the function in (19) is positive definite if and only if $-1/2 \leq a \leq 1/2$. His argument too is based on a calculation of Fourier transforms.

Two general questions are suggested by our work. Let $\mathcal{L}_{\pm}$ be the classes of all differentiable functions from $[0, \infty)$ into itself for which all matrices of the form

$$\left[\frac{f(\lambda_i) \pm f(\lambda_j)}{\lambda_i \pm \lambda_j}\right]$$

are positive definite. Let $\mathcal{M}_{\pm}$ be the classes consisting of those f for which all these matrices are infinitely divisible.

The class $\mathcal{L}_-$ is the Loewner class and consists of all operator monotone functions. Horn's theorem says that $\mathcal{M}_-$ consists of those functions in $\mathcal{L}_-$ whose analytic continuations map the upper half-plane into itself univalently. It is known that $\mathcal{L}_- \subset \mathcal{L}_+$. (See [K] or [BP].)

Question 1. Is $\mathcal{M}_- \subset \mathcal{M}_+$?

Question 2. Are there any good characterisations of the classes $\mathcal{L}_+$ and $\mathcal{M}_+$? (The theroems of Loewner and Horn give interesting descriptions of $\mathcal{L}_-$ and $\mathcal{M}_-$, respectively.)

4. Appendix

We have the well-known formula

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}\,dx}{\cosh^r x} = \frac{2^{r-1}|\Gamma((r+i\xi)/2)|^2}{\Gamma(r)}. \tag{26}$$

See e.g. [O, p.33] and [HK3, p.138]. On the other hand, putting $t=0$ in (22) we see that this is also equal to

$$\frac{2\sin \pi r}{\sinh \pi\xi}\left[\int_0^{\pi/2} \frac{\sinh \alpha\xi}{\cos^r \alpha}\,d\alpha + \int_0^{\infty} \frac{\sin \alpha\xi}{\cosh^r \alpha}\,d\alpha\right]. \tag{27}$$

In this appendix we clarify the relation between these two expressions.

We set

$$D = \begin{cases} \{z \in \mathbf{C}; \mathrm{Re}\, z > 0 \,\mathrm{and}\, |\mathrm{Im}\, z| < \arccos\, t\} & \text{if } t \in [0,1), \\ \{z \in \mathbf{C}; \mathrm{Re}\, z > 0 \,\mathrm{and}\, -\pi/2 < \mathrm{Im}\, z < \arccos t\} & \text{if } t \in (-1,0). \end{cases}$$

Then, $(\cosh\, z - t)^r$ $(= \exp(r\log(\cosh\, z-t))$ makes sense as a (single-valued) holomorphic function on D: We note

$$\cosh\, z - t = \cosh\, a\cos b - t + i\sinh\, a\, \sin\, b \quad (\text{for } z = a+ib \in D).$$

(i) Case $t \geq 0$: Since $\cos b > t \geq 0$, we have

$$\mathrm{Re}\,(\cosh\, z - t) = \cosh\, a\cos b - t \geq \cos b - t > 0.$$

(ii) Case $t < 0$: For $b \in (-\pi/2, \pi/2)$ we have $\cos b > 0$ and hence $\mathrm{Re}\,(\cos z - t) > 0$ as above. On the other hand, for $b \in [\pi/2, \arccos\, t)$ we have

$$\mathrm{Im}\,(\cosh\, z - t) = \sinh\, a\, \sin\, b > 0.$$

In either case the range of $\cosh\, z - s$ stays in $\mathbf{C} \setminus (-\infty, 0]$ so that $\log(\cosh\, z - t)$ indeed makes sense on D in the standard way.

Note $\cosh\,(i\arccos\, t) - t = 0$ but $i\arccos\, t \notin D$, and $\cosh\, z - t$ does not have a zero in D. Therefore, (for a fixed real number ξ) the function

$$f(z) = \frac{\sin\, z\xi}{(\cosh\, z - t)^r}$$

is holomorphic on D. We note that

$$|\cosh(a+ib)-t|^2 = (\cosh\, a\cos b - t)^2 + (\sinh\, a\, \sin\, b)^2 \tag{28}$$
$$= \sinh^2\, a + \cos^2 b - 2t\cosh\, a\cos b + t^2.$$

Lemma A.1. For each $t \in (-1,1)$ and $r \in (0,1)$ we have

$$\int_0^{\arccos t} \frac{\sinh\, \alpha\xi}{(\cos\, \alpha - t)^r}\, d\alpha + \int_0^\infty \frac{\sin\, \alpha\xi}{(\cosh\, \alpha - t)^r}\, d\alpha$$
$$= \int_0^\infty \frac{\cosh(\xi \arccos\, t)\sin(\xi s) + i\sinh(\xi\arccos\, t)\cos(\xi s)}{\left(t(\cosh\, s-1) + i\sqrt{1-t^2}\, \sinh\, s\right)^r}\, ds.$$

Proof. We fix an $\varepsilon > 0$ sufficiently small and a large $N > 0$. Let $R\ (\subseteq D)$ be the rectangular region with vertices $\varepsilon, N, N+i(\arccos\, t-\varepsilon)$ and $\varepsilon+i(\arccos\, t-\varepsilon)$ so that ∂R is the contour (oriented counterclockwise) consisting of the four oriented edges

$$C_1 \ :\ \varepsilon \to N,$$
$$C_2 \ :\ N \to N + i(\arccos\, t - \varepsilon),$$
$$C_3 \ :\ N + i(\arccos\, t - \varepsilon) \to \varepsilon + i(\arccos\, t - \varepsilon),$$
$$C_4 \ :\ \varepsilon + i(\arccos\, t - \varepsilon) \to \varepsilon.$$

Cauchy's theorem says

$$\sum_{i=1}^4 \int_{C_i} f(z)dz = \int_{\partial R} f(z)dz = 0, \tag{29}$$

and we will let $\varepsilon \to 0$ here.

From the definition we directly compute

$$\int_{C_3} f(z)dz =$$
$$-\int_\varepsilon^N \frac{\cosh((\arccos\, t-\varepsilon)\xi)\sin(\xi s) + i\sinh((\arccos\, t-\varepsilon)\xi)\cos(\xi s)}{\left(\cosh s\cos(\arccos\, t-\varepsilon) - t + i\sinh s\, \sin(\arccos\, t-\varepsilon)\right)^r}\, ds.$$

We use the dominated convergence theorem to see its behavior as $\varepsilon \to 0$. The numerator of the integrand obviously stays bounded, and we need to estimate the (reciprocal of) denominator. We have

$$
\begin{aligned}
&|\cosh s\cos(\arccos\, t-\varepsilon)-t+i\sinh s\, \sin(\arccos\, t-\varepsilon)|^2\\
&=\sinh^2 s+\big(\cos^2(\arccos\, t-\varepsilon)-2t\cos(\arccos\, t-\varepsilon)\cosh s+t^2\big)\\
&\geq (1-t^2)\sinh^2 s.
\end{aligned}
$$

Here, the first equality is a consequence of (28), and for the second inequality we note that the difference of the two sides is

$$
\begin{aligned}
&\big(\cos^2(\arccos\, t-\varepsilon)-2t\cos(\arccos\, t-\varepsilon)\cosh s+t^2\big)+t^2\sinh^2 s\\
&\qquad=\cos^2(\arccos\, t-\varepsilon)-2t\,\cosh\, s\cos(\arccos\, t-\varepsilon)+t^2\cosh^2 s\\
&\qquad=\big(\cos(\arccos\, t-\varepsilon)-t\cosh s\big)^2\geq 0.
\end{aligned}
$$

Consequently, the modulus of the above integrand is majorized by a constant multiple of $\sinh^{-r} s$, which is integrable over the interval $[0,N]$. The dominated convergence theorem thus guarantees

$$
\begin{aligned}
(30)\quad &\lim_{\varepsilon\to 0}\int_{C_3} f(z)dz\\
&\quad=-\int_0^N \frac{\cosh(\xi\arccos\, t)\sin(\xi s)+i\sinh(\xi\arccos\, t)\cos(\xi s)}{\big(t(\cosh s-1)+i\sqrt{1-t^2}\sinh s\big)^r}\,ds.
\end{aligned}
$$

Secondly, from the definition we have

$$
\int_{C_4} f(z)dz=-i\int_0^{\arccos\, t-\varepsilon}\frac{\sin(\varepsilon\xi)\cosh(\xi s)+i\cos(\varepsilon\xi)\sinh(\xi s)}{(\cosh\varepsilon\cos s-t+i\sinh\,\varepsilon\,\sin\, s)^r}\,ds.
$$

In this case we estimate

$$
\begin{aligned}
&|\cosh\,\varepsilon\cos s-t+i\sinh\,\varepsilon\,\sin\, s|^2\\
&\qquad=\sinh^2\varepsilon+\cos^2 s-2t\cosh\,\varepsilon\cos s+t^2\geq(\cos s-t)^2,
\end{aligned}
$$

or equivalently,

$$
\sinh^2\varepsilon-2t\cos s\,(\cosh\,\varepsilon-1)=\cosh^2\varepsilon-2t\cos s\,\cosh\,\varepsilon+2t\cos s-1\geq 0.
$$

Indeed, the quadratic polynomial $g(X)=X^2-2(t\cos s)X+2t\cos s-1$ takes a minimum value at $X=t\cos s$ (<1 for $s\in[0,\arccos t]$) and

$g(X) \geq g(1) = 0$ for $X = \cosh \varepsilon \geq 1$. Thus, the integrand is majorized by a constant multiple of $(\cos s - t)^{-r}$. The integrability of this majorant over the interval $[0, \arccos t]$ (together with the dominated convergence theorem again) yields

$$\lim_{\varepsilon \to 0} \int_{C_4} f(z)dz = \int_0^{\arccos t} \frac{\sinh s\xi}{(\cos s - t)^r} ds. \tag{31}$$

We obviously have

$$\lim_{\varepsilon \to 0} \int_{C_1} f(z)dz = \int_0^{N} \frac{\sin s\xi}{(\cosh s - t)^r} ds, \tag{32}$$

and the sum of (30),(31),(32) and $\lim_{\varepsilon \to 0} \int_{C_2} f(z)dz$ is zero (due to (29)). Then, by letting $N \to \infty$, we get the result since $\lim_{N \to \infty}$ of the last quantity disappears thanks to the obvious estimate $|\int_{C_2} f(z)dz| = O(e^{-rN})$ (based on (28)). ■

When $t = 0$, Lemma A.1 says

$$\begin{aligned} &\int_0^{\pi/2} \frac{\sinh \alpha\xi}{\cos^r \alpha}\, d\alpha + \int_0^{\infty} \frac{\sin \alpha\xi}{\cosh^r \alpha}\, d\alpha \\ &= \int_0^{\infty} \frac{\cosh(\pi\xi/2)\sin(\xi s) + i \sinh(\pi\xi/2)\cos(\xi s)}{(i \sinh s)^r}\, ds \\ &= e^{-i\pi r/2} \Big[\cosh(\pi\xi/2) \int_0^{\infty} \frac{\sin \xi s}{\sinh^r s}\, ds \\ &\qquad\qquad + i \sinh(\pi\xi/2) \int_0^{\infty} \frac{\cos \xi s}{\sinh^r s}\, ds \Big] \end{aligned} \tag{33}$$

thanks to $(i \sinh s)^r = \left(e^{i\pi/2} \sinh s\right)^r = e^{i\pi r/2} \sinh^r s$.

Lemma A.2.

$$\int_0^{\infty} \frac{\sin \xi s}{\sinh^r s}\, ds = \frac{2^{r-1} |\Gamma\left((r+i\xi)/2\right)|^2 \Gamma(1-r)}{\pi} \cdot \cos(\pi r/2) \sinh(\pi\xi/2),$$

$$\int_0^{\infty} \frac{\cos \xi s}{\sinh^r s}\, ds = \frac{2^{r-1} |\Gamma\left((r+i\xi)/2\right)|^2 \Gamma(1-r)}{\pi} \cdot \sin(\pi r/2) \cosh(\pi\xi/2).$$

With this lemma (whose proof is postponed) the quantity (33) is

$$
\begin{aligned}
&e^{-i\pi r/2}\sinh(\pi\xi/2)\cosh(\pi\xi/2)\\
&\qquad\times\frac{2^{r-1}|\Gamma((r+i\xi)/2)|^2\Gamma(1-r)}{\pi}\times\big(\cos(\pi r/2)+i\sin(\pi r/2)\big)\\
&\quad=\frac{\sinh\,\pi\xi}{2}\times\frac{2^{r-1}|\Gamma((r+i\xi)/2)|^2\Gamma(1-r)}{\pi}.
\end{aligned}
$$

Consequently, the quantity given by (27) is equal to

$$
\begin{aligned}
&\frac{2\sin\,\pi r}{\sinh\,\pi\xi}\times\frac{\sinh\,\pi\xi}{2}\times\frac{2^{r-1}|\Gamma((r+i\xi)/2)|^2\Gamma(1-r)}{\pi}\\
&\qquad=\frac{\sin\,\pi r\,\Gamma(1-r)}{\pi}\times 2^{r-1}|\Gamma((r+i\xi)/2)|^2,
\end{aligned}
$$

which is exactly (26) since $\Gamma(r)\Gamma(1-r)=\pi/\sin\,\pi r$.

Proof of Lemma A.2. We set $t=-\frac{1}{2}\times\log(1-x)$ so that

$$
e^{-2t}=1-x\quad\text{and}\quad\sinh x=\frac{1}{2}\left(\frac{1}{\sqrt{1-x}}-\sqrt{1-x}\right)=\frac{x}{2\sqrt{1-x}}.
$$

Since $dt=dx/2(1-x)$, this change of variables gives us

$$
\begin{aligned}
\int_0^\infty\frac{\sin\,\xi s}{\sinh^r s}\,ds &= \int_0^1\frac{\sin\left(-\frac{\xi}{2}\log(1-x)\right)}{\left(x/2\sqrt{1-x}\right)^r}\,\frac{dx}{2(1-x)}\\
&= 2^{r-1}\int_0^1(1-x)^{\frac{r}{2}-1}x^{-r}\sin\left(-\tfrac{\xi}{2}\log(1-x)\right)dx\\
&= 2^{r-1}\mathrm{Im}\left(\int_0^1(1-x)^{\frac{r}{2}-1-\frac{i\xi}{2}}x^{-r}dx\right),\\
\int_0^\infty\frac{\cos\xi s}{\sinh^r s}\,ds &= 2^{r-1}\mathrm{Re}\left(\int_0^1(1-x)^{\frac{r}{2}-1-\frac{i\xi}{2}}x^{-r}dx\right).
\end{aligned}
$$

With these expressions we get the lemma from the following:

$$
\begin{aligned}
&\int_0^1(1-x)^{\frac{r}{2}-1-\frac{i\xi}{2}}x^{-r}dx=B((r-i\xi)/2,1-r)=\frac{\Gamma\left((r-i\xi)/2\right)\Gamma(1-r)}{\Gamma\left(1-(r+i\xi)/2\right)}\\
&\quad=\frac{\sin\left(\pi(r+i\xi)/2\right)\Gamma\left((r+i\xi)/2\right)\Gamma\left((r-i\xi)/2\right)\Gamma(1-r)}{\pi}\\
&\quad=\frac{|\Gamma\left((r+i\xi)/2\right)|^2\,\Gamma(1-r)}{\pi}\times\sin\left(\pi(r+i\xi)/2\right),
\end{aligned}
$$

where we have used the identities $\Gamma(z)\Gamma(1-z)=\pi/\sin\,\pi z$ and $\Gamma(\overline{z})=\overline{\Gamma(z)}$ (a consequence of Schwarz' reflection principle). ■

References

[B1] R. Bhatia, *Matrix Analysis*, Springer, 1997.

[B2] R. Bhatia, *Infinitely divisible matrices*, Amer. Math. Monthly, **113** (2006), 221-235.

[B3] R. Bhatia, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl., **413** (2006), 355-363.

[BD1] R. Bhatia and C. Davis, *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. Appl., **14** (1993), 132-136.

[BD2] R. Bhatia and D. Drissi, *Generalized Lyapunov equations and positive definite functions*, SIAM J. Matrix Anal. Appl., **27** (2006), 103-114.

[BP] R. Bhatia and K. R. Parthasarathy, *Positive definite functions and operator inequalities*, Bull. London Math. Soc., **32** (2000), 214-228.

[D] D. Drissi, *Sharp inequalities for some operator means*, preprint.

[HJ] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[HK1] F. Hiai and H. Kosaki, *Comparison of various means for operators*, J. Funct. Anal., **163** (1999), 300-323.

[HK2] F. Hiai and H. Kosaki, *Means of matrices and comparison of their norms*, Indiana Univ. Math. J., **48** (1999), 899-936.

[HK3] F. Hiai and H. Kosaki, *Means of Hilbert space operators*, Lecture Notes in Mathematics 1820, Springer, 2003.

[H1] R. A. Horn, *On boundary values of a schlicht mapping*, Proc. Amer. Math. Soc., **18** (1967), 782-787.

[H2] R. A. Horn, *On infinitely divisible matrices, kernels, and functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gabiete, **8** (1967), 219-230.

[H3] R. A. Horn, *The theory of infinitely divisible matrices and kernels*, Trans. Amer. Math. Soc., **136** (1969), 269-286.

[K] M. K. Kwong, *Some results on matrix monotone functions*, Linear Algebra Appl., **118** (1989), 129-153.

[K1] H. Kosaki, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal., **156** (1998), 429-451.

[K2] H. Kosaki, in preparation.

[O] F. Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer, 1990.

[Z] X. Zhan, *Inequalities for unitarily invariant norms*, SIAM J. Matrix Anal. Appl., **20** (1998), 466-470.