Proc. Indian Acad. Sci. (Math. Sci.) Vol. 116, No. 2, May 2006, pp. 193–220.

Minimal Cuntz–Krieger dilations and representations of Cuntz–Krieger algebras

B V RAJARAMA BHAT[1], SANTANU DEY[2] and JOACHIM ZACHARIAS[3]

[1]Indian Statistical Institute, R.V. College Post, Bangalore 560 059, India
[2]Institut für Mathematik und Informatik, Ernst-Moritz-Arndt-Universität, Friedrich-Ludwig-Jahn-Str. 15a, 17487 Greifswald, Germany
[3]School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD UK
E-mail: bhat@isibang.ac.in; dey@uni-greifswald.de; joachim.zacharias@nottingham.ac.uk

MS received 31 August 2005; revised 23 February 2006

Abstract. Given a contractive tuple of Hilbert space operators satisfying certain A-relations we show that there exists a unique minimal dilation to generators of Cuntz–Krieger algebras or its extension by compact operators. This Cuntz–Krieger dilation can be obtained from the classical minimal isometric dilation as a certain maximal A-relation piece. We define a maximal piece more generally for a finite set of polynomials in n noncommuting variables. We classify all representations of Cuntz–Krieger algebras $\mathcal{O}_A$ obtained from dilations of commuting tuples satisfying A-relations. The universal properties of the minimal Cuntz–Krieger dilation and the WOT-closed algebra generated by it is studied in terms of invariant subspaces.

Keywords. Dilation; commuting tuples; complete positivity; Cuntz algebras; Cuntz–Krieger algebras.

1. Introduction

Cuntz–Krieger algebras were introduced by Cuntz and Krieger in [CK] as examples of simple purely infinite C^*-algebras not stably isomorphic to Cuntz algebras [Cu]. Let $A = (a_{ij})_{n\times n}$ be a square 0–1-matrix, i.e. $a_{ij} \in \{0, 1\}$ and each row and column has at least one nonzero entry. The Cuntz–Krieger algebra $\mathcal{O}_A$ is defined as follows:

DEFINITION 1

$\mathcal{O}_A$ is the universal C^*-algebra generated by n partial isometries $s_1, \ldots, s_n$ with orthogonal ranges satisfying

$$s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^*,$$

$$I = \sum_{i=1}^{n} s_i s_i^*. \tag{1.1}$$

We denote the tuple $(s_1, \ldots, s_n)$ by $\underline{s}$.

Notice that $s_is_j = a_{ij}s_is_j$ for all s_i and s_j in $\mathcal{O}_A$. In this paper we study dilations related to these algebras. Equations (1.1) are called *Cuntz–Krieger relations.* An n-tuple of bounded operators $\underline{T} = (T_1, \ldots, T_n)$ on a Hilbert space is said to be a *contractive n-tuple* if $T_1T_1^* + \cdots + T_nT_n^* \leq I$. Such tuples are also called row contractions as the condition is equivalent to saying that the operator $(T_1, \ldots, T_n)$ from $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n-times) to $\mathcal{H}$ is a contraction. We will only consider contractive tuples. For such tuples, Davis [Da], Bunce [Bu], Frazho [Fr] and more extensively Popescu [Po1–Po6,AP] constructed dilations consisting of isometries with orthogonal ranges. Under natural minimality conditions this dilation is unique up to unitary equivalence. We refer to it as the *minimal isometric dilation* or the *standard (noncommuting) dilation* $\hat{\underline{T}}$ acting on $\hat{\mathcal{H}}$ (see [BBD]). In case $T_1T_1^* + \cdots + T_nT_n^* = I$ we will sometimes call $\underline{T}$ *unital.* $\underline{T}$ is unital iff its standard dilation is unital. The standard noncommuting dilation has similar characterizations as classical dilations of single operators notably the minimal normal extension of subnormal operators. The universal role of the unilateral shift for single operators is played by the tuple of creation operators on the full Fock space.

However, tuples are more complex than single operators and one may impose symmetry conditions on the tuple and study dilations within this restricted class of tuples. For instance if all operators in the tuple commute i.e. $\underline{T}$ is a commuting tuple, Arveson [Ar] showed that there is a unique minimal commuting dilation with similar properties. Crucial in his approach is the tuple of creation operators on symmetric Fock space playing the role of the shift for single operators. In [BBD] the relation between the minimal commuting and the standard dilation has been investigated. It was shown that for every contractive tuple there is a maximal subspace on which it forms a commuting tuple and that the minimal commuting dilation is precisely this maximal commuting piece of the standard noncommuting dilation.

In this article we consider the following class of tuples and investigate dilations within this class and their connection with the commuting and noncommuting standard dilation.

DEFINITION 2

Let $A = (a_{ij})_{n\times n}$ be a 0–1-matrix. A contractive n-tuple $\underline{T}$ is an *A-relation tuple* or is said to satisfy *A-relations* if $T_iT_j = a_{ij}T_iT_j$ for $1 \leq i, j \leq n$.

Given such a tuple $\underline{T}$ there is a unique minimal dilation to partial isometries satisfying A-relations, i.e. generators of a Cuntz–Krieger algebras if $\underline{T}$ is unital or an extension of it by compact operators if $\underline{T}$ is contractive. We call this the minimal Cuntz–Krieger dilation of $\underline{T}$. It will be denoted by $\tilde{\underline{T}}$ and acts on $\tilde{\mathcal{H}}$.

For an arbitrary tuple we define a maximal A-relation piece and compare the maximal A-relation piece of the standard isometric dilation with the minimal Cuntz–Krieger dilation. As for commuting tuples both turn out to be the same. We also prove similar results for the maximal commuting A-relation piece.

We begin in §2 by defining the maximal piece of a tuple of operators with respect to a finite set of polynomials in n-noncommuting variables. Similarly to results by Arias and Popescu [AP] there is a canonical homomorphism between the WOT-closed (nonselfadjoint) algebra generated by them and the WOT-closed algebra generated by the original tuple modulo a two-sided ideal. The maximal A-relation piece of a tuple of n-isometries with orthogonal ranges is a special case of this. In fact the maximal commuting piece and maximal q-commuting piece (see [BBD,De,AP]) can all be treated using this approach.

In §3 we show that the maximal A-relation piece of the standard dilation of an n-tuple satisfying A-relations is the minimal Cuntz–Krieger dilation. The section begins with two different constructions of minimal Cuntz–Krieger dilations, one using positive definite kernels and the other using a modification of Popescu's Poisson transform.

In §4 we study the minimal Cuntz–Krieger dilation of commuting A-relation tuples from a representation theoretical point of view. If such a tuple is also unital then it determines a unique representation of the Cuntz–Krieger algebra $\mathcal{O}_A$. Generalizing results from [BBD] for $\mathcal{O}_n$ we are able to show that these representations are determined by the GNS-representations of analogues of Cuntz states.

Based on the ideas of Bunce, Popescu [Po5] showed that the minimal isometric dilation can be characterized by a universal property of the C^*-algebra generated by it. In §5 we first point out that minimal Cuntz–Krieger dilations can be characterized in a similar way. We study the structure of the WOT-closed algebra generated by the operators constituting the minimal Cuntz–Krieger dilation and describe this algebra by making use of its invariant and wandering subspaces. We use techniques of Davidson, Kribs and Shpigel [DKS] and Davidson and Pitts [DP2] to understand the structure of 'free semigroup algebras', i.e. WOT-closed algebras generated by a finite number of isometries with orthogonal ranges. In this section many proofs are only sketched or omitted.

All Hilbert spaces in this paper are complex and separable. We denote the full Fock space over $\mathcal{L}$ by $\Gamma(\mathcal{L})$ which is defined as

$$\Gamma(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes^2} \oplus \mathcal{L}^{\otimes^3} \oplus \cdots .$$

Let the vacuum vector $1 \oplus 0 \oplus 0 \oplus \cdots$ be denoted by ω. $\mathbb{C}^n$ is the n-dimensional complex Euclidean space with standard orthonormal basis $\{e_1, \ldots, e_n\}$. The left creation operator L_i on $\Gamma(\mathbb{C}^n)$ is defined by

$$L_i x = e_i \otimes x,$$

where $1 \leq i \leq n$ and $x \in \Gamma(\mathbb{C}^n)$. The L_i's are clearly isometries with orthogonal ranges. We denote the tuple $(L_1, \ldots, L_n)$ by $\underline{L}$. Also $\sum_i L_i L_i^* = I - P_0$ where P_0 is the projection onto the vacuum space.

Let Λ be the set $\{1, 2, \ldots, n\}$ and Λ^m the m-fold Cartesian product of Λ for $m \in \mathbb{N}$. For an operator tuple $(T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$ and for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) = \alpha_1\alpha_2 \ldots \alpha_m$ in Λ^m, the operator $T_{\alpha_1} T_{\alpha_2} \ldots T_{\alpha_m}$ will be denoted by $\underline{T}^\alpha$. Let $\tilde{\Lambda}$ denote $\cup_{m=0}^{\infty} \Lambda^m$, where Λ^0 is $\{0\}$ and $\underline{T}^0$ is the identity operator. We may think of elements in $\tilde{\Lambda}$ as words with concatenation as product written $\alpha\beta$. $\tilde{\Lambda}$ is the free semigroup with n generators. Given a 0–1-matrix A as above we can define $\Lambda_A^m = \{\alpha_1\alpha_2 \ldots \alpha_m : a_{\alpha_i, \alpha_{i+1}} = 1$ for $i = 1, \ldots, n-1\}$ and the subsemigroup $\tilde{\Lambda}_A = \cup_{m=0}^{\infty} \Lambda_A^m$. $o(\alpha)$ and $t(\alpha)$ denote the first and last letter (i.e. index) of α.

DEFINITION 3

Let $\mathcal{H}$ and $\mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ is a closed subspace of $\mathcal{L}$ and $\underline{T}$, $\underline{R}$ be n-tuples of operators on $\mathcal{H}$, $\mathcal{L}$ respectively. Then $\underline{R}$ is a *dilation* of $\underline{T}$ or $\underline{T}$ a *piece* of $\underline{R}$ if

$$R_i^* h = T_i^* h$$

for all $h \in \mathcal{H}$, $1 \leq i \leq n$. A dilation is said to be *minimal dilation* if

$$\overline{\text{span}}\{\underline{R}^\alpha h : \alpha \in \tilde{\Lambda}, h \in \mathcal{H}\} = \mathcal{L}.$$

(1) A dilation $\underline{R}$ of $\underline{T}$ is said to be *isometric* if $\underline{R}$ consists of isometries with orthogonal ranges.

(2) When $\underline{T}$ satisfies A-relations, a dilation $\underline{R}$ of $\underline{T}$ is said to be a *Cuntz–Krieger dilation* if $\underline{R}$ consists of partial isometries with orthogonal ranges satisfying A-relations and

$$R_i^* R_i = I - \sum_{j=1}^{n} (1 - a_{ij}) R_j R_j^* = P_0 + \sum_{j=1}^{n} a_{ij} R_j R_j^*, \tag{1.2}$$

where $P_0 = 1 - \sum_{j=1}^{n} R_j R_j^*$.

Thus, like for isometric dilations, $C^*(\underline{R}) = \overline{\text{span}}\{\underline{R}^\alpha (\underline{R}^\beta)^*\colon \alpha, \beta \in \tilde{\Lambda}\}$ for any Cuntz–Krieger dilation $\underline{R}$ of $\underline{T}$. Moreover, since

$$(\underline{R}^\alpha)^* \underline{R}^\beta = \delta_{\alpha,\beta} R_{t(\alpha)}^* R_{t(\alpha)} = \delta_{\alpha,\beta} \left(I - \sum_{j=1}^{n} (1 - a_{t(\alpha),j}) R_j R_j^* \right) \tag{1.3}$$

whenever $\alpha \neq 0$, it follows that if $\underline{R}_1$ and $\underline{R}_2$ are any two minimal Cuntz–Krieger dilations then $\sum \underline{R}_1^{\alpha_i} h_i \mapsto \sum \underline{R}_2^{\alpha_i} h_i$ extends to a unitary equivalence.

Given any dilation $\underline{R}$ of $\underline{T}$ all R_i^* leave $\mathcal{H}$ invariant and if p, q are polynomials in n-noncommuting variables then

$$\underline{T}^\alpha (\underline{T}^\beta)^* = P_{\mathcal{H}} \underline{R}^\alpha (\underline{R}^\beta)^*|_{\mathcal{H}} \quad \text{and} \quad p(\underline{T})(q(\underline{T}))^* = P_{\mathcal{H}} p(\underline{R})(q(\underline{R}))^*|_{\mathcal{H}}.$$

It follows that if $\underline{R}$ is a Cuntz–Krieger dilation, then there is a unique completely positive map $\rho\colon C^*(\underline{R}) \to C^*(\underline{T})$ mapping $\underline{R}^\alpha (\underline{R}^\beta)^*$ to $\underline{T}^\alpha (\underline{T}^\beta)^*$.

Finally let us recall a concept we need later. For an n-tuple $\underline{R}$ of bounded operators on $\mathcal{L}$, a subspace $\mathcal{K}$ of $\mathcal{L}$ is said to be *wandering* for the tuple if $\underline{R}^\alpha \mathcal{K}$ are pairwise orthogonal for all $\alpha \in \tilde{\Lambda}$.

2. Maximal A-relation piece and A-Fock space

We begin with an n-tuple of bounded operators $\underline{R}$ on a Hilbert space $\mathcal{L}$ and a finite set of polynomials $\{p_\xi\}_{\xi \in \mathcal{I}}$ in n-noncommuting variables with finite index set $\mathcal{I}$. Consider

$$\mathcal{C}(\underline{R}) = \{\mathcal{M}\colon R_i^* \mathcal{M} \subseteq \mathcal{M} \quad \text{and} \quad (p_\xi(\underline{R}))^* h = 0, \\ \forall h \in \mathcal{M}, 1 \leq i \leq n, \xi \in \mathcal{I}\}.$$

$\mathcal{C}(\underline{R})$ consists of all co-invariant subspaces of $\underline{R}$ such that the compressions form a tuple $\underline{R}^p = (R_1^p, \ldots, R_n^p)$ satisfying $p_\xi(\underline{R}^p) = 0$ for all $\xi \in \mathcal{I}$. It is a complete lattice, in the sense that arbitrary intersections and closed spans of arbitrary unions of such spaces are again in this collection. Its maximal element is denoted by $\mathcal{L}^p(\underline{R})$ (or by $\mathcal{L}^p$ when the tuple under consideration is clear). Since $(p_\xi(\underline{R}))^*(\underline{R}^\alpha)^*\mathcal{M} = 0$ for all $\mathcal{M} \in \mathcal{C}(\underline{R})$, $\alpha \in \tilde{\Lambda}$ and $\xi \in \mathcal{I}$ we have $\mathcal{L}^p(\underline{R}) \subseteq \bigcap_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} \ker(p_\xi(\underline{R})^*(\underline{R}^\alpha)^*)$. On the other hand, this intersection lies in $\mathcal{C}(\underline{R})$ and hence

$$\mathcal{L}^p(\underline{R}) = \bigcap_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} \ker(p_\xi(\underline{R})^*(\underline{R}^\alpha)^*) = \left[\bigvee_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} \underline{R}^\alpha p_\xi(\underline{R})(\mathcal{L}) \right]^\perp.$$

Therefore we have the following lemma.

Lemma 4. *Let $\underline{R}$ be an n-tuple of operators on a Hilbert space $\mathcal{L}$ and $\mathcal{K} = \overline{\mathrm{span}}\{\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta h\colon h \in \mathcal{L},\ \xi \in \mathcal{I}$ and $\alpha, \beta \in \tilde{\Lambda}\}$. Then $\mathcal{L}^p(\underline{R}) = \mathcal{K}^\perp = \{h \in \mathcal{L}\colon (\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta)^* h = 0, \forall \xi \in \mathcal{I}$ and $\alpha, \beta \in \tilde{\Lambda}\}$.*

$\mathcal{L}^p(\underline{R})$ can also be thought of as follows: Let $\mathcal{R}$ be the (nonself-adjoint) WOT-closed algebra generated by $\underline{R}$. Then $\mathcal{L}^p(\underline{R}) = (\mathcal{J}\mathcal{L})^\perp$, where

$$\mathcal{J} = \overline{\mathrm{span}}^w\{\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta\colon \alpha, \beta \in \tilde{\Lambda}, \xi \in \mathcal{I}\} \subseteq \mathcal{R}$$

is the WOT-closed ideal generated by $\{p_\xi(\underline{R})\colon \xi \in \mathcal{I}\}$.

DEFINITION 5

The *maximal piece* of $\underline{R}$ with respect to $\{p_\xi\}_{\xi\in\mathcal{I}}$ is defined as the piece obtained by compressing $\underline{R}$ to the maximal element $\mathcal{L}^p(\underline{R})$ of $\mathcal{C}(\underline{R})$ denoted by $\underline{R}^p = (R_1^p, \ldots, R_n^p)$. The maximal piece is said to be *trivial* if the space $\mathcal{L}^p(\underline{R})$ is the zero space.

Let $\mathcal{R}^p$ be the WOT-closed algebra generated by $\underline{R}^p$. By co-invariance the map

$$\Psi_{\mathcal{L}^p(\underline{R})}\colon \mathcal{R} \to \mathcal{R}^p, \quad X \mapsto P_{\mathcal{L}^p(\underline{R})}X = P_{\mathcal{L}^p(\underline{R})}XP_{\mathcal{L}^p(\underline{R})}$$

is a WOT-continuous homomorphism of $\mathcal{R}$ whose kernel is a WOT-closed ideal of $\mathcal{R}$. Since $\Psi_{\mathcal{L}^p}(p_\xi(\underline{R})) = p_\xi(\underline{R}^p) = 0$ for all $\xi \in \mathcal{I}$ we certainly have $p_\xi(\underline{R}) \in \ker \Psi_{\mathcal{L}^p}$, i.e. $\mathcal{J} \subseteq \ker \Psi_{\mathcal{L}^p}$. Thus there is a canonical surjective and contractive homomorphism

$$\Psi\colon \mathcal{R}/\mathcal{J} \to \mathcal{R}^p.$$

When the polynomials are $p_{(l,m)} = z_l z_m - a_{lm} z_l z_m$, $(l, m) \in \{1, \ldots, n\} \times \{1, \ldots, n\} = \mathcal{I}$ we call $\mathcal{L}^p(\underline{R})$ the *maximal A-relation subspace* and the corresponding piece the *maximal A-relation piece* $\underline{R}^A$. The maximal A-relation subspace is explicitly given by

$$\mathcal{L}_A(\underline{R}) = \{\underline{R}^\alpha(R_i R_j - a_{ij} R_i R_j)h\colon h \in \mathcal{L},\ \alpha \in \Lambda,\ i, j = 1, \ldots, n\}^\perp.$$

When the noncommuting polynomials are $q_{(l,m)} = z_l z_m - z_m z_l$ then we obtain the maximal commuting subspace

$$\mathcal{L}^c(\underline{R}) = \{\underline{R}^\alpha(R_i R_j - R_j R_i)h\colon h \in \mathcal{L},\ \alpha \in \Lambda,\ i, j = 1, \ldots, n\}^\perp$$

studied in [BBD].

Lemma 6. *Let $\underline{R}$ and $\underline{T}$ be two n-tuples of bounded operators on $\mathcal{M}$ and $\mathcal{H}$ respectively.*

(1) *The maximal A-relation piece of $(R_1 \oplus T_1, \ldots, R_n \oplus T_n)$ is $(R_1^A \oplus T_1^A, \ldots, R_n^A \oplus T_n^A)$ acting on $\mathcal{M}_A \oplus \mathcal{H}_A$ and the maximal A-relation piece of $(R_1 \otimes I, \ldots, R_n \otimes I)$ acting on $\mathcal{M} \otimes \mathcal{H}$ is $(R_1^A \otimes I, \ldots, R_n^A \otimes I)$ on $\mathcal{M}_A \otimes \mathcal{H}$.*
(2) *Suppose $\mathcal{H} \subseteq \mathcal{M}$ and $\underline{R}$ is a dilation of $\underline{T}$ then $\underline{R}^A$ is a dilation of $\underline{T}^A$ with $\mathcal{H}_A(\underline{T}) = \mathcal{M}_A(\underline{R}) \cap \mathcal{H}$.*

Proof. Follows from Lemma 4 (compare with [BBD] for part (2)). □

Cuntz–Krieger relations are naturally related to A-Fock space, a variant of the usual Fock space.

DEFINITION 7

For a given $A = (a_{ij})_{n\times n}$ as above, the *A-Fock space* is defined as the maximal A-relation subspace $(\Gamma(\mathbb{C}^n))_A(\underline{L})$ with respect to the left creation operators. It is denoted by Γ_A. We also define the n-tuple $\underline{S} = (S_1, \ldots, S_n)$, where the S_i's are the compressions of left creation operators L_i onto Γ_A.

The A-Fock space has a very concrete description justifying the terminology.

PROPOSITION 8

$\Gamma_A = \overline{\text{span}}\{e^\alpha\colon \alpha \in \tilde{\Lambda}_A\}$ *and* $\underline{L}^A = \underline{S}$.

Proof. Let $\alpha \in \Lambda^m$ be such that there exist $1 \le k \le m-1$ for which $a_{\alpha_k\alpha_{k+1}} = 0$. Denoting α_k, α_{k+1} by s, t, it is clear that

$$e^\alpha \in \overline{\text{span}}\{\underline{L}^\gamma(L_sL_t - a_{st}L_sL_t)h\colon h \in \Gamma(\mathbb{C}^n), \quad \gamma \in \tilde{\Lambda}\},$$

which implies that such e^α are orthogonal to $\Gamma(\mathbb{C}^n)_A(\underline{L})$, whereas if for all $1 \le k \le m-1$, $a_{\alpha_k\alpha_{k+1}} = 1$ then for all $1 \le i, j \le m-1$, $\beta \in \tilde{\Lambda}$, $h \in \Gamma(\mathbb{C}^n)$

$$\langle e^\alpha, \underline{L}^\beta(L_iL_j - a_{ij}L_iL_j)h\rangle = 0,$$

and thus such $e^\alpha \in \Gamma_A$. By taking completions the proposition follows. □

Similar Fock spaces were also studied by Muhly and Solel [Mu,MS].

Now suppose that $\alpha = \alpha_1 \ldots \alpha_m \in \Lambda_A^m$ and $m > 0$, then

$$S_ie^\alpha = P_{\Gamma_A}L_ie^\alpha = \begin{cases} e_i, & \text{if } |\alpha| = 0 \\ a_{i\alpha_1}e_i \otimes e^\alpha, & \text{if } |\alpha| \ge 1 \end{cases},$$

$$S_i^*e^\alpha = L_i^*e^\alpha = \begin{cases} 0, & \text{if } |\alpha| = 0 \\ \delta_{i\alpha_1}\omega, & \text{if } |\alpha| = 1 \\ \delta_{i\alpha_1}e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m}, & \text{if } |\alpha| > 1 \end{cases},$$

$$S_i^*S_ie^\alpha = \begin{cases} \omega, & \text{if } |\alpha| = 0 \\ a_{i\alpha_1}e^\alpha, & \text{if } |\alpha| \ge 1 \end{cases} \quad \text{and} \quad S_iS_i^*e^\alpha = \begin{cases} 0, & \text{if } |\alpha| = 0 \\ \delta_{i\alpha_1}e^\alpha, & \text{if } |\alpha| = 1 \end{cases}.$$

PROPOSITION 9

The maximal A-relation piece of an n-tuple of isometries with orthogonal ranges is an n-tuple of partial isometries with orthogonal ranges.

Proof. Let $\underline{V} = (V_1, \ldots, V_n)$ be an n-tuple of isometries with orthogonal ranges on a Hilbert space $\mathcal{K}$. Fix a matrix $A = (a_{ij})_{n\times n}$ as above and denote the projection onto $\mathcal{K}_A(\underline{V})$ by P. Any k_A in $\mathcal{K}_A(\underline{V})$ can be written as $k_A = \oplus_{p=1}^n V_pk_p \oplus k_0$ for some $k_p \in \mathcal{K}$, $1 \le p \le n$ and some $k_0 \in (I - \sum_{p=1}^n V_pV_p^*)\mathcal{K}$. (Any $k \in \mathcal{K}$ can be written in this form.) Clearly k_0 is in $\mathcal{K}_A(\underline{V})$ using Lemma 4, since the ranges of the V_q's and $I - \sum_i V_iV_i^*$ are all mutually orthogonal. Similarly one observes that the other k_p's also belong to $\mathcal{K}_A(\underline{V})$ as for $k \in \mathcal{K}, \alpha \in \tilde{\Lambda}$,

$$\begin{aligned}\langle k_p, \underline{V}^\alpha(V_iV_j - a_{ij}V_iV_j)k\rangle &= \langle V_pk_p, V_p\underline{V}^\alpha(V_iV_j - a_{ij}V_iV_j)k\rangle \\ &= \langle \oplus_{q=1}^n V_qk_q \oplus k_0, V_p\underline{V}^\alpha(V_iV_j - a_{ij}V_iV_j)k\rangle \\ &= \langle k_A, V_p\underline{V}^\alpha(V_iV_j - a_{ij}V_iV_j)k\rangle = 0,\end{aligned}$$

where again we use that the ranges of the V_q's and $I - \sum_i V_iV_i^*$ are all mutually orthogonal. Next we show that

$$PV_ik_0 = V_ik_0 \tag{2.1}$$

and

$$PV_iV_pk_p = a_{ip}V_iV_pk_p. \tag{2.2}$$

Equation (2.1) follows from $\langle V_ik_0, \underline{V}^\beta(V_sV_t - a_{st}V_sV_t)k\rangle = 0$, for all $\beta \in \tilde{\Lambda}$, $1 \le s, t \le n$, $k \in \mathcal{K}$ (since k_0 is orthogonal to the range of V_t, $1 \le t \le n$). When $a_{ip} = 0$, we have $PV_iV_pk_p = P(V_iV_p - a_{ip}V_iV_p)k_p = 0 = a_{ip}V_iV_pk_p$. So it is enough to show that for $a_{ip} = 1$, $V_iV_pk_p \in \mathcal{K}_A(\underline{V})$. When $|\alpha| > 1$ or $|\alpha| = 0$, it is easy to see that for $1 \le s, t \le n, k \in \mathcal{K}$,

$$\langle V_iV_pk_p, \underline{V}^\alpha(V_sV_t - a_{st}V_sV_t)k\rangle = 0 \tag{2.3}$$

as the V_i's are isometries with orthogonal ranges and $k_p \in \mathcal{K}_A(\underline{V})$. When $|\alpha| = 1$,

$$\begin{aligned}\langle V_iV_pk_p, V_i(V_pV_t - a_{pt}V_pV_t)k\rangle &= \langle V_pk_p, (V_pV_t - a_{pt}V_pV_t)k\rangle \\ &= \langle \oplus_{s=1}^n V_sk_s \oplus k_0, (V_pV_t - a_{pt}V_pV_t)k\rangle \\ &= 0.\end{aligned}$$

Clearly eq. (2.3) holds in all other cases when $|\alpha| = 1$. So eq. (2.2) is true in general and we have

$$\begin{aligned}V_i^A(V^A)_i^*V_i^Ak_A &= PV_iV_i^*PV_ik_A \\ &= PV_iV_i^*P(\oplus_pV_iV_pk_p \oplus V_ik_0) \\ &= PV_iV_i^*(\oplus_{p=1}^n a_{ip}V_iV_pk_p \oplus V_ik_0) \\ &= \oplus_{p=1}^n a_{ip}PV_iV_pk_p \oplus PV_ik_0 \\ &= PV_ik_A = V_i^Ak_A.\end{aligned}$$

Thus the V_i^A's are partial isometries. Now the assertion of the proposition that for $1 \le i \neq j \le n$, the range of V_i^A is orthogonal to the range of V_j^A can be proved in the following way:

$$\begin{aligned}(V_j^A)^*V_i^Ak_A &= V_j^*PV_ik_A \\ &= V_j^*PV_i(\oplus_{p=1}^n V_pk_p \oplus k_0) \\ &= V_j^*(\oplus_pa_{ip}V_iV_pk_p \oplus V_ik_0) = 0.\end{aligned}$$

Alternatively this follows since $\underline{V}^A$ is contractive. □

COROLLARY 10

The following holds for $\underline{S}$:

(1) $I - \sum_{i=1}^{n} S_i S_i^* = P_0$ *where* P_0 *is the projection onto the vacuum space.*
(2) *The* S_i*'s are partial isometries with orthogonal ranges.*
(3) $S_i^* S_i = I - \sum_{j=1}^{n}(1 - a_{ij}) S_j S_j^* = P_0 + \sum_{j=1}^{n} a_{ij} S_j S_j^*$.

Proof.

(1) $I - \sum S_i S_i^* = P_{\Gamma_A}(I - \sum L_i L_i^*) P_{\Gamma_A} = P_0$.
(2) Follows from Proposition 9 or can be checked directly from the relations before Proposition 9.
(3) Suppose $e^\alpha \in \Gamma_A$, and when $|\alpha| > 0$, let $\alpha = \alpha_1 \dots \alpha_m$. Then

$$\left[I - \sum_j (1 - a_{ij}) S_j S_j^* \right] e^\alpha = \begin{cases} \omega, & \text{if } |\alpha| = 0 \\ a_{i\alpha_1} e^\alpha, & \text{if } |\alpha| \geq 1 \end{cases}.$$

□

3. Minimal Cuntz–Krieger dilations and standard noncommuting dilations

The main purpose of this section is to prove the following result.

Theorem 11. *Let* $\underline{T}$ *be a contractive* A*-relation tuple on* $\mathcal{H}$*. Then there exists a minimal Cuntz–Krieger dilation* $\underline{\tilde{T}}$ *on* $\tilde{\mathcal{H}}$ *unique up to unitary equivalence.* $\underline{\tilde{T}}$ *is unital iff* $\underline{T}$ *is unital.*

Uniqueness has already been pointed out and the last part follows since $\sum_{|\alpha|=k} \underline{\tilde{T}}^\alpha (\underline{\tilde{T}}^\alpha)^*$ form a decreasing sequence of projections converging weakly to a limit projection P. If $\underline{T}$ is unital then $P_{\mathcal{H}} \leq P$ and so $P = 1$ by minimality.

We will give two proofs of the existence. The first is direct and uses positive definite kernels. It gives an explicit construction of the dilation Hilbert space and the dilated tuple. The second construction is an adaptation of Popescu's Poisson transform method which uses completely positive maps. Though elegant it is less direct. Then we show that the minimal Cuntz–Krieger dilation can also be obtained as the maximal A-relation piece of the standard dilation.

First proof via positive definite kernels

Let $\underline{T} = (T_1, \dots, T_n)$ be a contractive n-tuple on a Hilbert space $\mathcal{H}$ satisfying A-relations. Assume that we have already found the minimal Cuntz–Krieger dilation $\underline{\tilde{T}}$ on the Hilbert space $\tilde{\mathcal{H}}$. Then

$$K((\alpha, u), (\beta, v)) := \langle \underline{\tilde{T}}^\alpha u, \underline{\tilde{T}}^\beta v \rangle$$

clearly defines a positive definite kernel on the set

$$X = \tilde{\Lambda}_A \times \mathcal{H}.$$

By minimality $\tilde{\mathcal{H}} = \overline{\text{span}}\{\underline{\tilde{T}}^\alpha u \colon u \in \mathcal{H}, \alpha \in \tilde{\Lambda}_A\}$ which is precisely the kernel Hilbert space. Moreover $\tilde{T}_i$ corresponds to the map $(\alpha, u) \mapsto (i\alpha, u)$.

Using co-invariance of $\mathcal{H}$ under $\underline{\tilde{T}}$ and the relation $\tilde{T}_i^*\tilde{T}_i = I - \sum_j (1 - a_{ij})\tilde{T}_j\tilde{T}_j^*$ we find that

$$K((\alpha, u), (\beta, v)) = \begin{cases} \langle u, v\rangle, & \text{if } \alpha = \beta = 0 \\ \langle u, [I - \sum_k (1 - a_{t(\alpha)k})T_kT_k^*]v\rangle, & \text{if } \alpha = \beta \neq 0 \\ \langle u, \underline{T}^\gamma v\rangle, & \text{if } \beta = \alpha\gamma \\ \langle u, (\underline{T}^\gamma)^* v\rangle, & \text{if } \alpha = \beta\gamma \\ 0, & \text{otherwise} \end{cases}$$

and this kernel depends only on $\underline{T}$. We will show directly by induction that the kernel K thus defined is always positive definite.

To simplify the calculations we assume that $\underline{T}$ is unital, i.e. $\sum_i T_iT_i^* = I$. There is no loss in doing so since positivity for the kernel defined by the $(n+1)$-tuple $(T_1, \ldots, T_n, (I - \sum_i T_iT_i^*)^{1/2})$ implies positivity of K. Under this assumption $T_iQ_i = T_i$, where $Q_i = I - \sum_k (1 - a_{ik})T_kT_k^* = \sum_{j=1}^n a_{i,j}T_jT_j^*$ and $K((\alpha, u), (\alpha, v)) = \langle u, Q_{t(\alpha)}v\rangle$, whenever $\alpha \neq 0$.

Now let $A^{(m)}$ denote operator matrices with entries in $B(\mathcal{H})$ indexed by $\alpha, \beta \in \tilde{\Lambda}$, where $|\alpha|, |\beta| \leq m$ and define $K^{(m)} = (K^{(m)}_{\alpha,\beta})$ by

$$K^{(m)}_{\alpha,\beta} := \begin{cases} I, & \text{if } \alpha = \beta = 0 \\ Q_{t(\alpha)}, & \text{if } \alpha = \beta \neq 0 \\ \underline{T}^\gamma, & \text{if } \beta = \alpha\gamma \\ (\underline{T}^\gamma)^*, & \text{if } \alpha = \beta\gamma \\ 0, & \text{otherwise} \end{cases}$$

i.e. $K^{(m)}$ is a compression of K. Clearly it suffices to show that all $K^{(m)}$ are positive. For $m = 1$ this follows from the equation

$$\begin{bmatrix} I & T_1 & T_2 & \ldots & T_n \\ T_1^* & Q_1 & 0 & \ldots & 0 \\ T_2^* & 0 & Q_2 & \ldots & 0 \\ \vdots & & & & \vdots \\ T_n^* & 0 & & \ldots & Q_n \end{bmatrix}$$

$$= \begin{bmatrix} I & T_1 & T_2 & \ldots & T_n \\ 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \ldots & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & Q_1 & 0 & \ldots & 0 \\ 0 & 0 & Q_2 & \ldots & 0 \\ 0 & & & & \vdots \\ 0 & 0 & & \ldots & Q_n \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \ldots & 0 \\ T_1^* & I & 0 & \ldots & 0 \\ T_2^* & 0 & I & \ldots & 0 \\ \vdots & & & & \vdots \\ T_n^* & 0 & & \ldots & I \end{bmatrix}.$$

If $m > 1$ define matrices $L_1, L_2, \ldots, L_{m-1}$ by

$$L_{k;\alpha,\beta} := \begin{cases} T_i, & \text{if } \beta = \alpha i, |\beta| = k \\ I, & \text{if } \alpha = \beta \text{ and } |\beta| \geq k \\ 0, & \text{otherwise} \end{cases}.$$

Finally let

$$Q^{(m)}_{\alpha,\beta} := \begin{cases} Q_{t(\alpha)}, & \text{if } \alpha = \beta \text{ and } |\alpha| = m \\ 0, & \text{otherwise} \end{cases}.$$

Then it is not hard to check that

$$K^{(m)} = L_1 L_2 \dots L_{m-1} Q^{(m)} L^*_{m-1} \dots L^*_2 L^*_1$$

which shows that K is positive. By Kolmogorov's theorem there exists a Hilbert space $\tilde{\mathcal{H}}$ and an injective map $\lambda\colon X \to \tilde{\mathcal{H}}$ such that $\overline{\text{span}}\{\lambda(\alpha, u)\colon 1 \le i \le n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \tilde{\mathcal{H}}$ and

$$K((\alpha, u), (\beta, v)) = \langle \lambda(\alpha, u), \lambda(\beta, v)\rangle.$$

It remains to show that $\underline{\tilde{T}} = (\tilde{T}_1, \dots, \tilde{T}_n)$ consisting of maps $\tilde{T}_i\colon \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ defined by

$$\tilde{T}_i \lambda(\alpha, u) = \lambda(i\alpha, u),$$

constitute a tuple $\underline{\tilde{T}}$ which is the minimal Cuntz–Krieger dilation of $\underline{T}$. First note that $\tilde{T}_i$ is a well-defined contraction. Indeed, thinking of K as a block matrix we have

$$K_{i\alpha,i\beta} = \begin{cases} Q_i, & \text{if } \alpha = \beta = 0 \\ a_{io(\alpha)} a_{io(\beta)} K_{\alpha,\beta}, & \text{otherwise} \end{cases},$$

where we define $a_{io(\alpha)} = 1$ for all i if $\alpha = 0$. So if $F \subseteq X$ is a finite subset then

$$\begin{aligned} \left\| \sum_{(\alpha,u)\in F} \lambda(i\alpha, u) \right\|^2 &= \left\| \sum_{(\alpha,u)\in F} a_{io(\alpha)} \lambda(i\alpha, u) \right\|^2 \\ &= \sum_{\alpha=\beta=0} \langle u, Q_i v\rangle + \sum_{\alpha\neq 0 \text{ or } \beta\neq 0} a_{io(\alpha)} a_{io(\beta)} \langle u, K_{\alpha,\beta} v\rangle \\ &\le \sum_{\alpha=\beta=0} \langle u, v\rangle + \sum_{\alpha\neq 0 \text{ or } \beta\neq 0} \langle u, K_{\alpha,\beta} v\rangle \\ &= \left\| \sum_{(\alpha,u)\in F} \lambda(\alpha, u) \right\|^2. \end{aligned}$$

For $i \neq j$

$$\begin{aligned} \langle \tilde{T}_i \lambda(\alpha, u), \tilde{T}_j \lambda(\beta, v)\rangle &= \langle \lambda(i\alpha, u), \lambda(j\beta, v)\rangle \\ &= K((i\alpha, u), (j\beta, v)) = 0 \end{aligned}$$

since neither $i\alpha = j\beta\gamma$ nor $j\beta = i\alpha\gamma$ is possible. As required for dilations we have $\tilde{T}_i^* \lambda(0, u) = \lambda(0, T_i^* u)$ which may be seen as follows:

$$\begin{aligned} \langle \tilde{T}_i^* \lambda(0, u), \lambda(\beta, v)\rangle &= \langle \lambda(0, u), \tilde{T}_i \lambda(\beta, v)\rangle \\ &= \langle \lambda(0, u), \lambda(i\beta, v)\rangle \\ &= K((0, u), (i\beta, v)) \\ &= K((0, T_i^* u), (\beta, v)) \\ &= \langle \lambda(0, T_i^* u), \lambda(\beta, v)\rangle. \end{aligned}$$

Next we show that $\tilde{T}_i$ is a partial isometry by evaluating $\tilde{T}_i^*\tilde{T}_i\lambda(\alpha, u)$. By definition of K we have

$$\langle\lambda(\alpha, u), \tilde{T}_i^*\tilde{T}_i\lambda(\beta, v)\rangle = \begin{cases} a_{i,o(\alpha)}a_{i,o(\beta)}\langle\lambda(\alpha, u), \lambda(\beta, v)\rangle, & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\ \langle u, Q_i v\rangle, & \text{if } \alpha = \beta = 0 \end{cases}.$$

Thus $\tilde{T}_i^*\tilde{T}_i\lambda(\beta, u) = a_{i,o(\beta)}\lambda(\beta, u)$ if $\beta \neq 0$ and

$$\begin{aligned}\langle\lambda(0, u), \tilde{T}_i^*\tilde{T}_i\lambda(0, v)\rangle &= \langle u, Q_i v\rangle \\ &= \left\langle u, \sum_k a_{ik} T_k T_k^* v\right\rangle \\ &= \left\langle \lambda(0, u), \sum_k a_{ik}\lambda(k, T_k^* v)\right\rangle.\end{aligned}$$

Since $\langle\lambda(\alpha, u), \lambda(k, T_k^* v)\rangle = \delta_{ko(\alpha)}\langle u, (\underline{T}^\alpha)^* v\rangle$ for $\alpha \neq 0$ we have

$$\begin{aligned}\langle\lambda(\alpha, u), \tilde{T}_i^*\tilde{T}_i\lambda(0, v)\rangle &= a_{i,o(\alpha)}\langle\lambda(\alpha, u), \lambda(0, v)\rangle \\ &= \left\langle \lambda(\alpha, u), \sum_k a_{ik}\lambda(k, T_k^* v)\right\rangle\end{aligned}$$

and therefore

$$\tilde{T}_i^*\tilde{T}_i\lambda(\alpha, u) = \begin{cases} \sum_k a_{ik}\lambda(k, T_k^* u), & \text{if } \alpha = 0 \\ a_{io(\alpha)}\lambda(\alpha, u), & \text{otherwise} \end{cases}.$$

It follows that $\tilde{T}_i\tilde{T}_i^*\tilde{T}_i = \tilde{T}_i$, i.e. $\tilde{T}_i$ are partial isometries. Finally minimality holds as

$$\overline{\text{span}}\{\underline{\tilde{T}}^\alpha\lambda(0, u) \colon \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \overline{\text{span}}\{\lambda(\alpha, u) \colon \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \tilde{\mathcal{H}}.$$

Second proof using Popescu's method

Recall that $\underline{L}$, $\underline{S}$ denote the n-tuples of creation operators on $\Gamma(\mathbb{C}^n)$, Γ_A respectively.

DEFINITION 12

For a contractive tuple $\underline{T} = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$ the operator $\Delta_{\underline{T}} = (I - \sum_{i=1}^n T_i T_i^*)^{1/2}$ is called *defect operator* of $\underline{T}$. If $\sum_{\alpha\in\Lambda^m} \underline{T}^\alpha(\underline{T}^\alpha)^*$ converges to zero in the strong operator topology as m tends to infinity then this tuple is said to be *pure*.

Now let $\underline{T}$ be a pure tuple on $\mathcal{H}$ satisfying A-relations. Similarly as in [Po1],

$$Kh = \sum_{\alpha\in\tilde{\Lambda}_A} e^\alpha \otimes \Delta_{\underline{T}}(\underline{T}^\alpha)^* h \tag{3.1}$$

defines an isometry $K \colon \mathcal{H} \to \Gamma_A \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ such that $\underline{T}^\alpha = K^*(\underline{S}^\alpha \otimes I)K$. Moreover for all $\alpha \in \tilde{\Lambda}$, $S_i^* \otimes I$ leaves the range of K invariant and

$$\overline{\text{span}}\{(S_i \otimes I)Kh \colon i = 1, \ldots, n,\ h \in \mathcal{H}\} = \Gamma_A \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$$

since $((I - \sum S_i S_i^*) \otimes I)Kh = \omega \otimes \Delta_{\underline{T}} h$ and $\overline{\text{span}}\{\underline{S}^\alpha \omega \colon \alpha \in \tilde{\Lambda}_A\} = \Gamma_A$, the tuple $(S_1 \otimes I, \ldots, S_n \otimes I)$ is the minimal Cuntz–Krieger dilation of $\underline{T}$. Note that for $\alpha, \beta \in \tilde{\Lambda}_A$ and $P_0 = I - \sum_i S_i S_i^*$ we have

$$K^*[\underline{S}^\alpha P_0 (\underline{S}^\beta)^* \otimes I]Kh = K^*[\underline{S}^\alpha P_0 (\underline{S}^\beta)^* \otimes I] \left(\sum_\gamma \underline{S}^\gamma \omega \otimes \Delta_{\underline{T}} (\underline{T}^\gamma)^* h \right)$$

$$= K^* \left(\sum_\gamma \underline{S}^\alpha P_0 (\underline{S}^\beta)^* \underline{S}^\gamma \omega \otimes \Delta_{\underline{T}} (\underline{T}^\gamma)^* h \right)$$

$$= K^* (\underline{S}^\alpha \omega \otimes \Delta_{\underline{T}} (\underline{T}^\beta)^* h) = \underline{T}^\alpha \Delta_{\underline{T}}^2 (\underline{T}^\beta)^* h. \quad (3.2)$$

As in [Po1] starting with a contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{H}$, the tuple $r\underline{T} = (rT_1, \ldots, rT_n)$ is pure for $0 < r < 1$. By (3.1) there is an isometry $K_r \colon \mathcal{H} \to \Gamma_A \otimes \overline{\Delta_r(\mathcal{H})}$ defined by

$$K_r h = \sum_\alpha e^\alpha \otimes \Delta_r ((r\underline{T})^\alpha)^* h, \quad (3.3)$$

where $\Delta_r = (I - r^2 \sum T_i T_i^*)^{1/2}$. From this we obtain a unital completely positive map $\psi_r \colon C^*(\underline{S}) \to B(\mathcal{H})$ defined by $\psi_r(X) = K_r^*(X \otimes I)K_r$, $X \in C^*(\underline{S})$. As the family of maps ψ_r, where $0 < r < 1$, is uniformly bounded, ψ_r converges pointwise. Taking the limit as r increases to 1, we get a unique unital completely positive map θ from $C^*(\underline{S})$ to $B(\mathcal{H})$ satisfying

$$\theta(\underline{S}^\alpha (\underline{S}^\beta)^*) = \underline{T}^\alpha (\underline{T}^\beta)^* \quad \text{for } \alpha, \beta \in \tilde{\Lambda}_A. \quad (3.4)$$

Once we have this map we can use a minimal Stinespring dilation $\pi_1 \colon C^*(\underline{S}) \to B(\tilde{\mathcal{H}})$ of θ such that

$$\theta(X) = P_{\mathcal{H}} \pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S})$$

and $\overline{\text{span}}\{\pi_1(X)h \colon X \in C^*(\underline{S}), h \in \mathcal{H}\} = \tilde{\mathcal{H}}$. The tuple $\underline{\tilde{T}} = (\tilde{T}_1, \ldots, \tilde{T}_n)$ where $\tilde{T}_i = \pi_1(S_i)$, is the minimal Cuntz–Krieger dilation of $\underline{T}$ which is unique up to unitary equivalence. $\underline{\tilde{T}}$ consists of partial isometries with orthogonal ranges satisfying A-relations.

We remark that if $\underline{R}$ is a tuple consisting of partial isometries with orthogonal ranges satisfying the condition

$$R_i^* R_i = I - \sum_{j=1}^n (1 - a_{ij}) R_j R_j^*$$

(e.g., a Cuntz–Krieger dilation), then the completely positive map Θ in (3.4) mapping $\underline{S}^\alpha (\underline{S}^*)^\beta$ to $\underline{R}^\alpha (\underline{R}^*)^\beta$ is $*$-homomorphisms because the S_i's and R_i's have orthogonal ranges and for $1 \leq i \leq n$,

$$\Theta(S_i^* S_i) = \Theta \left(I - \sum_j (1 - a_{ij}) S_j S_j^* \right) = I - \sum_j (1 - a_{ij}) R_j R_j^*$$

$$= R_i^* R_i = \Theta(S_i^*) \Theta(S_i).$$

The algebra generated by a Cuntz–Krieger dilation

The tuple obtained from the above constructions satisfies Cuntz–Krieger relations, that is,

$$\tilde{T}_i^* \tilde{T}_i = I - \sum_j (1 - a_{ij}) \tilde{T}_j \tilde{T}_j^*. \tag{3.5}$$

We consider the C^*-algebra generated by such tuples.

First consider the C^*-algebra generated by left creation operators on A-Fock space. Since for any $\alpha, \beta \in \tilde{\Lambda}_A$ the rank-one operator $\eta \mapsto \langle \underline{S}^\beta \omega, \eta \rangle \underline{S}^\alpha \omega$ on Γ_A can be written as $\underline{S}^\alpha (I - \sum S_i S_i^*)(\underline{S}^\beta)^* = \underline{S}^\alpha P_0 (\underline{S}^\beta)^*$ and they span the subalgebra of compact operators in $C^*(\underline{S})$, we conclude that $C^*(\underline{S})$ also contains all compact operators. For $\underline{\tilde{T}} = \pi_1(\underline{S})$ the Hilbert space $\tilde{\mathcal{H}}$ can be decomposed as $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_C \oplus \tilde{\mathcal{H}}_N$, where

$$\tilde{\mathcal{H}}_C := \overline{\text{span}}\{\pi_1(X)h \colon h \in \tilde{\mathcal{H}},\, X \in C^*(\underline{\tilde{T}}) \text{ and compact}\}$$

is bi-invariant with respect to the $\tilde{T}_i$'s, that is, invariant with respect to $\tilde{T}_i$ and $\tilde{T}_i^*$ for all i. Thus π_1 can be decomposed as $\pi_{1C} \oplus \pi_{1N}$, where $\pi_{1C}(X) = P_{\tilde{\mathcal{H}}_C} \pi_1(X) P_{\tilde{\mathcal{H}}_C}$ and $\pi_{1N}(X) = P_{\tilde{\mathcal{H}}_N} \pi_1(X) P_{\tilde{\mathcal{H}}_N}$. As π_{1N} annihilates compacts, $\pi_{1N}(I - \sum S_i S_i^*) = \pi_{1N}(P_0) = 0$, hence $(\pi_{1N}(S_1), \dots, \pi_{1N}(S_n))$ satisfy Cuntz–Krieger relations (in particular A-relations) and generate a Cuntz–Krieger algebra.

Let $\mathcal{K}$ be the range of $\pi_1(P_0)$ then $\pi_{1C}(\underline{S}^\alpha)k \mapsto e^\alpha \otimes k$ extends to a unitary equivalence between $\pi_{1C}(\underline{S})$ and $\underline{S} \otimes I$ on $\Gamma_A \otimes \mathcal{K}$ so that $\mathcal{K}$ is a wandering subspace for $\underline{\tilde{T}}$ generating $\tilde{\mathcal{H}}_C$.

The isometry in Stinespring's theorem is of the form $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ such that V_1 maps $\mathcal{H}$ to $\Gamma_A \otimes \mathcal{K}$ and V_2 maps $\mathcal{H}$ to $\tilde{\mathcal{H}}_N$. Now for $\alpha, \beta \in \tilde{\Lambda}$,

$$\begin{aligned} \underline{T}^\alpha \Delta_{\underline{T}}^2 (\underline{T}^\beta)^* h &= \theta \left(\underline{S}^\alpha \left(I - \sum_i S_i S_i^* \right) (\underline{S}^\beta)^* \right)(h) \\ &= V_1^* [\underline{S}^\alpha P_0 (\underline{S}^\beta)^* \otimes I] V_1(h) + V_2^* [\pi_{1N}(\underline{S}^\alpha P_0 (\underline{S}^\beta)^*)] V_2(h) \\ &= V_1^* [\underline{S}^\alpha P_0 (\underline{S}^\beta)^* \otimes I] V_1(h) \end{aligned}$$

as π_{1N} annihilate compacts. Comparison with identity (3.2) shows that V_1 may be taken to be K. Hence $\mathcal{K} := \overline{\Delta_{\underline{T}}(\mathcal{H})}$.

In fact given just a tuple $\underline{R}$ verifying

$$R_i^* R_i = I - \sum_{j=1}^n (1 - a_{ij}) R_j R_j^*,$$

it is clear that we will always obtain a decomposition of this type as the minimal Cuntz–Krieger dilation of such a tuple is the tuple itself. Such a decomposition is called *Wold decomposition.*

Using arguments similar to Theorem 1.3 in [Po1] we conclude that

$$\tilde{\mathcal{H}}_N = \bigcap_{m=0}^{\infty} \overline{\text{span}}\{\underline{\tilde{T}}^\alpha h \colon h \in \tilde{\mathcal{H}},\, |\alpha| = m\}.$$

COROLLARY 13

Suppose $\underline{\hat{T}}$ is the minimal isometric dilation of a contractive tuple $\underline{T}$ satisfying A-relations.

(1) $\operatorname{rank}(I - \sum_i \hat{T}_i \hat{T}_i^*) = \operatorname{rank}(I - \sum_i \tilde{T}_i \tilde{T}_i^*) = \operatorname{rank}(I - \sum_i T_i T_i^*)$.
(2) $\lim_{k\to\infty} \sum_{|\alpha|=k} \underline{\tilde{T}}^{\alpha} (\underline{\tilde{T}}^{\alpha})^* = P_{\tilde{\mathcal{H}}_N}$.

Proof. Clear. □

Now we would like to see how the minimal Cuntz–Krieger dilation and the minimal isometric dilation are related. The following is an analogue of Theorem 13 in [BBD] for the maximal A-relation piece.

Theorem 14. *Let $\underline{T}$ be a contractive n-tuple on a Hilbert space $\mathcal{H}$ satisfying A-relations. Then the maximal A-relation piece of the standard noncommuting dilation $\underline{\hat{T}}$ of $\underline{T}$ is a realization of the minimal Cuntz–Krieger dilation $\underline{\tilde{T}}$ of $\underline{T}$.*

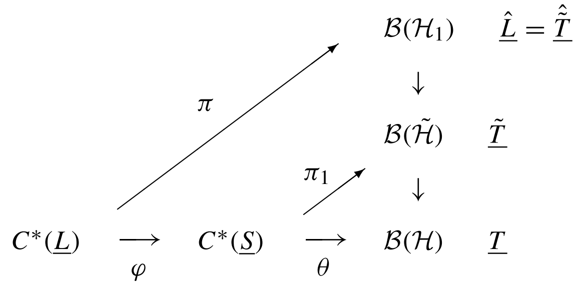
Proof. Let $\theta\colon C^*(\underline{S}) \to B(\mathcal{H})$ be the unital completely positive map as in eq. (3.4), π_1 the corresponding minimal Stinespring dilation and $\tilde{T}_i = \pi_1(S_i)$ as before. Since the standard tuple $\underline{S}$ on Γ_A is also a contractive tuple, there is a unique completely positive map φ from the C^*-algebra $C^*(\underline{L})$ generated by the left creation operators to $C^*(\underline{S})$, satisfying

$$\varphi(\underline{L}^{\alpha}(\underline{L}^{\beta})^*) = \underline{S}^{\alpha}(\underline{S}^{\beta})^* \quad \text{for } \alpha, \beta \in \tilde{\Lambda}.$$

Thus ψ as defined before on $C^*(\underline{L})$ satisfies $\psi = \theta \circ \varphi$. Let the minimal Stinespring dilation of $\pi_1 \circ \varphi$ be the $*$-homomorphism $\pi\colon C^*(\underline{L}) \to B(\mathcal{H}_1)$ for some Hilbert space $\mathcal{H}_1 = \overline{\operatorname{span}}\{\pi(X)h\colon X \in C^*(\underline{L}), h \in \tilde{\mathcal{H}}\}$ such that

$$\pi_1 \circ \varphi(X) = P_{\tilde{\mathcal{H}}} \pi(X)|_{\tilde{\mathcal{H}}} \ \forall X \in C^*(\underline{L}).$$

In the following commuting diagram



all horizontal arrows are unital completely positive maps, down arrows are compressions and diagonal arrows are minimal Stinespring dilations. Let $\hat{L}_i = \pi(L_i)$ and $\underline{\hat{L}} = (\hat{L}_1, \ldots, \hat{L}_n) = \underline{\hat{\hat{T}}}$. We will first show that $\underline{\tilde{T}}$ is the maximal A-relation piece of $\underline{\hat{L}}$ and then show that $\underline{\hat{L}}$ is the standard noncommuting dilation of $\underline{T}$.

To this end we use the presentation of the minimal isometric dilation $\underline{\hat{L}}$ which was given by Popescu [Po1]. In the one-variable case it was given by Schäffer (see also [BBD]). Define $D\colon \underbrace{\tilde{\mathcal{H}} \oplus \cdots \oplus \tilde{\mathcal{H}}}_{n\text{-copies}} \to \underbrace{\tilde{\mathcal{H}} \oplus \cdots \oplus \tilde{\mathcal{H}}}_{n\text{-copies}}$ by

$$D^2 = [\delta_{ij} I - \tilde{T}_i^* \tilde{T}_j]_{n\times n} = [\delta_{ij}(I - \tilde{T}_i^* \tilde{T}_i)]_{n\times n}$$

as used by Popescu. Note that D^2 is a projection as all $\tilde{T}_i$'s are partial isometries and hence $D^2 = D$. Let $\mathcal{D}$ denote the range of D. We identify $\underbrace{\tilde{\mathcal{H}} \oplus \cdots \oplus \tilde{\mathcal{H}}}_{n\text{-copies}}$ with $\mathbb{C}^n \otimes \tilde{\mathcal{H}}$ and hence $(h_1, \ldots, h_n)$ with $\sum_{i=1}^n e_i \otimes h_i$ and $\mathbb{C}\omega \otimes \mathcal{D}$ with $\mathcal{D}$.

$$D(h_1, \ldots, h_n) = D\left(\sum_{i=1}^n e_i \otimes h_i\right) = \sum_{i=1}^n e_i \otimes (I - \tilde{T}_i^* \tilde{T}_i) h_i.$$

For $h \in \tilde{\mathcal{H}}$, $d_\alpha \in \mathcal{D}$, $1 \leq i \leq n$, the standard noncommuting dilation $\underline{\hat{L}} = (\hat{L}_1, \ldots, \hat{L}_n)$ is given by

$$\hat{L}_i\left(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha\right) = \tilde{T}_i h \oplus D(e_i \otimes h) \oplus e_i \otimes \left(\sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha\right)$$

on the dilation space $\mathcal{H}_1 = \tilde{\mathcal{H}} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathcal{D})$. As $\underline{\tilde{T}}$ satisfies A-relations and $\hat{L}_i^*$ leaves $\tilde{\mathcal{H}}$ invariant, $\tilde{\mathcal{H}} \subseteq (\mathcal{H}_1)_A(\underline{\hat{L}})$. To show the reverse inclusion suppose that there exists a nonzero $z \in \tilde{\mathcal{H}}^\perp \cap (\mathcal{H}_1)_A(\underline{\hat{L}})$. z can be written as $0 \oplus \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes z_\alpha$ such that $z_\alpha \in \mathcal{D}$. Since $\langle \omega \otimes z_\alpha, (\underline{\hat{L}}^\alpha)^* z\rangle = \langle e^\alpha \otimes z_\alpha, z\rangle = \langle z_\alpha, z_\alpha\rangle$ and $(\underline{\hat{L}}^\alpha)^* z \in (\mathcal{H}_1)_A(\underline{\hat{L}})$, we can assume $\|z_0\| = 1$ without loss of generality. Also $z_0 = D(h_1, \ldots, h_n)$ for some $h_i \in \tilde{\mathcal{H}}$ as projections have closed ranges. Now consider

$$\begin{aligned}
\sum_{i,j=1}^n (\hat{L}_i \hat{L}_j - a_{ij} \hat{L}_i \hat{L}_j) \tilde{T}_j^* h_i &= \sum_{i,j=1}^n (\tilde{T}_i \tilde{T}_j - a_{ij} \tilde{T}_i \tilde{T}_j) \tilde{T}_j^* h_i \\
&\quad + \sum_{i=1}^n D\left(e_i \otimes \sum_{j=1}^n (1 - a_{ij}) \tilde{T}_j \tilde{T}_j^* h_i\right) \\
&\quad + \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{T}_j^* h_i) \\
&= 0 + \sum_{i=1}^n D[e_i \otimes (I - \tilde{T}_i^* \tilde{T}_i) h_i] + x \\
&= D^2(h_1, \ldots, h_n) + x = \tilde{z}_0 + x,
\end{aligned}$$

where $x = \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{T}_j^* h_i)$. Thus $\langle z, \tilde{z}_0 + x\rangle = 0$ by Lemma 4. Moreover,

$$\begin{aligned}
\|x\|^2 &= \left\| \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{T}_j^* h_i) \right\|^2 \\
&= \sum_{i=1}^n \left\langle \sum_{j=1}^n (1 - a_{ij}) D(e_j \otimes \tilde{T}_j^* h_i), \sum_{j'=1}^n (1 - a_{ij'}) e_{j'} \otimes \tilde{T}_{j'}^* h_i \right\rangle
\end{aligned}$$

$$= \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} (1 - a_{ij})(I - \tilde{T}_j^* \tilde{T}_j) \tilde{T}_j^* h_i, \sum_{j=1}^{n} (1 - a_{ij}) \tilde{T}_j^* h_i \right\rangle$$

$$= \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} (1 - a_{ij})(\tilde{T}_j^* - \tilde{T}_j^*) h_i, \sum_{j=1}^{n} (1 - a_{ij}) \tilde{T}_j^* h_i \right\rangle = 0,$$

i.e. $x = 0$ and therefore $\|\tilde{z}_0\|^2 = \langle z, \tilde{z}_0 \rangle = 0$ which is a contradiction. Hence $z = 0$ which implies $\mathcal{H} = (\mathcal{H}_1)_A(\hat{\underline{L}})$.

To finally show that $\hat{\underline{L}}$ is standard note that

$$\mathcal{H}_1 = \overline{\text{span}}\{\hat{\underline{L}}^\alpha x : \alpha \in \tilde{\Lambda}_A,\ x \in \tilde{\mathcal{H}}\}$$

and

$$\tilde{\mathcal{H}} = \overline{\text{span}}\{\tilde{\underline{T}}^\alpha z : \alpha \in \tilde{\Lambda}_A,\ z \in \mathcal{H}\},$$

moreover, $P_{\tilde{\mathcal{H}}} \hat{\underline{L}}^\alpha = \tilde{\underline{T}}^\alpha$ and $P_{\mathcal{H}} \hat{\underline{L}}^\alpha = \underline{T}^\alpha$ for $\alpha \in \tilde{\Lambda}_A$ by assumption. Hence

$$\begin{aligned}
\mathcal{H}_1 &= \overline{\text{span}}\{\hat{\underline{L}}^\alpha x : \alpha \in \tilde{\Lambda}_A,\ x \in \tilde{\mathcal{H}}\} \\
&= \overline{\text{span}}\{\hat{\underline{L}}^\alpha \tilde{\underline{T}}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A,\ z \in \mathcal{H}\} \\
&= \overline{\text{span}}\{\hat{\underline{L}}^\alpha P_{\tilde{\mathcal{H}}} \hat{\underline{L}}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A,\ z \in \mathcal{H}\} \\
&\subseteq \overline{\text{span}}\{\hat{\underline{L}}^\alpha \hat{\underline{L}}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A,\ z \in \mathcal{H}\} = \mathcal{H}_1.
\end{aligned}$$

□

In the same way one can show that similar results hold even for q-commuting tuples considered in [BB] (see [BBD]). To keep the presentation simpler we have worked with the above special case. The following example illustrates the foregoing results.

Example 15. For $\mathcal{H} = \mathbb{C}^2$, let $T_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then one observes that $\underline{T}$ satisfies A-relations for the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $T_1 T_1^* + T_2 T_2^* = I$ and the T_i's are partial isometries. Further D used in the above theorem turns out to be

$$D = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

Let us denote the two basis vectors in the range of D corresponding to the entries 1 appearing in D by f_1 and f_2 such that

$$D(e_1 \otimes (a_1, a_2) + e_2 \otimes (b_1, b_2)) = a_1 f_1 + b_2 f_2$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{C}$. The dilation space for the minimal isometric dilation $\underline{V} = (\tilde{V}_1, \tilde{V}_2)$ of $\underline{T}$ is $\mathcal{H} \oplus \Gamma(\mathbb{C}^n) \otimes \mathcal{D}$ where $\mathcal{D}$ is the range of D.

$$\tilde{V}_1 \tilde{V}_1(a_1, a_2) = (0, 0) + \omega \otimes (a_2, 0) + e_1 \otimes (a_1, 0),$$

$$\tilde{V}_2 \tilde{V}_2(a_1, a_2) = (0, 0) + \omega \otimes (0, a_1) + e_2 \otimes (0, a_2).$$

As a_1, a_2 are arbitrary using the above equations together with the description of the maximal A-relation piece from Lemma 4 we get $\tilde{\mathcal{H}} = \mathcal{H}$.

4. Representations of Cuntz–Krieger algebras

In general a Cuntz–Krieger algebra $\mathcal{O}_A$ admits many inequivalent representations. When $\underline{T} = (T_1, \ldots, T_n)$ is a tuple on the Hilbert space $\mathcal{H}$ satisfying A-relations and $\sum_{i=1}^n T_i T_i^* = I$, the minimal Cuntz–Krieger dilation $\underline{\tilde{T}} = (\tilde{T}_1, \ldots, \tilde{T}_n)$ is such that $C^*(\underline{\tilde{T}})$ is a Cuntz–Krieger algebra. If $\underline{T}$ is commuting then the unital completely positive map $\rho_{\underline{T}}\colon \mathcal{O}_A \to C^*(\underline{\tilde{T}})$ given by $\rho_{\underline{T}}(s_i) = \tilde{T}_i$ is a representation of $\mathcal{O}_A$. We will classify such representations here. In brief, the classification results that we prove here says any representation of $\mathcal{O}_A$ can be decomposed as the direct sum of a spherical representation (Definition 16) and a representation corresponding to trivial maximal commuting piece. Further, any spherical representation turns out to be the direct integral of GNS representations of certain pure states on $\mathcal{O}_A$.

For a tuple $\underline{R} = (R_1, \ldots, R_n)$ on a Hilbert space $\mathcal{K}$, we use the concept of *maximal commuting piece* and the space $\mathcal{K}^c(\underline{R})$ as defined before Lemma 6 and §2 of [BBD]. We refer to $\mathcal{K}^c(\underline{R})$ as the *maximal commuting subspace.*

DEFINITION 16

(1) A commuting tuple $\underline{T} = (T_1, \ldots, T_n)$ is called *spherical unitary* if $\sum T_i T_i^* = I$ and all T_i's are normal.
(2) A representation ρ of $\mathcal{O}_A$ on $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$, is said to be *spherical* if $R_i = \rho(s_i)$, $1 \le i \le n$ and $\mathcal{K} = \{\underline{R}^\alpha k\colon k \in \mathcal{K}^c(\underline{R})$ and $\alpha \in \tilde{\Lambda}\}$.

If $\underline{T}$ is a spherical unitary then by Fuglede's theorem $C^*(\underline{T})$ is a commutative C^*-algebra, i.e. T_i^* and T_j also commute for all i and j.

DEFINITION 17

The *maximal commuting A-subspace* of a n-tuple of isometries $\underline{V}$ with orthogonal ranges is defined as the intersection of its maximal commuting subspace and maximal A-relation subspace. The n-tuple obtained by compressing each V_i to the maximal commuting A-subspace is called *maximal commuting A-piece.*

Remark 18. Making use of Lemma 4, it follows that

$$\mathcal{K}_A \cap \mathcal{K}^c = (\mathcal{K}_A)^c = (\mathcal{K}^c)_A,$$

i.e. the maximal commuting A-subspace of a n-tuple is in fact the maximal commuting subspace of the maximal A-relation piece or the maximal A-relation subspace of the maximal commuting piece.

Let $P_0 = 1$ on $\mathbb{C}$ and P_m be the projection $\frac{1}{m!}\sum_{\sigma \in \mathcal{S}_m} U_\sigma^m$ acting on $(\mathbb{C}^n)^{\otimes^m}$ where

$$U_\sigma^m(y_1 \otimes \cdots \otimes y_m) = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(m)},$$

$y_i \in \mathbb{C}^n$. We denote $\oplus_{m=0}^\infty P_m$ by P^s. Given A-relations define $\Lambda_{sA}^m = \{\alpha \in \tilde{\Lambda}$: either $|\alpha| = m > 1$ and $a_{\alpha_i \alpha_j} = 1$ for $1 \le i \ne j \le m$, or $|\alpha| \le 1\} \subseteq \Lambda_A^m$.

DEFINITION 19

The subspace of Γ_A defined by $\overline{\text{span}}\{P^s e^\alpha\colon \alpha \in \Lambda_{sA}^m\}$ is called *commuting A-Fock space* and denoted by Γ_{sA}.

To see that Γ_{sA} is the maximal commuting A-subspace of $\underline{L}$ we first note that the maximal commuting A-subspace of $\underline{L}$ is the intersection of the symmetric Fock space $\Gamma_s(\mathbb{C}^n)$ (see [BBD]) and the maximal A-relation subspace of $\underline{L}$. Also

$$\Gamma_s(\mathbb{C}^n) = \overline{\text{span}}\{P^s e^\alpha \colon \alpha \in \tilde{\Lambda}\}.$$

Suppose $\alpha \in \Lambda^m$ and for all $1 \leq k \neq l \leq m$, $a_{\alpha_k \alpha_l} = 1$ then for $h \in \Gamma(\mathbb{C}^n)$ and all i, j

$$\langle P^s e^\alpha, \underline{L}^\beta (L_i L_j - a_{ij} L_i L_j) h \rangle = 0.$$

So, from the definition it is clear that

$$\Gamma_{sA} \subseteq \Gamma_s(\mathbb{C}^n) \cap \Gamma_A.$$

Let $\hat{P}$ denote the projection onto $\Gamma_s(\mathbb{C}^n) \cap \Gamma_A$ and let $z \in \Gamma_s(\mathbb{C}^n) \cap \Gamma_A$ be arbitrary. Suppose $\alpha \in \Lambda^m$ is such that $\langle e^\alpha, z\rangle$ is not equal to 0. As $z \in \Gamma_A$, it follows that $\alpha \in \Lambda_A^m$. Further, for any $\sigma \in S_m$

$$\begin{aligned} \langle U_\sigma^m e^\alpha, z\rangle &= \langle U_\sigma^m e^\alpha, \hat{P} z\rangle = \langle \hat{P} U_\sigma^m e^\alpha, z\rangle \\ &= \langle \hat{P} e^\alpha, z\rangle = \langle e^\alpha, z\rangle. \end{aligned}$$

Thus $\langle U_\sigma^m e^\alpha, z\rangle$ is not equal to 0. This implies that $\alpha \in \Lambda_{sA}^m$ and hence $z \in \Gamma_{sA}$. We conclude that Γ_{sA} is the maximal commuting A-subspace of $\underline{L}$.

Also we would like to remark that the fermionic Fock space (see [De]) $\Gamma_a(\mathbb{C}^n)$ is the intersection of the maximal q-commuting subspace (defined in [De]) and the maximal A-relation subspace with respect to the following $q = (q_{ij})_{n\times n}$ and $A = (a_{ij})_{n\times n}$:

$$q_{ij} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{otherwise} \end{cases} \quad \text{and} \quad a_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{otherwise} \end{cases}.$$

It is easy to see this using arguments similar to that we use to show that Γ_{sA} is the maximal commuting A-subspace with respect to $\underline{L}$. In other words, the fermionic Fock space $\Gamma_a(\mathbb{C}^n)$ is the maximal piece for the set of polynomials:

$$p_{1ij}(\underline{z}) = z_j z_i - q_{ij} z_i z_j \quad \text{and} \quad p_{2ij}(\underline{z}) = z_i z_j - a_{ij} z_i z_j \quad \forall 1 \leq i, j \leq n.$$

Notice that L_i^* leaves Γ_{sA} invariant as S_i^* leaves Γ_{sA} invariant. Let the compression of L_i onto Γ_{sA} be denoted by W_i, i.e. $\underline{W}$ is the maximal commuting A-relation piece of $\underline{L}$. Suppose $\alpha \in \Lambda_{sA}^m$, and when $|\alpha| > 0$, let $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $m = |\alpha|$. The operator W_i turns out to be

$$W_i P^s e^\alpha = \begin{cases} e_i, & \text{if } |\alpha| = 0 \\ P^s e_i \otimes e^\alpha, & \text{if } a_{i\alpha_j} a_{\alpha_j i} = 1, \quad \forall 1 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}.$$

From this it follows that $\underline{W} = (W_1, \ldots, W_n)$ is the maximal commuting piece satisfying A-relations of $\underline{L}$. Let us denote the maximal commuting piece of $\underline{L}$ on $\Gamma(\mathbb{C}^n)$ by $\underline{C} =$

$(C_1, \ldots, C_n)$. (These are just the creation operators on the symmetric Fock space.) Then for $\alpha \in \Lambda^m_{sA}$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $m > 1$ the commutators verify

$$[W_i, W_i^*]P^s e^\alpha = \begin{cases} [C_i, C_i^*]P^s e^\alpha, & \text{if } a_{i\alpha_j} a_{\alpha_j i} = 1, \quad \forall 1 \le j \le m \text{ or if } \alpha = 0 \\ \frac{1}{m!} P^s e^\alpha, & \text{if } \alpha_j = i \quad \text{for some } 1 \le j \le m \text{ and } a_{ii} = 0\ . \\ 0, & \text{otherwise} \end{cases}$$

It is known that $[C_i, C_i^*]$ is compact for all i (see Proposition 5.3 of [Ar]), so the $[W_i, W_i^*]$'s are compact. Clearly, the vacuum vector is contained in Γ_{sA} and $I - \sum W_i W_i^*$ is the projection onto the vacuum space. It contains all the rank-one operators of the type $\mu \to \langle \underline{W}^\alpha \omega, \mu \rangle \underline{W}^\beta \omega$ on Γ_{sA} as those can be written as $\underline{W}^\alpha (I - \sum W_i W_i^*)(\underline{W}^\beta)^*$. As these rank-one operators span the subalgebra of compact operators, we conclude that $C^*(\underline{W})$ contains the subalgebra of all compact $\mathcal{K}$ as an ideal. Since the image of $\underline{W}$ in $C^*(\underline{W})/\mathcal{K}$ is a spherical unitary it follows from Fuglede's theorem that $[W_i, W_j^*]$ where $i \neq j$ must be compact and we also conclude that

$$C^*(\underline{W}) = \overline{\text{span}}\{\underline{W}^\alpha (\underline{W}^\beta)^* \colon \alpha, \beta \in \tilde{\Lambda}\}.$$

For a commuting pure tuple $\underline{T}$ satisfying A-relations, with easy computations it can be seen that the range of the isometry $K_r \colon \mathcal{H} \to \Gamma_A \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$, $1 \le r \le 1$, defined in eq. (3.4) is contained in $\Gamma_{sA} \otimes \overline{\Delta_{\underline{T}} \mathcal{H}}$ and we obtain a unital completely positive map $\phi \colon C^*(\underline{W}) \to B(\mathcal{H})$ defined as strong operator topology limit of $K_r^*(.\otimes I)K_r$ as r increases to 1. Let $\pi_0 \colon C^*(\underline{W}) \to B(\mathcal{H}_0)$ be the minimal Stinespring dilation of ϕ for some Hilbert space $\mathcal{H}_0$ and $\check{W}_i = \pi_0(W_i)$ where $\mathcal{H}_0 = \overline{\text{span}}\{\underline{\check{W}}^\alpha h \colon \alpha \in \tilde{\Lambda}, h \in \mathcal{H}\}$.

DEFINITION 20

The above-defined tuple $\underline{\check{W}} = (\check{W}_1, \ldots, \check{W}_n)$ is said to be the *standard commuting A-dilation of $\underline{T}$.*

Remark 21. It follows from Theorem 15 in [BBD] that for spherical unitaries $\underline{T}$ satisfying A-relation the maximal commuting piece of the standard noncommuting dilation is $\underline{T}$. As $\underline{T}$ satisfies A-relations, it is clear that $\underline{T}$ is also the maximal commuting A-piece.

So far for a commuting contractive tuple satisfying A-relations we have four types of standard minimal dilations: the isometric dilation, the Cuntz–Krieger dilation (or A-dilation), the commuting dilation and the commuting A-dilation. These are obtained by considering Stinespring dilations of suitable completely positive maps on $C^*(\underline{L})$, $C^*(\underline{S})$, $C^*(\underline{C})$, and $C^*(\underline{W})$ respectively. The last dilation is in a certain sense the intersection of the previous two. The next lemma, which is a generalization of Theorem 13 in [BBD] makes this statement rigorous. This will be crucial for classifying certain types of representations of Cuntz–Krieger algebras.

Lemma 22. *The maximal commuting piece of the minimal Cuntz–Krieger dilation of a commuting tuple $\underline{T}$ satisfying A-relations is the standard commuting A-dilation.*

Proof. Let the unital completely positive map $\phi \colon C^*(\underline{W}) \to B(\mathcal{H})$, π_0 and $\mathcal{H}_0$ be as above. We denote the operator $\pi_0(W_i)$ by $\check{W}_i$ and denote the n-tuple $(\check{W}_1, \ldots, \check{W}_n)$ by $\underline{\check{W}}$. As $\underline{W}$ is a contractive tuple satisfying A-relation, there is a unital completely positive map $\eta \colon C^*(\underline{S}) \to C^*(\underline{W})$ such that $\eta(\underline{S}^\alpha(\underline{S}^\beta)^*) = \underline{W}^\alpha(\underline{W}^\beta)^*$. The completely positive map θ

as in eq. (3.4) is equal to $\phi \circ \eta$. Let $\tilde{\pi}_1$ be the minimal Stinespring dilation of $\pi_0 \circ \eta$ and $V_i = \tilde{\pi}_1(S_i)$. We have the following commuting diagram.

$$\begin{array}{ccccc} & & & & \mathcal{B}(\tilde{\mathcal{H}}_1) \\ & \overset{\tilde{\pi}_1}{\nearrow} & & & \downarrow \\ & & & & \mathcal{B}(\mathcal{H}_0)\,. \\ & & & \overset{\pi_0}{\nearrow} & \downarrow \\ C^*(\underline{S}) & \underset{\eta}{\longrightarrow} & C^*(\underline{W}) & \underset{\phi}{\longrightarrow} & \mathcal{B}(\mathcal{H}) \end{array}$$

As before, horizontal arrows are completely positive maps, diagonal arrows are $*$-homomor-phism and down arrows are compressions.

Since $C^*(\underline{W})$ contains all compact operators, $\mathcal{H}_0$ can again be decomposed as $\mathcal{H}_{0C} \oplus \mathcal{H}_{0N}$ where $\mathcal{H}_{0C} = \overline{\text{span}}\{\pi_0(X)h \colon h \in \mathcal{H}, X \in C^*(\underline{W}), X \text{ compact}\}$ and $\mathcal{H}_{0N} = \mathcal{H}_0 \ominus \mathcal{H}_{0C}$. Correspondingly,

$$\pi_0(X) = \begin{pmatrix} \pi_{0C}(X) & \\ & \pi_{0N}(X) \end{pmatrix},$$

where $\pi_{0C}(X)$ and $\pi_{0N}(X)$ are compressions of $\pi_0(X)$ to $\mathcal{H}_{0C}$ and $\mathcal{H}_{0N}$ respectively. Furthermore $\mathcal{H}_{0C} = \Gamma_{sA} \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ and $\pi_{0C}(X) = X \otimes I$. Let $E_i = \pi_{0N}(W_i)$ and $\underline{E} = (E_1, \ldots, E_n)$. As $[W_i, W_i^*]$ and $I - \sum W_i W_i^*$ are compacts, clearly $\underline{E}$ consists of pairwise commuting normal operators, i.e. $\underline{E}$ is a spherical unitary satisfying A-relations.

From the properties of Popescu's Poisson transform and Γ_{sA}, it follows that $(W_1 \otimes I, \ldots, W_n \otimes I)$ is the maximal commuting A-piece of its standard noncommuting dilation $(L_1 \otimes I, \ldots, L_n \otimes I)$. Also from Remark 21 we conclude that $\underline{E}$ is the maximal commuting A-piece of its standard noncommuting dilation. Using Remark 18 and Theorem 14 we observe that each of them is the maximal commuting piece of their minimal Cuntz–Krieger dilation. Hence by Lemma 6, $\underline{\check{W}}$ is the maximal commuting piece of $\underline{V}$. From this using arguments similar to Theorem 13 in [BBD] it can be shown that $\underline{V}$ is the minimal Cuntz–Krieger dilation of $\underline{\check{W}}$. Hence the lemma follows. □

If a commuting contractive tuple $\underline{T}$ also satisfies A-relations for $A = (a_{ij})_{n \times n}$, then without loss of generality we may assume A to be symmetric, i.e., $A = A^t$. In this case A is the adjacency matrix of a (nondirected) graph G with vertex set $\{1, 2, \ldots, n\}$ and set of edges $E = \{(i, j) \colon a_{ij} = 1, 1 \le i < j \le n\}$. A vertex i is said to be a *zero vertex* if $a_{ii} = 0$. Let us associate to this graph a subset M of $\{(z_1, \ldots, z_n) \colon \sum_{i=1}^n |z_i|^2 = 1\}$ defined as the set of elements satisfying A-relations, that is

$$M = \left\{ (z_1, \ldots, z_n) \colon \sum_{i=1}^{n} |z_i|^2 = 1, z_i z_j = a_{ij} z_i z_j, 1 \le i, j \le n \right\}.$$

The set M can be described in the following way: For a zero vertex i, the corresponding z_i of any element of M will always be taken as zero. For any element $(z_1, \ldots, z_n)$ of M, some elements $z_{i_1}, \ldots, z_{i_k}$ for different $1 \le i_k \le n$ can be simultaneously chosen to be nonzero if and only if $i_1, \ldots, i_k$ are nonzero vertices and form vertices of an induced subgraph of G which is also complete.

Let C_n^M be the C^*-algebra of continuous complex-valued functions on M. Consider the tuple $\underline{z} = (z_1, \dots, z_n)$ of co-ordinate functions z_i in C_n^M. To any spherical unitary $\underline{R} = (R_1, \dots, R_n)$ satisfying A-relations there corresponds a unique representation of C_n^M mapping z_i to R_i. As for any commuting tuple $\underline{T}$ satisfying A-relations with $\sum T_i T_i^* = I$, the standard commuting dilation $\underline{\tilde{T}} = (\tilde{T}_1, \dots, \tilde{T}_n)$ is a spherical unitary (see §3 of [BBD]), we have a representation $\eta_{\underline{T}}$ of C_n^M such that $\eta_{\underline{T}}(z_i) = \tilde{T}_i$. From Theorem 14, it is easy to see that if $\underline{D}$ and $\underline{E}$ are two commuting n-tuples of operators satisfying the same A-relations (on not necessarily the same Hilbert space), the corresponding representations $\rho_{\underline{D}}$ and $\rho_{\underline{E}}$ of $\mathcal{O}_A$ are unitarily equivalent if and only if the representations $\eta_{\underline{D}}$ and $\eta_{\underline{E}}$ of C_n^M are unitarily equivalent.

Any $z = (z_1, \dots, z_n) \in M$ satisfying A-relations as operator tuple on $\mathbb{C}$ is a spherical unitary. We can obtain a one-dimensional representation η_z of C_n^M which maps f to $f(z)$. Let $(V_1^z, \dots, V_n^z)$ and $(S_1^z, \dots, S_n^z)$ be the standard noncommuting dilation and the minimal Cuntz–Krieger dilation respectively of this operator tuple $z = (z_1, \dots, z_n)$. The dilation space of the standard noncommuting dilation is

$$\mathcal{H}^z = \mathbb{C} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}_z^n),$$

where $\mathbb{C}_z^n$ is the $(n-1)$-dimensional subspace of $\mathbb{C}^n$ orthogonal to $(\bar{z}_1, \dots, \bar{z}_n)$ and

$$V_i^z \left(h \oplus \sum_\alpha e^\alpha \otimes d_\alpha \right) = a_i \oplus D(e_i \otimes h) \oplus e_i \otimes \left(\sum_\alpha e^\alpha \otimes d_\alpha \right).$$

Using the minimal Cuntz–Krieger dilation $\underline{S}^z$ we get a representation $\vartheta \colon \mathcal{O}_A \to C^*(\underline{S}^z)$ mapping s_i to S_i^z. This is the GNS representation of the Cuntz–Krieger state

$$\underline{s}^\alpha (\underline{s}^\beta)^* \to \underline{z}^\alpha \overline{(\underline{z}^\beta)}$$

which exists by (3.4). We call such states *Cuntz–Krieger states.*

Theorem 23. *Any spherical representation of* $\mathcal{O}_A$ (*on a separable Hilbert space*) *can be written as direct integral of GNS representations of Cuntz–Krieger states.*

Proof. An arbitrary representation of C_n^M is a countable direct sum of multiplicity free representations. Also any multiplicity free representation of C_n^M can be seen as a map which sends $g \in C_n^M$ to an operator which acts as multiplication by g on $L^2(M, \mu)$ for some finite Borel measure μ on M and the associated representation ϑ of $\mathcal{O}_A$ can be expressed as direct integral of representations ϑ_z with respect to the measure μ acting on $\oint \mathcal{H}^z \mu(\mathrm{d}z)$. Thus the theorem follows. □

Example 24. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then any commuting contractive A-relation tuple also satisfies A'-relations, where A' is the symmetric matrix

$$A' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore the set of vertices of the corresponding graph is $\{1, 2, 3, 4\}$, the set of edges is $E = \{(1, 2), (1, 3)\}$ and 3 is a zero vertex. Hence $M = [(\mathbb{C}^2 \times \{0\}^2) \cup (\{0\}^3 \times \mathbb{C})] \cap \partial B_n$ where $\partial B_n = \{(z_1, \ldots, z_n)\colon \sum_{i=1}^n |z_i|^2 = 1\}$.

COROLLARY 25

Any representation of $\mathcal{O}_A$ can be decomposed as $\pi_s \oplus \pi_t$ where π_s is a spherical representation and $(\pi_t(s_1), \ldots, \pi_t(s_n))$ has trivial maximal commuting piece.

Proof. Similar to the proof of Theorem 19 in [BBD]. □

It also follows that for irreducible representations of $\mathcal{O}_A$, the maximal commuting piece of $(\pi(s_1), \ldots, \pi(s_n))$ is either one-dimensional or trivial.

5. Universal properties and WOT-closed algebras related to minimal Cuntz–Krieger dilation

Assume $\underline{\tilde{T}}$ to be the minimal Cuntz–Krieger dilation of a contractive tuple $\underline{T}$ satisfying A-relations. Define $C^*(\underline{\tilde{T}})$ to be the unital C^*-algebra generated by $\underline{\tilde{T}}$. Clearly the linear map from $C^*(\underline{\tilde{T}})$ to $B(\mathcal{H})$ such that $\underline{\tilde{T}}^\alpha (\underline{\tilde{T}}^\beta)^* \mapsto P_{\mathcal{H}} \underline{\tilde{T}}^\alpha (\underline{\tilde{T}}^\beta)^*|_{\mathcal{H}} = \underline{T}^\alpha (\underline{T}^\beta)^*$ is a unital completely positive map. We will investigate some universal properties of minimal Cuntz–Krieger dilations using methods employed by Popescu [Po5] for minimal isometric dilations. Proposition 26 is a nonspatial characterization of the minimal Cuntz–Krieger dilation and Theorem 27 describes functoriality and commutant lifting in this setting. The proofs are omitted as they are similar to those appearing in §2 of [Po5].

PROPOSITION 26

Suppose $\underline{\tilde{T}}$ is the minimal Cuntz–Krieger dilation of a contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{H}$ satisfying A-relations with respect to some matrix A.

(1) *Consider a unital C^*-algebra $C^*(\underline{d})$ generated by the entries of the tuple $\underline{d} = (d_1, \ldots, d_n)$ where the entries satisfy*

$$d_i^* d_i = I - \sum_{j=1}^n (1 - a_{ij}) d_j d_j^*.$$

Assume that $\underline{d}$ also satisfies $d_i^* d_j = 0$ for $1 \leq i \neq j \leq n$. Let there be a completely positive map $\varrho\colon C^*(\underline{d}) \to B(\mathcal{H})$ such that $\varrho(\underline{d}^\alpha (\underline{d}^\beta)^*) = \underline{T}^\alpha (\underline{T}^\beta)^*$. Then there is a $*$-homomorphism form $C^*(\underline{d})$ to $C^*(\underline{\tilde{T}})$ such that $d_i \mapsto \tilde{T}_i$ for all $1 \leq i \leq n$.

(2) *Suppose $\pi\colon C^*(\underline{T}) \to B(\tilde{\mathcal{K}})$ is a $*$-homomorphism and $\theta\colon C^*(\underline{\tilde{T}}) \to C^*(\underline{T})$ the completely positive map obtained by restricting the compression map (to $B(\mathcal{H})$) of $B(\tilde{\mathcal{H}})$ to $C^*(\underline{\tilde{T}})$. Assume the minimal Stinespring dilation of $\pi \circ \theta$ to be $\tilde{\pi}$ i.e., $\pi \circ \theta(X) = P_{\tilde{\mathcal{K}}} \tilde{\pi}(X)|_{\tilde{\mathcal{K}}}$. Then $(\tilde{\pi}(\tilde{T}_1), \ldots, \tilde{\pi}(\tilde{T}_n))$ is the minimal Cuntz–Krieger dilation of $(\pi(T_1), \ldots, \pi(T_n))$.*

Theorem 27. *Let $\underline{T}$ be a contractive n-tuple on $\mathcal{H}$ satisfying A-relations and $\underline{\tilde{T}}$ be its minimal Cuntz–Krieger dilation.*

(1) *Suppose π_1 and π_2 are two $*$-homomorphism from $C^*(\underline{T})$ to $B(\mathcal{K}_1)$ and $B(\mathcal{K}_2)$ respectively, for some Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$. Let θ be as defined in the previous proposition. If X is an operator such that $X\pi_1(Y) = \pi_2(Y)X$ for all $Y \in C^*(\underline{T})$, and $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the minimal Stinespring dilations of $\pi_1 \circ \theta$ and $\pi_2 \circ \theta$ respectively then there exists an operator $\tilde{X}$ such that $\tilde{X}\tilde{\pi}_1 = \tilde{\pi}_2\tilde{X}$ and $\tilde{X}P_{\mathcal{K}_1} = P_{\mathcal{K}_2}\tilde{X}$.*

(2) *If $X \in C^*(\underline{T})'$ then there exists a unique $\tilde{X} \in C^*(\underline{\tilde{T}})' \cap \{P_{\mathcal{H}}\}'$ such that $P_{\mathcal{H}}\tilde{X}|_{\mathcal{H}} = X$. Also the map $X \mapsto \tilde{X}$ is a $*$-isomorphism.*

Many of the results and arguments for minimal Cuntz–Krieger dilation in the following part of this section are similar to those of [DKS] and [DP2] for standard noncommuting dilation. Using eq. (3.4) we observe that

$$\begin{aligned}\tilde{T}_j^*\tilde{T}_i^*\tilde{T}_i\tilde{T}_j &= \tilde{T}_j^*\left[I - \sum_k (1 - a_{ik})\tilde{T}_k\tilde{T}_k^*\right]\tilde{T}_j \\ &= a_{ij}\tilde{T}_j^*\tilde{T}_j\tilde{T}_j^*\tilde{T}_j = a_{ij}\tilde{T}_j^*\tilde{T}_j.\end{aligned}$$

From this it follows that for $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$\begin{aligned}(\underline{\tilde{T}}^{\alpha})^*\underline{\tilde{T}}^{\alpha} &= a_{\alpha_1\alpha_2}\ldots a_{\alpha_{m-1}\alpha_m}\tilde{T}_{\alpha_m}^*\tilde{T}_{\alpha_m} \\ &= a_{\alpha_1\alpha_2}\ldots a_{\alpha_{m-1}\alpha_m}\left[I - \sum_k (1 - a_{\alpha_m k})\tilde{T}_k\tilde{T}_k^*\right] \end{aligned} \tag{5.1}$$

and

$$\underline{\tilde{T}}^{\alpha}(\underline{\tilde{T}}^{\alpha})^*\underline{\tilde{T}}^{\alpha} = \underline{\tilde{T}}^{\alpha}$$

so that each $\underline{\tilde{T}}^{\alpha}$ is a partial isometry. Let $\hat{\mathcal{H}}$ and $\tilde{\mathcal{H}}$ be the dilation spaces associated with standard noncommuting dilation $\underline{\hat{T}}$ and $\underline{\tilde{T}}$ respectively as before and let us denote $\tilde{\mathcal{H}} \ominus \mathcal{H}$ by $\mathcal{E}$. We know that $\hat{T}_i$ and $\tilde{T}_i$ leaves $\mathcal{E}$ and $\hat{\mathcal{H}} \ominus \mathcal{H}$ respectively invariant. Let Φ: $B(\mathcal{H}_1) \to B(\mathcal{H}_1)$ be the completely positive map defined by

$$\Phi(X) = \sum_{i=1}^{n} \hat{T}_i P_{\mathcal{H}^\perp} X P_{\mathcal{H}^\perp} \hat{T}_i^*.$$

Thus, $\Phi(P_{\mathcal{E}}) \leq \Phi(I)$. Also let $Q_i := P_{\mathcal{E}}\tilde{T}_i P_{\mathcal{E}}$. Then for $h \in \mathcal{E}$,

$$\begin{aligned}\left\langle h, \sum_{|\alpha|=m} \underline{Q}^{\alpha}(\underline{Q}^{\alpha})^*h\right\rangle &= \left\langle h, \sum_{|\alpha|=m} \underline{\tilde{T}}^{\alpha} P_{\mathcal{E}}(\underline{\tilde{T}}^{\alpha})^*h\right\rangle \\ &= \left\langle h, \sum_{|\alpha|=m} \underline{\hat{T}}^{\alpha} P_{\mathcal{H}^\perp} P_{\mathcal{E}} P_{\mathcal{H}^\perp}(\underline{\hat{T}}^{\alpha})^*h\right\rangle \\ &= \langle h, \Phi^m(P_{\mathcal{E}})h\rangle \\ &\leq \langle h, \Phi^m(I)h\rangle.\end{aligned}$$

But as $\lim_{m\to\infty}\langle h, \Phi^m(I)h\rangle = 0$ we have $\lim_{m\to\infty}\langle h, \sum_{|\alpha|=m} \underline{Q}^{\alpha}(\underline{Q}^{\alpha})^*h\rangle = 0$ which implies that $\underline{Q}$ is pure. In the above computation we used $\hat{T}_i^*$ invariance of $\tilde{\mathcal{H}}$ for $1 \leq i \leq n$.

Here we are interested in understanding the structure of WOT-closed algebra generated by the minimal Cuntz–Krieger dilation $\underline{\tilde{T}}$ of some contractive tuple $\underline{T} = (T_1, \dots, T_n)$ satisfying A-relations where $T_i \in B(\mathcal{H})$. Let $\mathcal{A}$ denote the WOT-closed algebra generated by all $\tilde{T}_i$, $1 \leq i \leq n$.

Lemma 28.

(1) *If $\mathcal{A}$ has no wandering vector then every nontrivial invariant subspace with respect to $\mathcal{A}$ is also reducing.*
(2) $\mathcal{K} := \mathcal{E} \ominus [\sum_i^n \tilde{T}_i \mathcal{E}]$ *is a wandering subspace for* $\underline{\tilde{T}}$.

Proof. Let there be no wandering vector for $\mathcal{A}$ and if possible let $\mathcal{N}$ be a nontrivial invariant subspace for $\mathcal{A}$. If $\sum_{i=1}^n \tilde{T}_i \mathcal{N}$ is not equal to $\mathcal{N}$ then $\mathcal{N} \ominus \sum_{i=1}^n \tilde{T}_i \mathcal{N}$ would be wandering as seen using orthogonality of the ranges of the $\tilde{T}_i$'s, eq. (5.1) and the following: For $n_1, n_2 \in \mathcal{N} \ominus \sum_{i=1}^n \tilde{T}_i \mathcal{N}$,

$$\langle \tilde{T}_i^* \tilde{T}_i \tilde{T}_{\alpha_1} \dots \tilde{T}_{\alpha_m} n_1, n_2 \rangle = \langle a_{i\alpha_1} \tilde{T}_{\alpha_1} \dots \tilde{T}_{\alpha_m} n_1, n_2 \rangle = 0.$$

But this is not possible by our assumption. So

$$\mathcal{N} = \sum_{i=1}^n \tilde{T}_i \mathcal{N}. \tag{5.2}$$

Now let $h \in \mathcal{N}$ be arbitrary. From the above equation it follows that one can write h as $\sum_{i,j=1}^n \tilde{T}_i \tilde{T}_j n_{ij}$ for some $n_{ij} \in \mathcal{N}$. From this and eq. (5.1) it is clear that $\tilde{T}_k^* h \in \mathcal{N}$ for all $1 \leq k \leq n$. So $\mathcal{N}$ is reducing for $\mathcal{A}$. Hence (1) follows.

$\mathcal{E}$ is also an invariant subspace for $\mathcal{A}$. The nontrivial case is when $\mathcal{E}$ is nonzero. $\mathcal{E} \neq \sum_{i=1}^n \tilde{T}_i \mathcal{E}$ as otherwise $\mathcal{E}$ would be reducing which is not possible as $\mathcal{H}$ spans $\tilde{\mathcal{H}}$. It follows from above that $\mathcal{K}$ is a wandering subspace of $\mathcal{A}$. □

So we can write $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}' \oplus (\Gamma_A \otimes \mathcal{K})$ for some Hilbert space $\mathcal{H}'$. So $\sum_{\alpha \in \tilde{\Lambda}} \underline{\tilde{T}}^\alpha \mathcal{K} = \Gamma_A \otimes \mathcal{K}$ and this is left invariant by all $\tilde{T}_i$. Also $\tilde{T}_i P_{\Gamma_A \otimes \mathcal{K}}$ is $S_i \otimes I$ for $1 \leq i \leq n$.

Let us denote by $\mathcal{B}$ the WOT-closed algebra generated by $T_1, \dots, T_n$. In order to get reducing subspaces for $\mathcal{A}$ it is sufficient to demand for $\mathcal{B}^*$-invariant subspace as seen in the next lemma. ($\mathcal{A}[\mathcal{L}]$ denotes the closed linear span of $A\mathcal{L}$.)

Lemma 29. *Let $\mathcal{L}$ be a $\mathcal{B}^*$-invariant subspace of $\mathcal{H}$. Then $\mathcal{A}[\mathcal{L}]$ reduces $\mathcal{A}$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthogonal $\mathcal{B}^*$-invariant subspace of $\mathcal{H}$ then $\mathcal{A}[\mathcal{L}_1]$ and $\mathcal{A}[\mathcal{L}_2]$ are also mutually orthogonal.*

Proof. $\tilde{T}_i^*$ leaves $\mathcal{L}$ invariant as $\tilde{T}_i^*$ and T_i^* leaves $\mathcal{H}$ and $\mathcal{L}$ respectively invariant. Thus

$$\mathcal{A}[\mathcal{L}] = \overline{\text{span}}\{\underline{\tilde{T}}^\alpha h \colon \alpha \in \tilde{\Lambda}, h \in \mathcal{L}\}.$$

Now for any $x \in \mathcal{L}$ and $\alpha = (\alpha_1, \dots, \alpha_m)$, using eq. (5.1)

$$\tilde{T}_i^* \underline{\tilde{T}}^\alpha x = \begin{cases} [I - \sum_k (1 - a_{ik}) \tilde{T}_k \tilde{T}_k^*] x, & \text{if } \alpha_1 = i, |\alpha| = 1 \\ a_{\alpha_1 \alpha_2} \tilde{T}_{\alpha_2} \dots \tilde{T}_{\alpha_m} x, & \text{if } \alpha_1 = i, |\alpha| > 1 \\ 0, & \text{if } \alpha_1 \neq i \\ \tilde{T}_i^* x, & \text{if } |\alpha| = 0 \end{cases}.$$

As $\mathcal{L}$ is invariant for $\mathcal{A}^*$, $\tilde{T}_i^* \underline{\tilde{T}}^{\alpha} x \in \mathcal{A}[\mathcal{L}]$ and hence $\mathcal{A}[\mathcal{L}]$ reduces $\mathcal{A}$.

Further when $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthogonal $\mathcal{B}^*$-invariant subspaces, to establish that $\mathcal{A}[\mathcal{L}_1]$ and $\mathcal{A}[\mathcal{L}_2]$ are orthogonal it is sufficient to check if $|\alpha| \leq |\beta|$ and $l_1 \in \mathcal{L}_1, l_2 \in \mathcal{L}_2$, then $\langle \underline{\tilde{T}}^{\alpha} l_1, \underline{\tilde{T}}^{\beta} l_2 \rangle = 0$. This is checked easily for all cases by orthogonality of ranges of different $\tilde{T}_i$'s, $\mathcal{B}^*$-invariance of $\mathcal{L}_i$ and eq. (5.1) except $\alpha = (\alpha_1, \ldots, \alpha_m) = \beta$. In this case

$$\langle (\underline{\tilde{T}}^{\alpha})^* (\underline{\tilde{T}}^{\alpha}) l_1, l_2 \rangle = \left\langle a_{\alpha_1\alpha_2} \ldots a_{\alpha_{m-1}\alpha_m} \left[I - \sum_k (1 - a_{\alpha_m k}) \tilde{T}_k \tilde{T}_k^* \right] l_1, l_2 \right\rangle$$

$$= a_{\alpha_1\alpha_2} \ldots a_{\alpha_{m-1}\alpha_m} \left\{ \langle l_1, l_2 \rangle - \sum_k (1 - a_{\alpha_m k}) \langle \tilde{T}_k^* l_1, \tilde{T}_k^* l_2 \rangle \right\} = 0,$$

hence the lemma follows. □

Recall that $\tilde{\mathcal{H}}_N$ denotes the summand on which the compact operators in $C^*(\underline{\tilde{T}})$ act trivially. Let $\mathcal{H}_N := \tilde{\mathcal{H}}_N \cap \mathcal{H}$. In the next proposition we assume $\mathcal{H}$ to be finite-dimensional.

PROPOSITION 30

Let $\underline{T}$ be a contractive tuple satisfying A-relations of operators on a finite-dimensional Hilbert space $\mathcal{H}$.

(1) *Let $\mathcal{K}$ be a reducing subspace of $\tilde{\mathcal{H}}_N$ with respect to $\mathcal{A}$ and let $h \in \tilde{\mathcal{H}}$ such that $P_{\mathcal{K}} h$ is nonzero. Then there exists $k \in \mathcal{A}^*[h] \cap \mathcal{H}_N$ such that $P_{\mathcal{K}} k$ is nonzero.*
(2) *Any nonzero subspace of $\tilde{\mathcal{H}}_N$ which is co-invariant with respect to $\tilde{T}_i$, $1 \leq i \leq n$ has a nontrivial intersection with $\mathcal{H}_N$.*
(3) *$\tilde{\mathcal{H}}_N = \mathcal{A}[\mathcal{H}_N]$. When $\sum T_i T_i^* = I$ and $\mathcal{B} = B(\mathcal{H})$ every co-invariant subspace of $\mathcal{H}$ with respect to all $\tilde{T}_i$'s contains $\mathcal{H}$.*

Proof. Proof is similar to the proof of Lemma 4.1, Corollary 4.2 and Corollary 4.3 in [DKS]. It uses the above two lemmas, pureness of $\underline{Q}$, Wold decomposition of $\tilde{T}$ and compactness of the unit ball of finite-dimensional Hilbert space $\mathcal{H}$. Proposition 30(2) and 30(3) are corollaries of Proposition 30(1). □

Now we consider the tuple $\underline{R}$ consisting of right creation operators on $\Gamma(\mathbb{C}^n)$ given by $R_i x = x \otimes e_i$. One can easily notice using methods similar to the proof of Lemma 4 that for polynomials $p_{(l,m)} = z_l z_m - a_{ml} z_l z_m$, $(l, m) \in \{1, \ldots, n\} \times \{1, \ldots, n\} = \mathcal{I}$, we get $(\Gamma(\mathbb{C}^n))^p(\underline{R}) = \Gamma_A$. Let X_i denote the compression of R_i to Γ_A, i.e. $X_i = P_{\Gamma_A} R_i|_{\Gamma_A}$. Suppose $e^{\alpha} \in \Gamma_A$, and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \ldots, \alpha_m)$. Then

$$X_i e^{\alpha} = P_{\Gamma_A} R_i e^{\alpha} = \begin{cases} e_i, & \text{if } |\alpha| = 0 \\ a_{\alpha_m i} e^{\alpha} \otimes e_i, & \text{if } |\alpha| \geq 1 \end{cases}.$$

Moreover from Proposition 9 it follows that $\underline{X}$ consists of isometries with orthogonal ranges satisfying A^t-relations, where A^t is the transpose of A. Let $\mathcal{S}$ and $\mathcal{X}$ denote the WOT-closed algebras generated by $S_1, \ldots, S_n$ and $X_1, \ldots, X_n$ respectively. Now we shall analyze the structure of these WOT-closed algebras. Let Q_k denote the projection onto $\text{span}\{e^{\alpha} \colon \alpha \in \tilde{\Lambda}_A, |\alpha| = k\}$.

PROPOSITION 31

(1) *$\mathcal{S}$ coincides with the commutant of $\mathcal{X}$ in $B(\Gamma_A)$, that is $\mathcal{S} = \mathcal{X}'$. Also $\mathcal{X} = \mathcal{S}'$ and hence $\mathcal{S}$ and $\mathcal{X}$ are double commutants of themselves.*
(2) *$\mathcal{S}$ and $\mathcal{X}$ are inverse closed and the only normal elements in $\mathcal{S}$ and $\mathcal{X}$ are scalars.*

Proof. Any element in $\mathcal{S}$ can be written as a formal sum $\sum_\alpha b_\alpha \underline{S}^\alpha$, where $b_\alpha \in \mathbb{C}$ are given by $X\omega = \sum b_\alpha e^\alpha$. For $\beta = (\beta_1, \ldots, \beta_m)$, let β' denote $(\beta_m, \ldots, \beta_1)$.

$$\underline{S}^\alpha \underline{X}^{\beta'} e^\gamma = \begin{cases} a_{\alpha_{|\alpha|}\gamma_1} a_{\gamma_{|\gamma|}\beta_1} e^\alpha \otimes e^\gamma \otimes e^\beta, & \text{if } |\gamma| > 0 \\ a_{\alpha_{|\alpha|}\beta_1} e^\alpha \otimes e^\beta, & \text{if } |\gamma| = 0 \end{cases} = \underline{X}^{\beta'} \underline{S}^\alpha e^\gamma.$$

So, $\mathcal{S} \subseteq \mathcal{X}'$. The converse is similar to the proof of Theorem 1.2 in [DP2] after noticing that $X_i Q_k = Q_{k+1} X_i$ and considering the Cesáro sums

$$p_k(L) = \sum_{|\alpha|<k} \left(1 - \frac{|\alpha|}{k}\right) d_\alpha \underline{S}^\alpha$$

for $L\omega = \sum_{\alpha \in \tilde{\Lambda}_A} d_\alpha e^\alpha$.

$\mathcal{S}$ and $\mathcal{X}$ are inverse closed as this is the case for any algebra which is a commutant. This and (2) can be proved by taking the same approach as that of the proof of Corollary 1.4 and Corollary 1.5 in [DP2]. □

PROPOSITION 32

Any element $A \in \mathcal{L}$ leaves the range of $\underline{X}^\alpha (\underline{X}^\alpha)^*$ invariant.

Proof. Note that one can argue as we did for $\underline{S}^\alpha$ and show that $\underline{X}^\alpha$ are partial isometries. Further as $\mathcal{L} = \mathcal{X}'$,

$$\begin{aligned} \underline{X}^\alpha (\underline{X}^\alpha)^* A \underline{X}^\alpha (\underline{X}^\alpha)^* &= \underline{X}^\alpha (\underline{X}^\alpha)^* \underline{X}^\alpha A (\underline{X}^\alpha)^* \\ &= \underline{X}^\alpha A (\underline{X}^\alpha)^* = A \underline{X}^\alpha (\underline{X}^\alpha)^*, \end{aligned}$$

the proposition follows. □

In these algebras the wandering subspace description is much simpler than the general case as can be seen from the next result.

PROPOSITION 33

(1) *If $\mathcal{N}$ is an invariant subspace of $\mathcal{L}$ then $\mathcal{M} = \mathcal{N} \ominus \sum_{i=1}^n S_i \mathcal{N}$ is a wandering subspace and $\mathcal{L}[\mathcal{M}] = \mathcal{N}$.*
(2) *A subspace is cyclic and invariant with respect to $\mathcal{L}$ if and only if it is the range of some element in $\mathcal{X}$.*

Proof. Follows from the Wold decomposition using methods similar to the proof of Theorem 2.1 in [DP2]. □

Remark 34. Another notion of minimal partial isometric dilation exists in [JK], [Mu] and [MS]. The class of partial isometric dilation tuples of Jury and Kribs [JK] include the extension of Cuntz–Krieger algebras by compacts or Cuntz–Krieger–Toeplitz algebras. They assume the existence of a family of projections 'stabilizing' the initial contractive tuple $\underline{T}$. Such stabilizing tuple is not assumed for our initial tuple, but we put a different condition on our $\underline{T}$, namely, it has to satisfy A-relations. If the family of projection stabilizing $\underline{T}$ is taken to be the trivial one consisting of identity and zeros, then the minimal partial isometric dilation in the sense of [JK] gives nothing new but just the minimal isometric dilation. Muhly and Solel [Mu,MS] have given the construction of dilations somewhat similar to [JK], for quiver and tensor algebras in the language of Hilbert C^*-modules. In other words, the notions of dilation in these papers are quite different from ours. Our construction of the partial isometric dilation is crucial for the classification results in this article. We have also come to know of some work on graph algebras by Katsoulis and Kribs [KK1,KK2]. It is to be noted that these graph algebras include algebras generated by A-relations as a special case.

Acknowledgements

The first author is funded by the Department of Science and Technology (India) under the Swarnajayanthi Fellowship scheme. The second author is supported by Deutscher Akademischer Austausch Dienst Fellowship.

References

[AP] Arias A and Popescu G, Noncommutative interpolation and Poisson transforms, *Israel J. Math.* **115** (2000) 205–234, MR 2001i:47021

[Ar] Arveson W, Subalgebras of C^*-algebras III, Multivariable operator theory, *Acta Math.* **181(2)** (1998) 159–228, MR 2000e:47013

[BB] Bhat B V R and Bhattacharyya T, A model theory for q-commuting contractive tuples, *J. Operator Theory* **47(1)** (2002) 97–116, MR 2003c:47018

[BBD] Bhat B V R, Bhattacharyya T and Dey S, Standard noncommuting and commuting dilations of commuting tuples, *Trans. Am. Math. Soc.* **356** (2004) 1551–1568, MR 2005b:47011

[Bu] Bunce J W, Models for n-tuples of noncommuting operators, *J. Funct. Anal.* **57** (1984) 21–30, MR 85k:47019

[Cu] Cuntz J, Simple C^*-algebras generated by isometries, *Commun. Math. Phys.* **57** (1977) 173–185, MR 57#7189

[CK] Cuntz J and Krieger W, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **63** (1981) 25–40, MR 82f:46073a

[Da] Davis C, Some dilation and representation theorems, Proceedings of the Second International Symposium in West Africa on Functional Analysis and its Applications (Kumasi) (1979) pp. 159–182, MR 84e:47012

[De] Dey S, Standard dilations of q-commuting tuples (Bangalore: Indian Statistical Institute) (2003) preprint oai: arXiv.org: math/0310389

[DKS] Davidson K R, Kribs D W and Shpigel M E, Isometric dilations of non-commuting finite rank n-tuples, *Canad. J. Math.* **53** (2001) 506–545, MR 2002f:47010

[DP1] Davidson K R and Pitts D R, The algebraic structure of non-commutative analytic Toeplitz algebras, *Math. Ann.* **311** (1998) 275–303, MR 2001c:47082

[DP2] Davidson K R and Pitts D R, Invariant subspaces and hyper-reflexivity for free semigroup algebras, *Proc. London Math. Soc.* **78** (1999) 401–430, MR 2000k:47005

[Fr] Frazho A E, Models for noncommuting operators, *J. Funct. Anal.* **48** (1982) 1–11, MR 84h:47010

[JK] Jury M T and Kribs D W, Partially isometric dilations of noncommuting N-tuples of operators, *Proc. Am. Math. Soc.* **133(1)** (2005) 213–222, MR 2005d:47011

[KK1] Katsoulis E and Kribs D W, Applications of the Wold decomposition to the study of row contractions associated with directed graphs, *Trans. Am. Math. Soc.* **357** (2005) 3739–3755

[KK2] Katsoulis E and Kribs D W, Isomorphisms of algebras associated with directed graphs, *Math. Ann.* **330** (2004) 709–728

[Mu] Muhly P S, A finite-dimensional introduction to operator algebra, Operator algebras and applications (Samos) (1996) pp. 313–354, *NATO Adv. Sci. Inst.* **495** (Dordrecht: Kluwer Acad. Publ.) (1997), MR 98h:46062

[MS] Muhly P S and Solel B, Tensor algebras over C^*-correspondences: Representations, dilations, and C^*-envelopes, *J. Funct. Anal.* **158(2)** (1998) 389–457, MR 99j:46066

[Po1] Popescu G, Isometric dilations for infinite sequences of noncommuting operators, *Trans. Am. Math. Soc.* **316** (1989) 523–536, MR 90c:47006

[Po2] Popescu G, Characteristic functions for infinite sequences of noncommuting operators, *J. Operator Theory* **22** (1989) 51–71, MR 91m:47012

[Po3] Popescu G, von Neumann inequality for $(B(\mathcal{H})^n)_1$, *Math. Scand.* **68(2)** (1991) 292–304, MR 92k:47073

[Po4] Popescu G, Multi-analytic operators on Fock spaces, *Math. Ann.* **303(1)** (1995) 31–46, MR 96k:47049

[Po5] Popescu G, Universal operator algebras associated to contractive sequences of non-commuting operators, *J. London Math. Soc. (2)* **58(2)** (1998) 469–479, MR 99m:47054

[Po6] Popescu G, Poisson transforms on some C^*-algebras generated by isometries, *J. Funct. Anal.* **161** (1999) 27–61, MR 2000m:46117