Pramāṇa, Vol. 15, No. 4, October 1980, pp. 357-369. © Printed in India.

SU(3) representation for the polarisation of light

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MS received 5 June 1980; revised 10 September 1980

Abstract. A new mathematical representation for discussing the state of polarisation of an arbitrary beam of partially polarised light is described which makes use of the generators of the group SU(3). This representation is sufficiently general to describe not only physical photons which are transverse but also virtual photons. The correspondence between our representation and the conventional Stokes parameter representation is established and this leads to an equivalent geometrical description of partially polarised light in terms of diametrically opposite points on a Poincarè sphere with radius equal to the degree of polarisation. The connection with the spherical tensor representation is also discussed and this leads to a simple geometrical interpretation of the bounds on the parameters characterizing an arbitrary beam of partially polarised light.

Keywords. Photons; polarisation; density matrix, Stokes parameters; SU(3) representation; bounds.

1. Introduction

New mathematical representations for the state of polarisation of light or photons are of considerable interest in several areas of physics like crystal optics, nuclear theory or elementary particles. The well-known review article by Ramachandran and Ramaseshan (1961) discusses exhaustively several methods, starting with the Poincarè sphere and its connection with the Stokes parameters. The review articles by Fano (1957) and McMaster (1961) based on quantum mechanical ideas show how polarisation of light can be represented using the concept of the density matrix. Although the spin of the photon is one, it is found sufficient here, to use 2×2 matrices in view of the fact that light is a transverse wave and consequently the longitudinal state of polarisation is physically absent. However, in dealing with interactions between charged particles, it is well-known from quantum electrodynamics (Feynman 1962) that longitudinal state is also involved along with the two transverse states for the photons; in fact, the well-known Coulomb law between two charged particles is the result of an exchange of a ‘ longitudinal photon ’. Therefore, in several physical problems as in the case of electroproduction of pions (Dombey 1971), for example, it is advantageous to use 3×3 density matrices to represent the state of photon polarisation. A representation using the Kemmar algebra has been proposed by Roman (1959a, b) to describe the (3×3) density matrix for a stationary quasi-monochromatic field which is not a plane wave.

The distinct advantage which the density matrix formalism shares with the description in terms of Stokes parameters is that it can describe partially polarised as well as completely polarised systems with equal facility. Moreover the density matrix formalism lends itself to an elegant discussion of the behaviour of the system under coordinate transformations; in particular, rotations. While the formalism in terms of 2×2 matrices (Fano 1957; McMaster 1961) is quite adequate to discuss coordinate rotations with respect to an axis coinciding with the direction of propagation, a description in terms of 3×3 matrices is basically necessary for discussing the behaviour under rotations, in general.

A probably well-known but least mentioned fact while looking at the photon polarisation problem from the density matrix point of view is that a light beam characterised by the Stokes' parameters (Born and Wolf 1959) s_0, s_1, s_2 and s_3 is basically a non-oriented system† if $s_3 \neq 0$. More specifically, a 2×2 density matrix ρ written in the form

$$\rho = \frac{\mathrm{Tr}\,(\rho)}{2}\,[1 + \vec{\sigma} \cdot \vec{P}], \tag{1}$$

in terms of the Pauli spin matrices σ_x, σ_y and σ_z can be diagonalised purely through rotations alone if ρ denotes, for example, a system of spin $\frac{1}{2}$ particles; this is simply a consequence of the isomorphism between the group SU(2) and the rotation group in three dimensions R_3. However, the form (1) is not, in general, diagonalisable purely through rotations when ρ describes a system of photons. This feature, arising out of the fact that the spin of the photon is 1, comes out naturally when the density matrix for the system of photons is expanded in terms of the generators of the group SU(3) rather than the generators of the group SU(2). Moreover, a representation of the system in terms of the generators of the group SU(3) is capable of describing photons not only when they are physical (*i.e.*, transverse) but also when they are longitudinal as in problems where they are exchanged between two charged particles.

The purpose of this paper is thus to discuss a representation for the density matrix of the photons using the generators λ_i, $i=1,\ldots, 8$ of the group SU(3) introduced by Gell-Mann (1962) in the context of the quark model (see for example Gell-Mann and Neeman 1964; Lichtenberg 1978). Such a representation has already been used successfully to describe the spin states of the deuterons by Ramachandran and Murthy (1978). In §2 we indicate the 3×3 density matrix formalism for physical photons, introduce the SU(3) parameters characterising the light beam and express them in terms of Stokes parameters. Observing that the 3×3 density matrix has the so-called checker-board form (Capps 1961, Dalitz 1966, Ramachandran and Murthy 1979), we diagonalise the matrix and show that it leads to the characterisation of a partially polarised beam by specifying the intensities $I_{\alpha,\,\beta}$ and $I_{\alpha\,+\,\pi/2,\,-\,\beta}$ of two orthogonal states of polarisation denoted by the parameters $(2\alpha, 2\beta)$ and $(2\alpha+\pi, -2\beta)$ on the Poincarè sphere. The four parameters α, β, $I_{\alpha,\,\beta}$ and $I_{\alpha+\pi/2,-\beta}$ provide a complete description of a partially polarised beam as do the Stokes parameters s_1, s_3, s_0 and s_0. In §3, we generalise the density matrix description to virtual photons including the longitudinal state of polarisation. In particular, we write down explicitly the density matrix for a photon emitted at a vertex when the initial and

†A system is said to be non-oriented if the density matrix cannot be diagonalised through any rotation of the coordinate axis.

final spin states of the electron are specified either with respect to an external z-axis or in terms of its helicities. We also discuss the special form which the density matrix takes in the interesting case of the Breit frame (Perl 1974). These ideas will be applied to some problems in a sequel to this paper. In § 4, we express the 3×3 density matrix ρ in terms of the conventional spherical tensor parameters t_{kq} and discuss the bounds on the t_{kq} as well as the SU(3) parameters. The absence of the longitudinal state of polarisation for the photons leads to certain additional constraints on the parameters and consequently the bounds are more restrictive. The eight SU(3) parameters Λ_i introduced in § 2, the eight spherical tensor parameters t_{kq} and the eight generalised Stokes parameters r_i of Roman are related to each other. Explicit expressions for our Λ_i are given in the Appendix in terms of t_{kq} and r_i.

2. SU(3) formalism for physical photons

We choose a right handed frame of reference with the z-axis along the direction of propagation. If the density matrix is written in the form (1), with

$$\mathrm{Tr}\,(\rho) = s_0 \tag{2}$$

(where Tr denotes the trace), the Stokes parameters s_1, s_2 and s_3 are given by (Fano 1957)

$$s_1 = s_0 P_z;\; s_2 = s_0 P_x;\; s_3 = s_0 P_y, \tag{3}$$

if the rows and columns of the matrix are labelled by the two linearly polarised states along x- and y- axes respectively. On the other hand, if the basis states are chosen to be the left circular and right circular states respectively,

$$s_1 = s_0 P_x;\; s_2 = s_0 P_y;\; s_3 = s_0 P_z, \tag{4}$$

where the left circular (LC) and right circular (RC) states are related to the linearly polarised states through

$$|\mathrm{LC}\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle), \tag{5}$$

$$|\mathrm{RC}\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle). \tag{6}$$

If $|x'\rangle$ and $|y'\rangle$ denote linearly polarised states along the axes with respect to a coordinate system obtained on rotation through an angle α, *i.e.*,

$$|x'\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle, \tag{7}$$

$$|y'\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle, \tag{8}$$

the elliptically polarised state

$$|\alpha, \beta\rangle = \frac{1}{(a^2 + b^2)^{1/2}} [a|x'\rangle + ib\,|y'\rangle], \tag{9}$$

where a and b are real and

$$\beta = \tan^{-1}(b/a), \tag{10}$$

is represented by a point on the Poincarè sphere with unit radius and whose spherical polar coordinates (θ, ϕ) are given by

$$\theta = \frac{\pi}{2} - 2\beta, \tag{11}$$

$$\phi = 2\alpha. \tag{12}$$

Noting that

$$|\alpha, \beta\rangle = (\cos\beta\cos\alpha - i\sin\beta\sin\alpha)|x\rangle + (\cos\beta\sin\alpha + i\sin\beta\cos\alpha)|y\rangle$$
$$= c_1|x\rangle + c_2|y\rangle, \tag{13}$$

where c_1 and c_2 are complex, we have

$$\tan\alpha = \frac{\operatorname{Re}(c_2)}{\operatorname{Re}(c_1)} = -\frac{\operatorname{Im}(c_1)}{\operatorname{Im}(c_2)}, \tag{14}$$

$$\tan\beta = \frac{\operatorname{Im}(c_2)}{\operatorname{Re}(c_1)} = -\frac{\operatorname{Im}(c_1)}{\operatorname{Re}(c_2)}. \tag{15}$$

The state orthogonal to (13) is therefore

$$|\alpha + \frac{\pi}{2}, -\beta\rangle = -c_2^*|x\rangle + c_1^*|y\rangle, \tag{16}$$

which is represented by a diametrically opposite point (to $|\alpha, \beta\rangle$) on the Poincarè sphere (Pancharathnam 1956a). The two orthogonal states (13) and (16) may also be expressed in terms of the circularly polarised states as

$$|\alpha, \beta\rangle = \frac{1}{\sqrt{2}}[\exp(-i\alpha)(\cos\beta + \sin\beta)|\mathrm{RC}\rangle$$
$$+ \exp(i\alpha)(\cos\beta - \sin\beta)|\mathrm{LC}\rangle], \tag{17}$$

$$|\alpha + \frac{\pi}{2}, -\beta\rangle = \frac{-i}{\sqrt{2}}[\exp(-i\alpha)(\cos\beta - \sin\beta)|\mathrm{RC}\rangle$$
$$- \exp(i\alpha)(\cos\beta + \sin\beta)|\mathrm{LC}\rangle] \tag{18}$$

which would be useful later.

The 3×3 form for the density matrix for physical photons is now easily written down from equations (1) and (4) as

$$\rho = \frac{1}{2}\begin{bmatrix} s_0 - s_3 & 0 & -(s_1 + is_2) \\ 0 & 0 & 0 \\ -(s_1 - is_2) & 0 & s_0 + s_3 \end{bmatrix}, \tag{19}$$

where the rows and columns are labelled for convenience, in terms of the $|+1\rangle$, $|0\rangle$ and $|-1\rangle$ states which behave under rotations like the components of a spherical tensor (Racah 1961) of rank 1. More specifically, the state $|0\rangle$ denotes the longitudinal polarisation and

$$|+1\rangle \equiv -|\mathrm{R\cdot C}\rangle; \quad |-1\rangle \equiv |\mathrm{L\cdot c}\rangle. \tag{20}$$

Expressing now the density matrix (19) in terms of the generators of the group SU(3) (Ramachandran and Murthy 1978, 1979),

$$\rho = \frac{\mathrm{Tr}\,(\rho)}{3}\left(1 + \sum_{i=1}^{8} \Lambda_i\, \lambda_i\right), \tag{21}$$

where the generators λ_i satisfy

$$\mathrm{Tr}\,(\lambda_i\, \lambda_j) = 3\, \delta_{ij}, \tag{22}$$

and the parameters

$$\Lambda_i = \mathrm{Tr}\,(\lambda_i\, \rho)/\mathrm{Tr}\,(\rho), \tag{23}$$

denote the average expectation values. The parameters Λ_i, $i = 1, \ldots, 8$ are expressible in terms of the Stokes parameters and are given by

$$\Lambda_1 = 0 = \Lambda_2, \tag{24}$$

$$\Lambda_3 = \frac{1}{2}\sqrt{\frac{3}{2}}\left(1 - \frac{s_3}{s_0}\right), \tag{25}$$

$$\Lambda_4 = -\sqrt{\tfrac{3}{2}}\; s_0\, s_1, \tag{26}$$

$$\Lambda_5 = \sqrt{\tfrac{3}{2}}\; s_0\, s_2, \tag{27}$$

$$\Lambda_6 = 0 = \Lambda_7, \tag{28}$$

$$\Lambda_8 = -\frac{1}{2\sqrt{2}}\left(1 + \frac{3s_3}{s_0}\right), \tag{29}$$

where it may be noted that Λ_3 and Λ_8 satisfy the constraint

$$\Lambda_3 + \sqrt{3}\,\Lambda_8 = -\sqrt{6}\, s_3/s_0, \tag{30}$$

for transverse photons.

Observing that the density matrix in (19) is in the checker board form (Capps 1961), it may be diagonalised, for instance, using the procedure outlined earlier

(Ramachandran and Murthy 1979). The diagonal density matrix is then given by

$$\rho^0 = \frac{1}{3}\Big[1 + \sum_{i=3,8} \Lambda_i^0 \lambda_i\Big] = \frac{s_0}{2}\begin{bmatrix} 1+P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-P \end{bmatrix}, \tag{31}$$

where $\quad s_0^2 P^2 = s_1^2 + s_2^2 + s_3^2,$ (32)

and $\quad P = \dfrac{I_{\alpha,\beta} - I_{\alpha+\pi/2,-\beta}}{I},$ (33)

denotes the degree of partial polarisation (when I denotes the total intensity). The rows and columns in (31) are labelled by elliptically polarised states $|\alpha, \beta\rangle$, $|0\rangle$ and $|\alpha+\pi/2, -\beta\rangle$ where

$$\alpha = \tan^{-1}(s_2/s_1), \tag{34}$$

$$\beta = \tfrac{1}{2}\sin^{-1}(s_3/Ps_0). \tag{35}$$

An arbitrary beam of light with intensity $I=\mathrm{Tr}(\rho)=s_0$ whose state of polarisation is specified by the Stokes parameters s_1, s_2 and s_3 may therefore also be described in terms of either the SU(3) parameters $\Lambda_1, \ldots, \Lambda_8$ (of which only 3 are independent) or the parameters α, β and P or the parameters Λ_3^0, Λ_8^0, α, β where

$$\sqrt{3}\,\Lambda_3^0 - \Lambda_8^0 = \sqrt{2}, \tag{36}$$

$$P = s_3 / s_0 \sin 2\beta \tag{37}$$

Equivalently, the state of polarisation of an arbitrary partially polarised beam of light may also be specified completely in a geometrical way by the point $(2\alpha, 2\beta)$ on a Poincarè sphere whose radius is taken as P given by equation (37). This is in contrast to the normal practice of choosing the radius of the sphere to be either 1 or proportional to the intensity $I=s_0$ which is appropriate only to describe a pure state for which $P = 1$.

3. SU(3) formalism for virtual photons

The maximum utility of the SU(3) description is realised, when we consider a photon emitted at a vertex by a charged particle, say an electron. If $\vec{\epsilon}$ denotes the polarisation of the photon and we choose the z-axis to be along the three momentum transfer $\vec{q}=\vec{k}_1-\vec{k}_2$ (see figure 1), the density matrix for the emitted photon can by defined by the elements

$$\rho_{ij} = \epsilon_i\,\epsilon_j^*;\; i, j = x, y, z, \tag{38}$$

where z corresponds to the longitudinal state. If the fermion line in figure 1 denotes an electron, then ϵ_μ are given by†

$$\epsilon_\mu \propto \bar{u}(k_2)\, \gamma_\mu\, u(k_1), \tag{39}$$

where u denotes the electron spinor. These ϵ_μ are explicitly given in table 1, where the time-like component of ϵ_μ is not linearly independent of the spatial components $\vec{\epsilon}$, since

$$q_\mu\, \epsilon_\mu = 0. \tag{40}$$

The symbols ↑ and ↓ in table 1 correspond to the spin of the electron along the z-axis in terms of the standard solutions of the Dirac equation.

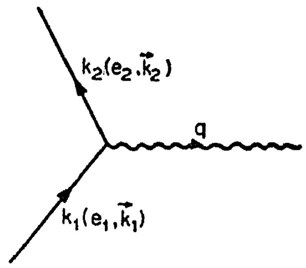

Figure 1. Feynman diagram corresponding to virtual photon creation. Solid lines denote Fermions and the wavy line denotes the virtual photon.

Table 1. The components ϵ_x, ϵ_y and ϵ_z are given in terms of the initial and final electron momenta k_1 $(\vec{k}_1, e_1)$ and k_2 $(\vec{k}_2, e_2)$ when the reaction plane is the $x-z$ plane and the Breit frame $(\vec{k}_1 + \vec{k}_2 = 0)$. N is the product of spinor normalizations.

		ϵ_x	ϵ_y	ϵ_z
↑ → ↑	$x-z$	$N\left(\frac{k_{1x}}{e_1+m}+\frac{k_{2x}}{e_2+m}\right)$	$-iN\left(\frac{k_{1x}}{e_1+m}-\frac{k_{2x}}{e_2+m}\right)$	$N\left(\frac{k_{1z}}{e_1+m}+\frac{k_{2z}}{e_2+m}\right)$
	Breit	0	0	0
↑ → ↓	$x-z$	$N\left(\frac{k_{1z}}{e_1+m}-\frac{k_{2z}}{e_2+m}\right)$	$iN\left(\frac{k_{1z}}{e_1+m}-\frac{k_{2z}}{e_2+m}\right)$	$-N\left(\frac{k_{1x}}{e_1+m}-\frac{k_{2x}}{e_2+m}\right)$
	Breit	k_1/m	ik_1/m	0
↓ → ↑	$x-z$	$-N\left(\frac{k_{1z}}{e_1+m}-\frac{k_{2z}}{e_2+m}\right)$	$iN\left(\frac{k_{1z}}{e_1+m}-\frac{k_{2z}}{e_2+m}\right)$	$N\left(\frac{k_{1x}}{e_1+m}-\frac{k_{2x}}{e_2+m}\right)$
	Breit	$-k_1/m$	ik_1/m	0
↓ → ↓	$x-z$	$N\left(\frac{k_{1x}}{e_1+m}+\frac{k_{2x}}{e_2+m}\right)$	$iN\left(\frac{k_{1x}}{e_1+m}-\frac{k_{2x}}{e_2+m}\right)$	$N\left(\frac{k_{1z}}{e_1+m}+\frac{k_{2z}}{e_2+m}\right)$
	Breit	0	0	0

†We have adopted the Bjorken and Drell (1965) conventions for the metric and the γ-matrices.

The density matrix when expressed with rows and columns labelled by the $|+1\rangle$, $|0\rangle$ and $|-1\rangle$ states taken the form

$$\rho = \frac{1}{2}\begin{bmatrix} \epsilon_x\epsilon_x^* + \epsilon_y\epsilon_y^* + 2\mathrm{Im}\,(\epsilon_x\epsilon_y^*) & -\sqrt{2}(\epsilon_x + i\epsilon_y)\epsilon_z^* & -(\epsilon_x\epsilon_x^* - \epsilon_y\epsilon_y^* + 2i\mathrm{Re}\,(\epsilon_x\epsilon_y^*)) \\ -\sqrt{2}\,\epsilon_z(\epsilon_x^* - i\epsilon_y^*) & 2\epsilon_z\epsilon_z^* & \sqrt{2}\epsilon_z(\epsilon_x^* + i\epsilon_y^*) \\ -(\epsilon_x\epsilon_x^* - \epsilon_y\epsilon_y^* - 2i\mathrm{Re}\,(\epsilon_x\epsilon_y^*)) & \sqrt{2}(\epsilon_x - i\epsilon_y)\epsilon_z^* & \epsilon_x\epsilon_x^* + \epsilon_y\epsilon_y^* - 2\mathrm{Im}\,(\epsilon_x\epsilon_y^*) \end{bmatrix} \tag{41}$$

which may again be expressed in the standard form (21) in terms of the SU(3) generators λ_i. The parameters Λ_i are now given by

$$\Lambda_1 = -\frac{1}{\sqrt{2}}\mathrm{Re}\,((\epsilon_x + i\epsilon_y)\epsilon_z^*), \tag{42}$$

$$\Lambda_2 = -\frac{1}{\sqrt{2}}\mathrm{Im}\,((\epsilon_x + i\epsilon_y)\epsilon_z^*), \tag{43}$$

$$\Lambda_3 = \frac{1}{2}\sqrt{\frac{3}{2}}(1 - 3\epsilon_z\epsilon_z^* + 2\mathrm{Im}\,\,(\epsilon_x\epsilon_y^*)\,), \tag{44}$$

$$\Lambda_4 = -\tfrac{1}{2}\,(\epsilon_x\epsilon_x^* - \epsilon_y\epsilon_x^*), \tag{45}$$

$$\Lambda_5 = -\mathrm{Re}\,(\epsilon_x\,\epsilon_y^*), \tag{46}$$

$$\Lambda_6 = \frac{1}{\sqrt{2}}\mathrm{Re}\,(\epsilon_z\,(\epsilon_x^* + i\epsilon_y^*)), \tag{47}$$

$$\Lambda_7 = \frac{1}{\sqrt{2}}\mathrm{Im}\,(\epsilon_z\,(\epsilon_x^* + i\epsilon_y^*)), \tag{48}$$

$$\Lambda_8 = \frac{3}{2\sqrt{2}}\left[\epsilon_z\,\epsilon_z^* + 2\,\mathrm{Im}\,(\epsilon_x\,\epsilon_y^*) - \frac{1}{3}\right]. \tag{49}$$

The density matrix (41) transforms under any arbitrary 3-dimensional rotation R of the coordinates characterised by the Euler angles (α, β, γ) according to

$$\rho \xrightarrow{R\,(\alpha,\,\beta,\,\gamma)} \rho' = R\,\rho\,R^{-1} \tag{50}$$

so that $$\rho'_{\mu'\nu'} = D^1_{\mu'\mu}\,(\alpha\beta\gamma)\rho_{\mu\nu}\;D^1_{\nu'\nu}(\alpha\beta\gamma)^*, \tag{51}$$

in terms of the well-known rotation matrices D (Rose 1957). [If the density matrix is oriented, *i.e.*, if it is diagonalisable through rotations, we should have $\Lambda_i'=0$ for all i except for $i = 3$, 8 for some suitable choice of α, β and γ.] The transformation for Λ_i, corresponding to an arbitrary rotation of the basis is explicitly given as

$$\Lambda_i' = M_{ij}\,\Lambda_j, \tag{52}$$

where the coefficients M_{ij} are given by

$$M_{ij} = \sum_{klmn} (\lambda_i)_{kl} (\lambda_j)_{mn} \left(\frac{(1+m)!\,(1-m)!\,(1+n)!\,(1-n)!}{(1+l)!\,(1-l)!\,(1+k)!\,(1-k)!}\right)^{1/2}$$

$$\exp\,[i\,(n-m)\,\gamma]\;\exp\,[i\,(k-l)\,\alpha]\left(\cos\frac{\beta}{2}\right)^{m+l+n+k}\left(\sin\frac{\beta}{2}\right)^{m+n-l-k}$$

$$p_{1-m}^{(m-l,\,m+l)}\;(\cos\beta)\;\;p_{1-n}^{(n-k,\,n+k)}\;\;(\cos\beta), \tag{53}$$

where $p_n^{(l,\,m)}$ $(\cos\beta)$ denote Jacobi polynomials.

In some problems in physics it is often advantageous to describe the initial and final spin states of the electrons as eigenstates of the helicity operator; but this leads to fairly complicated expressions for ϵ_μ. However, if we now go into the Breit frame, these simplify to the same as those given in table 1; the reason for this is obvious since in Breit frame the only momentum vector $\vec{q}$ is chosen to be along the z-axis.

4. Bounds on the polarisation parameters of the photon

One of the important problems in discussing polarisation phenomena is the discussion of bounds on the polarisation parameters. When the density matrix is represented in terms of the Stokes parameters s_1, s_2 and s_3 (see equation (19)) it is well-known (Pancharathnam 1956b) that any variation of s_1, s_2 and s_3 is constrained by the condition

$$s_1^2 + s_2^2 + s_3^2 \leqq s_0^2. \tag{54}$$

Interpreted geometrically, equation (54) means that the Stokes parameters s_1, s_2 and s_3 lie within a sphere of radius s_0 which is usually referred to as the Poincarè sphere (Born and Wolf 1959).

The density matrix for a spin 1 system in the spherical tensor representation is written as

$$\rho = \frac{\mathrm{Tr}(\rho)}{3}\left(1 + \sum_{k=1}^{2}\sum_{q=-k}^{k} t_{kq}\, T_{kq}^{+}\right), \tag{55}$$

where the spherical tensor operators T_{kq} are normalised (see for example Barschall and Haeberli 1970) such that

$$\tfrac{1}{3}\,\mathrm{Tr}(T_{kq}\; T_{k'q'}^{+}) = \delta_{kk'}\;\delta_{qq'}, \tag{56}$$

and the average expectation values t_{kq} are given by

$$t_{kq} = \frac{\mathrm{Tr}(\rho T_{kq})}{\mathrm{Tr}(\rho)}. \tag{57}$$

The density matrix written explicitly assumes the Checker-board form (Capps 1961)

$$\rho = \frac{\mathrm{Tr}(\rho)}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}\,t_{10}+\frac{1}{\sqrt{2}}\,t_{20} & 0 & \sqrt{3}\,t_{22} \\ 0 & 1-\sqrt{2}\,t_{20} & 0 \\ \sqrt{3}\,t_{2-2} & 0 & 1-\sqrt{\frac{3}{2}}\,t_{10}+\frac{1}{\sqrt{2}}\,t_{20} \end{bmatrix} \tag{58}$$

since t_{kq} for odd q goes to zero. Using the positive-definiteness of the eigenvalues of the density matrix ρ we obtain the following constraints (choosing $\mathrm{Tr}(\rho)=1$ for simplicity) on t_{kq} (Minnaert 1966; Dalitz 1966; Seiler and Roser 1977)

$$t_{10}^2+t_{20}^2+2\,|\,t_{22}|^2 \leqq 2, \tag{59}$$

$$3(t_{10}^2+2\,|\,t_{22}|^2)-(\sqrt{2}+t_{20})^2 \leqq 0. \tag{60}$$

However, for physical photons we have an additional constraint due to the absence of longitudinal states of polarisation, *viz.*,

$$\rho_{00} = 0, \tag{61}$$

which yields

$$t_{20} = 1/\sqrt{2}. \tag{62}$$

Conditions (59) and (60) are represented geometrically in figure 2 for a spin 1 system. It is obvious that condition (60) is more restrictive than condition (59) and the t_{kq} are constrained to be within the volume of a cone inscribed inside an ellipsoid. In the case of real photons, equation (62) imposes a further restriction and the relevant geometrical bounds are given by the base of the cone (which is an ellipse) whose centre is at

$$(t_{10}, t_{20}, |\,t_{22}\,|) = \left(0, \frac{1}{\sqrt{2}}, 0\right). \tag{63}$$

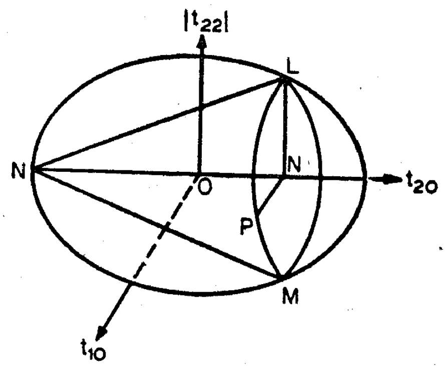

Figure 2. Schematic representation of the bounds on the spherical tensor parameters t_{10}, t_{20} and $|t_{22}|$. The bounds on t_{kq} for real photons are given by the base of the cone (inscribed inside the ellipsoid whose semimajor axis and semiminor axis are given by $NP=\sqrt{3/2}$ and $NL=\sqrt{3}/2$ respectively).

It is extremely interesting to note that the absence of longitudinal polarisation represented equivalently by equation (62) implies that physical photons considered as spin 1 particles are always tensor-polarised even when the light beam is in a random state of polarisation.

In order to provide a convenient geometrical representation of the bounds for the SU(3) parameters Λ_3, Λ_8, Λ_4 and Λ_5 for the photons, we represent the density matrix in terms of the Cartesian basis corresponding to the linearly polarised states of the photon. The density matrix is now written as (choosing Tr $(\rho) = 1$)

$$\rho = \tfrac{1}{3}\begin{bmatrix} 1+\sqrt{\frac{3}{2}}\,\Lambda_3'+\frac{1}{\sqrt{2}}\,\Lambda_8' & \sqrt{\frac{3}{2}}(\Lambda_1'-i\Lambda_2') & 0 \\ \sqrt{\frac{3}{2}}(\Lambda_1'+i\Lambda_2') & 1-\sqrt{\frac{3}{2}}\,\Lambda_3'+\frac{1}{\sqrt{2}}\,\Lambda_3' & 0 \\ 0 & 0 & 1-\sqrt{2}\,\Lambda'_8 \end{bmatrix}, \tag{64}$$

in general for any spin-1 system. The new set (Λ_8, Λ_8', Λ_1', Λ_2') is equivalent to the set (Λ_3, Λ_8, Λ_4, Λ_5). Using the positive definiteness of the eigenvalues of ρ, we obtain

$$\Lambda_3'^2+\Lambda_8'^2+|\,\Lambda_{12}\,|^2 \leqq 2, \tag{65}$$

and
$$3(\Lambda_3'^2+|\,\Lambda_{12}|^2)-(\sqrt{2}+\Lambda_8')^2 \leqq 0, \tag{66}$$

where
$$\Lambda_{12}=\Lambda_1'-i\Lambda_2'. \tag{67}$$

For physical photons, we obtain

$$\Lambda_8' = 1/\sqrt{2}, \tag{68}$$

since $\rho_{zz}=0$. Represented geometrically, condition (65) yields a sphere of radius $\sqrt{2}$ in the (Λ_3, Λ_8, $|\,\Lambda_{12}\lambda$) space. Condition (66), which is more restrictive than (65) yields a regular cone as shown in figure 3. As in the case of spherical tensor

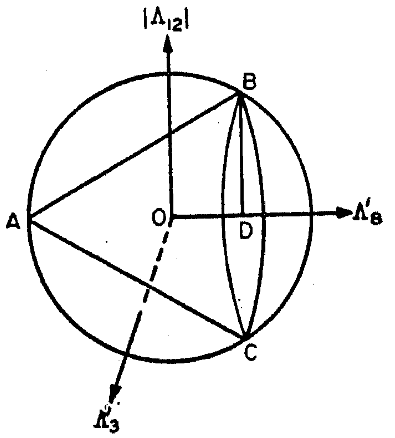

Figure 3. Schematic representation of the bounds on the SU(3) parameters Λ'_3, Λ'_8 and $|\,\Lambda_{12}\,|$. The bounds on Λ'_i for real photons are given by the base of the regular cone (inscribed inside a sphere whose radius is given by BD $= \sqrt{3/2}$).

representation, the allowed values of Λ'_3, Λ'_8 and $|\Lambda_{12}|$ for real photons should lie on the base of the regular cone which is a circle of radius $\sqrt{3/2}$ and whose centre is $(\Lambda'_3, \Lambda'_8, |\Lambda_{12}|) = (0, 1/\sqrt{2}, 0)$.

Acknowledgements

We would like to thank the referee for bringing to our notice the earlier work of Paul Roman. Two of us (MVNM and KSM) thank Prof. B Sanjeevaiah for providing facilities. MVNM thanks the Council of Scientific and Industrial Research, New Delhi for financial assistance through the award of fellowship and KSM thanks the Department of Atomic Energy for financial assistance through the award of a fellowship.

Appendix

The density matrix ρ in terms of the generalised Stokes parameters of Roman (1959a) can be written as

$$\rho = \frac{1}{2}\sum_{i=0}^{8} r_i \rho_i,$$

where the hermitian matrices ρ_i belong to a (3×3) dimensional irreducible representation of the Kemmer algebra (Roman 1959b). The nine expansion coefficients r_i (which are real) are the generalised Stokes parameters. The relationships between the generalised Stokes parameters and the SU(3) parameters and the spherical tensor parameters (see equations (21) and (55)) can be easily obtained by equating the corresponding density matrix elements in each of these three representations provided the basis states are the same, *i.e.*, either Cartesian or spherical basis. These relationships are given by

$$\Lambda_1 = \tfrac{1}{2}(t_{1-1} - t_{11} + t_{2-1} - t_{21}) = \sqrt{6}r_2/(3r_0 + 2r_4), \tag{1}$$

$$\Lambda_2 = \frac{i}{2}(t_{11} + t_{1-1} + t_{21} + t_{2-1}) = -\sqrt{6}r_3/(3r_0 + 2r_4), \tag{2}$$

$$\Lambda_3 = \tfrac{1}{2}(t_{10} + \sqrt{3}t_{20}) = \sqrt{\tfrac{3}{2}}(r_1 + r_4)/(3r_0 + 2r_4), \tag{3}$$

$$\Lambda_4 = \frac{1}{\sqrt{2}}(t_{22} + t_{2-2}) = \sqrt{6}r_7/(3r_0 + 2r_4), \tag{4}$$

$$\Lambda_5 = \frac{i}{\sqrt{2}}(t_{22} - t_{2-2}) = -\sqrt{6}r_8/(3r_0 + 2r_4), \tag{5}$$

$$\Lambda_6 = \tfrac{1}{2}(t_{1-1} - t_{11} - t_{2-1} + t_{21}) = \sqrt{6}(r_2 - r_5)/(3r_0 + 2r_4), \tag{6}$$

$$\Lambda_7 = \frac{i}{2}(t_{11} + t_{1-1} - t_{2-1} - t_{21}) = -\sqrt{6}(r_3 - r_6)/(3r_0 + 2r_4), \tag{7}$$

$$\Lambda_8 = \tfrac{1}{2}(\sqrt{3}t_{10} - t_{20}) = \frac{1}{\sqrt{2}}(3r_1 - r_4)/(3r_0 + 2r_4). \tag{8}$$

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