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A Stochastic Differential Equation with Time-Dependent and Unbounded Operator Coefficients

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Unitary solutions of a class of stochastic equations (SDE) in Fock space with time-dependent unbounded operator coefficients are constructed as a limit of a random Trotter Kato product. Some special cases of quantum stochastic differential equations are studied as an application. © 1993 Academic Press, Inc.

1. INTRODUCTION

Some attempts have been made [1–4] to study the existence and unitarity of solutions of quantum stochastic differential equations of the form

$$\begin{aligned} dU &= U[M\,dA^{\dagger} - M^{*}\,dA + (iH - \tfrac{1}{2}M^{*}M)\,dt] \\ U(0) &= I, \end{aligned} \tag{1.1}$$

where A, $A^{\dagger}$ are the annihilation and creation processes in the boson (symmetric) Fock space $\Gamma(L^2(\mathbb{R}_+))$ and the coefficients M, H are either bounded or closed or self-adjoint operators satisfying some further conditions in some initial Hilbert space h.

On the other hand, classical stochastic differential equations with operator coefficients of the form

$$\begin{aligned} dU &= U[-iL\,dw(t) + (iH - \tfrac{1}{2}L^2)\,dt] \\ U(0) &= I, \end{aligned} \tag{1.2}$$

where $w(t)$ is the standard Brownian motion and L and H are generators of contraction semigroups, have been discussed in the literature [5, 6].

We are interested here in considering equations of the type (1.2) and studying the unitarity of the solution in $h \otimes \Gamma(L^2(\mathbb{R}_+))$. This is a special case of Eq. (1.1) with M replaced by a self-adjoint operator L.

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In this section we start with some preliminaries of notation and basic results about quantum stochastic integrals, the details of which can be found in [2]. In Section 2, we prove an abstract theorem about unitary solutions of a stochastic differential equation with time-dependent operator coefficients, while Section 3 deals with the special case of equations like (1.2). Section 4 considers briefly a simple application to one-dimensional diffusion. As is common in quantum probability, we look upon this as an algebra homomorphism between an initial subalgebra of $\mathscr{B}(h)$ and that of $\mathscr{B}(h \otimes \Gamma(L^2(\mathbb{R}_+)))$.

In Fock space $\Gamma(L^2(\mathbb{R}_+))$, the exponential vectors $e(f)$, annihilation operators $a(g)$, and creation operators $a^\dagger(g)$ are defined for $f, g \in L^2(\mathbb{R}_+)$ as

$$\begin{aligned} e(f) &= 1 \oplus f \oplus \frac{f^{\otimes 2}}{\sqrt{2!}} \oplus \frac{f^{\otimes 3}}{\sqrt{3!}} \cdots \oplus \frac{f^{\otimes n}}{\sqrt{n!}} + \cdots \\ a(g)\, e(f) &= \left(\int_0^\infty \bar{g}(z) f(z)\, dz \right) (e(f)) \qquad (1.3) \\ a^\dagger(g)\, e(f) &= \operatorname*{s-lim}_{\varepsilon \to 0} \left(\frac{e(f+\varepsilon g) - e(f)}{\varepsilon} \right). \end{aligned}$$

We denote by $\mathscr{E}$ the linear manifold generated by exponential vectors. It can easily be seen that $\mathscr{E}$ is dense in $\Gamma(L^2(\mathbb{R}_+))$ and $a^\dagger(g)$ and $a(g)$ are adjoint to each other on $\mathscr{E}$. The annihilation and creation processes are defined as

$$A(t) = a(\chi_{[0,t]}) \qquad \text{and} \qquad A^\dagger(t) = a^\dagger(\chi_{[0,t]}),$$

respectively, for $0 \leqslant t < \infty$. It is also easy to see that the Fock space is isometrically isomorphic to the Wiener space $L^2(P)$ through the unitary operator $U\colon \Gamma(L^2(\mathbb{R}_+)) \to L^2(P)$ satisfying

$$(Ue(f))(w) = \exp\left(\int_0^\infty f\, dw - \tfrac{1}{2} \int_0^\infty f^2(t)\, dt \right).$$

For details concerning the definitions of adapted, square integrable processes and their integration with respect to the basic processes $A(t)$ and $A^\dagger(t)$ we refer to [2]. But we need from [2, Proposition 27.1] the estimate

$$\begin{aligned} &\left\| \left(\int_0^t F_1(s)\, dA(s) + F_2(s)\, dA^\dagger(s) + F_3(s)\, ds \right) f e(u) \right\|^2 \\ &\quad \leqslant 2 e^{\nu_u(t)} \sum_{i=1}^3 \int_0^t \|F_i(s) f e(u)\|^2\, d\nu_u(s), \qquad (1.4) \end{aligned}$$

where the $F_i(s)$ are adapted square integrable processes, $f \in h$, $u \in L^2(\mathbb{R}_+)$, and $v_u(t) = \int_0^t (1 + \|u(s)\|^2)\, ds$.

2. Unitary Solution of an Abstract Stochastic Differential Equation

We study the quantum stochastic differential equation

$$dV(t) = [-iL(t)\, dw(t) - \tfrac{1}{2}L(t)^2\, dt]\; V(t), \qquad V(0) = I \tag{2.1}$$

on $h \otimes \Gamma(L^2(\mathbb{R}_+))$, where h is a complex separable Hilbert space and $w(t)$ is the standard Wiener process viewed as an operator process in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. We construct unitary solutions when $L(t)$ is a self-adjoint (but possibly unbounded) operator for each $t \geqslant 0$ on h, under the following additional assumptions.

A2.1. (i) $\exists \mathscr{D} \subset \bigcap_{t \geqslant 0} D(L(t)^2)$, such that $\mathscr{D}$ is a core for every $L(t)^j$ $(j = 1, 2)$ and $e^{iL(t)\alpha}$ leaves $\mathscr{D}$ invariant for every positive t and real α.

(ii) $C_j(t, s) = (L(t) + i)^j\, (L(s) + i)^{-j} - I$ are everywhere defined bounded operators on h satisfying

$$\|C_j(t, s)\| \leqslant M_j |t - s| \qquad \text{for} \quad j = 1, 2 \qquad \text{and} \qquad t, s \geqslant 0.$$

$C_j(t) = \text{s-lim}_{s \uparrow t}(C_j(t, s)/(t - s))$ exists uniformly with respect to t in compact sets and $\{C_j(t)\}_{t \geqslant 0}$ forms a strongly continuous family of bounded operators on h.

In this section we make the assumption A2.1 without repeating it any further.

The method of solving (2.1) is very similar to what is done for the deterministic case with time-dependent (unbounded) coefficients [7]. We restrict to the case $0 \leqslant t \leqslant 1$ without loss of generality. For each positive integer k, consider the subdivision of [0, 1] into k equal parts of length $1/k$ and write

$$V_k(t, s) = e^{-i[w(t) - w(s)]L((j-1)/k)} \qquad \text{if} \quad \frac{j-1}{k} \leqslant s \leqslant t \leqslant \frac{j}{k}, \qquad 1 \leqslant j \leqslant k,$$

and extend it by the propagation property

$$V_k(t, r) = V_k(t, s)\, V_k(s, r) \qquad \text{if} \quad 0 \leqslant r \leqslant s \leqslant t \leqslant 1. \tag{2.2}$$

Before proving the strong convergence of $V_k(t, s)$ as $k \to \infty$, we prove a lemma. Set

$$W_k^j(t, s) = (L(t) + i)^j\, V_k(t, s)(L(s) + i)^{-j}, \qquad \text{for} \quad j = 1, 2, \quad 0 \leqslant s \leqslant t \leqslant 1. \tag{2.3}$$

LEMMA 2.1. *There exists constants N_j independent of t, s, k and the Brownian path w, such that*

$$\|W_k^j(t,s)\| \leqslant N_j \qquad \textit{for} \quad j=1,2.$$

Proof. Fix s, t, and k. Choose and fix a sample Brownian path w with $w \in C(\mathbb{R}_+)$ and $w(0)=0$. From assumption A2.1(i) it follows that $V_k(t,s)$ maps $\mathscr{D}$ onto $\mathscr{D}$ so that the W_k^j are well defined on $(L(s)+i)^j \mathscr{D}$, which is dense in h by assumption. Let $f \in h$, such that $(L(s)+i)^{-j} f \in \mathscr{D}$. Then for $j=1,2$, we have

$$\begin{aligned}
W_k^j(t,s)f &= (L(t)+i)^j\, V_k(t,s)(L(s)+i)^{-j} f \\
&= (L(t)+i)^j \left(L\left(\frac{[kt]}{k}\right)+i\right)^{-j} V_k\left(t, \frac{[kt]}{k}\right)\left(L\left(\frac{[kt]}{k}\right)+i\right)^{j} \\
&\quad \times \left(L\left(\frac{[kt]-1}{k}\right)+i\right)^{-j} V_k\left(\frac{[kt]}{k}, \frac{[kt]-1}{k}\right)\left(L\left(\frac{[kt]-1}{k}\right)+i\right)^{j} \\
&\quad \vdots \\
&\quad \times \left(L\left(\frac{[ks]}{k}\right)+i\right)^{-j} V_k\left(\frac{[ks]+1}{k}, s\right)\left(L\left(\frac{[ks]}{k}\right)+i\right)^{j} \\
&\quad \times (L(s)+i)^{-j} f \\
&= \left(1+C_j\left(t, \frac{[kt]}{k}\right)\right)\{V_k(t,s)+W_k^{j,1}(t,s)+W_k^{j,2}(t,s)+\cdots\} \\
&\quad \times \left(1+C_j\left(\frac{[ks]}{k}, s\right)\right) f,
\end{aligned} \tag{2.4}$$

where

$$W_k^{j,1}(t,s) = \sum_{ku=[ks]+1}^{[kt]} V_k(t,u)\, C_j\left(u, u-\frac{1}{k}\right) V_k(u,s)$$

and

$$W_k^{j,m+1}(t,s) = \sum_{ku=[ks]+1}^{[kt]} V_k(t,u)\, C_j\left(u, u-\frac{1}{k}\right) W_k^{j,m}(u,s),$$

for $1 \leqslant m \leqslant [kt]-1$.

Now, $\|C_j(u, u-1/k)\| \leqslant M_j/k$, by assumption A2.1(ii). Hence

$$\|W_k^{j,1}(t,s)f\| \leqslant \sum_{ku=[ks]+1}^{[kt]} \frac{M_j}{k}\|f\| \leqslant M_j(t-s)\|f\|,$$

$$\|W_k^{j,2}(t,s)f\| \leqslant \sum_{ku=[ks]+1}^{[kt]} \frac{M_j^2}{k}\|f\|(u-s) \leqslant M_j^2 \frac{(t-s)^2}{2!}\|f\|$$

and, by induction

$$\| W_k^{j,m}(t, s) \| \leqslant M_j^m \frac{(t-s)^m}{m!}.$$

Substituting this estimate in (2.4), we get

$$\begin{aligned} \| W_k^j(t, s) \| &\leqslant \left\| 1 + C_j\left(t, \frac{[kt]}{k}\right) \right\| \left\| 1 + C_j\left(\frac{[ks]}{k}, s\right) \right\| \left\{ 1 + \sum_{m \geqslant 1} \| W_k^{j,m}(t, s) \| \right\} \\ &\leqslant \left(1 + \frac{M_j}{k}\right)^2 \left(1 + \sum_{m \geqslant 1} \frac{(t-s)^m}{m!} M_j^m \right). \end{aligned}$$

For $k \geqslant \max(M_1, M_2)$, we obtain

$$\| W_k^j(t, s) \| \leqslant 4 \left(1 + \sum_{m=1}^{\infty} \frac{M_j^m}{m!}\right) \equiv N_j. \quad \blacksquare$$

Observe that $V_k(t, s)$ defined in (2.2) can also be written as

$$V_k(t, s) = \prod_{m=[ks]}^{[kt]} e^{-i[w(t \wedge (m+1)/k) - w(s \vee m/k)] L(m/k)} \tag{2.5}$$

for $0 \leqslant s \leqslant t \leqslant 1$, where the arrow $(\leftarrow)$ indicates that the product is computed keeping the factor with higher m-index on the left.

Proposition 2.2. *$V(t, s) \equiv \text{s-lim}_{k \to \infty} V_k(t, s)$ exists in $h \otimes \Gamma(L^2(\mathbb{R}_+))$ uniformly in t, s for t, s in $[0, 1]$.*

Proof. On computing the difference $V_k - V_n$ for $k > n$, one faces the difficulty of having future dependent quantities, making it impossible to apply Ito's formula directly. To avoid this we partition the interval (s, t) and rewrite the difference with the convention, $F(r)|_{r=a}^{b} = F(b) - F(a)$,

$$\begin{aligned} &V_k(t, s) - V_n(t, s) \\ &= \sum_{m=[ns]}^{[nt]} V_n\left(t, \frac{m}{n}\right) \left\{ e^{-i[w(m/n) - w(r)] L(m/n)} V_k(r, s) \right\} \Bigg|_{r = s \vee m/n}^{t \wedge (m+1)/n}. \end{aligned} \tag{2.6}$$

Now observe that one can apply Ito's formula for the terms inside the braces and get

$$\begin{aligned} &\left\{ e^{-i[w(m/n) - w(r)] L(m/n)} V_k(r, s) \right\} \Big|_{r = s \vee m/n}^{t \wedge (m+1)/n} \\ &= e^{-i[w(m/n) - w(r)] L(m/n)} e^{-i[w(r) - w(s \vee [kr]/k)] L([kr]/k)} \\ &\quad \times V_k\left(s \vee \frac{[kr]}{k}, s\right) \Bigg|_{r = s \vee m/n}^{t \wedge (m+1)/n} \end{aligned}$$

$$= -i\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}\left\{L\left(\frac{[kr]}{k}\right)-L\left(\frac{m}{n}\right)\right\}$$
$$\times V_k(r,s)\,dw(r)$$
$$-\frac{1}{2}\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}$$
$$\times\left\{L\left(\frac{m}{n}\right)^2+L\left(\frac{[kr]}{k}\right)^2-2L\left(\frac{m}{n}\right)L\left(\frac{[kr]}{k}\right)\right\}V_k(r,s)\,dr$$
$$= i\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}\left\{L\left(\frac{m}{n}\right)+i-\left(L\left(\frac{[kr]}{k}\right)+i\right)\right\}$$
$$\times V_k(r,s)\,dw(r)$$
$$-\frac{1}{2}\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}\left\{\left(L\left(\frac{m}{n}\right)+i\right)^2\right.$$
$$\left.+\left(L\left(\frac{[kr]}{k}\right)+i\right)^2-2\left(L\left(\frac{m}{n}\right)+i\right)\left(L\left(\frac{[kr]}{k}\right)+i\right)\right\}V_k(r,s)\,dr$$
$$= i\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)$$
$$\times\left\{C_1\left(\frac{[kr]}{k},r\right)+1\right\}(L(r)+i)\,V_k(r,s)\,dw(r)$$
$$-\frac{1}{2}\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}\left\{C_2\left(\frac{m}{n},\frac{[kr]}{k}\right)\right.$$
$$\left.-2C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)\right\}\left\{C_2\left(\frac{[kr]}{k},r\right)+1\right\}(L(r)+i)^2\,V_k(r,s)\,dr$$
$$= i\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)$$
$$\times\left\{C_1\left(\frac{[kr]}{k},r\right)+1\right\}W_k^1(r,s)(L(s)+i)\,dw(r)$$
$$-\frac{1}{2}\int_{s\vee m/n}^{t\wedge(m+1)/n} e^{-i[w(m/n)-w(r)]L(m/n)}\left\{C_2\left(\frac{m}{n},\frac{[kr]}{k}\right)\right.$$
$$\left.-2C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)\right\}\left\{C_2\left(\frac{[kr]}{k},r\right)+1\right\}W_k^2(r,s)(L(s)+i)^2\,dr. \tag{2.7}$$

Applying this expression on $fe(u)$ with $f\in\mathscr{D}$ and $u\in L^2(\mathbb{R}_+)$ and using (2.6) and (1.4) one gets the estimates

$$\|[V_k(t,s)-V_n(t,s)]\,fe(u)\|$$
$$\leqslant K(u)\sum_{m=[ns]}^{[nt]}\left\{\int_{s\vee m/n}^{t\wedge(m+1)/n} dv_u(r)\left\|C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)\left\{C_1\left(\frac{[kr]}{k},r\right)+1\right\}\right.\right.$$
$$\left.\times W_k^1(r,s)\right\|^2 \|L(s)+i)f\|^2$$
$$\times\int_{s\vee m/n}^{t\wedge(m+1)/n} dv_u(r)\left\|\left\{C_2\left(\frac{m}{n},\frac{[kr]}{k}\right)-2C_1\left(\frac{m}{n},\frac{[kr]}{k}\right)\right\}\right.$$
$$\left.\times\left\{C_2\left(\frac{[kr]}{k},r\right)+1\right\}W_k^2(r,s)\right\|^2$$
$$\left.\times\|(L(s)+i)^2 f\|^2\right\}^{1/2},$$

where $K(u)$ is a constant depending only on u. By assumption A2.1(ii),

$$\left\|C_j\left(\frac{m}{n},\frac{[kr]}{k}\right)\right\|\leqslant\left|\frac{m}{n}-\frac{[kr]}{k}\right|M_j\leqslant M_j\left(\frac{1}{n}\vee\frac{1}{k}\right)=\frac{M_j}{n}\qquad\text{since}\quad k>n.$$

Hence,

$$\|[V_k(t,s)-V_n(t,s)]\,fe(u)\|$$
$$\leqslant\frac{K_1(u)}{n}\left(\|(L(s)+i)f\|^2+\|(L(s)+i)^2 f\|^2\right)^{1/2}$$
$$\times\sum_{m=[ns]}^{[nt]}\left\{v_u\left(t\wedge\frac{m+1}{n}\right)-v_u\left(s\vee\frac{m}{n}\right)\right\}^{1/2}.$$

But

$$\sum_{m=[ns]}^{[nt]}\left\{v_u\left(t\wedge\frac{m+1}{n}\right)-v_u\left(s\vee\frac{m}{n}\right)\right\}^{1/2}\leqslant n^{1/2}[v_u(t)-v_u(s)]^{1/2}$$

by the Cauchy–Schwartz inequality, so that

$$\|[V_k(t,s)-V_n(t,s)]\,fe(u)\|$$
$$\leqslant\frac{K_1(u)}{\sqrt{n}}[v_u(t)-v_u(s)]^{1/2}\left(\|(L(s)+i)f\|^2+\|(L(s)+i)^2 f\|^2\right)^{1/2},\qquad(2.8)$$

thereby proving the strong convergence of $V_k(t,s)\,fe(u)$. Convergence is uniform in s and t as $\|(L(s)+i)f\|^2+\|(L(s)+i)^2 f\|^2$ are uniformly bounded with respect to s. ∎

The set $\{fe(u) \mid f \in \mathscr{D},\ u \in L^2(\mathbb{R}_+)\}$ is total in $h \otimes \Gamma(L^2(\mathbb{R}_+))$. Also for every k, the unitary operator family $V_k(t, s)$ is strongly continuous with respect to t and s. So it follows that $V(t, s)$ is an isometry and is strongly continuous with respect to both t and s in the compact interval $[0, 1]$.

Moreover, the $V(t, s)$ are adapted processes satisfying the propagator property

$$V(t, s) = V(t, r)\, V(r, s) \qquad \text{for} \quad 0 \leqslant s \leqslant r \leqslant t \leqslant 1 \quad \text{and} \quad V(s, s) = I.$$

In order to show that $V(t, s)$ satisfies a quantum stochastic differential equation, we need an elementary technical lemma.

LEMMA 2.3. *Let $\mathscr{H}$ be a Hilbert space and let A_k, $A: [0, 1] \to \mathscr{B}(\mathscr{H})$ be uniformly bounded for $k \in \mathbb{N}$. Assume furthermore that $f: [0, 1] \to \mathscr{H}$ is continuous and* s-$\lim_{k \to \infty} A_k(u) = A(u)$ *uniformly in u. Then* s-$\lim_{k \to \infty} A_k(u) f(u) = A(u) f(u)$ *uniformly in u.*

Proof. By the uniform continuity of f, for a given $\varepsilon > 0$ we can choose n such that if $x, y \in [0, 1]$ and $|x - y| < 1/n$, then

$$\|f(x) - f(y)\| < \frac{\varepsilon}{4B},$$

where B is a uniform bound of the family $\{A_k(u), A(u)\}$. Now choose k_0 such that, for $k > k_0$

$$\|(A_k(u) - A(u))\, f(i/n)\| < \frac{\varepsilon}{2}, \qquad \text{for} \quad 0 \leqslant i \leqslant n \text{ and for all } u.$$

We then have, for $k > k_0$,

$$\begin{aligned} \|A_k(u) f(u) - A(u) f(u)\| &= \left\|(A_k(u) - A(u)) \left(f(u) - f\left(\frac{[nu]}{n}\right)\right)\right\| \\ &\quad + \left\|(A_k(u) - A(u))\, f\left(\frac{[nu]}{n}\right)\right\| \\ &\leqslant \|(A_k(u) - A(u))\| \left\|f(u) - f\left(\frac{[nu]}{n}\right)\right\| + \frac{\varepsilon}{2} \\ &\leqslant 2B \cdot \frac{\varepsilon}{4B} + \frac{\varepsilon}{2} \leqslant \varepsilon, \qquad \text{uniformly in } u. \quad \blacksquare \end{aligned}$$

PROPOSITION 2.4. *The $V(t, s)$ defined in Proposition 2.2 maps $\mathscr{D}$ into $D(L(t)^2)$ and satisfies on $\mathscr{D}$ the stochastic differential equation*

$$d_t V(t, s) = [-iL(t)\, dw(t) - \tfrac{1}{2} L(t)^2\, dt]\, V(t, s) \qquad \text{for} \quad 0 \leqslant s \leqslant t \leqslant 1$$

with $V(s, s) = I$.

Proof. First observe that

$$W_k^{j,1}(t,s)=\sum_{i=[ks]+1}^{[kt]}\left\{V_k\left(t,\frac{i}{k}\right)\left(C_j\left(\frac{i}{k},\frac{i-1}{k}\right)k\right)V_k\left(\frac{i}{k},s\right)\right\}\frac{1}{k}$$

and, using the definition of strong Riemann integral,

$$\int_s^t V(t,r)\,C_j(r)\,V(r,s)\,dr=\operatorname{s-lim}_{k\to\infty}\sum_{i=[ks]+1}^{[kt]}\left\{V\left(t,\frac{i}{k}\right)C_j\left(\frac{i}{k}\right)V\left(\frac{i}{k},s\right)\right\}\frac{1}{k}.$$

Given a vector f in $h\otimes\Gamma(L^2(\mathbb{R}_+))$ and $\varepsilon>0$, choose $k_0(j)$ such that

$$\begin{aligned}&\left\|W_k^{j,1}(t,s)f-\int_s^t V(t,r)\,C_j(r)\,V(r,s)\,dr\,f\right\|\\&\quad\leqslant\frac{\varepsilon}{4}+\sum_{i=[ks]+1}^{[kt]}\left\{\left\|V_k\left(t,\frac{i}{k}\right)\left(C_j\left(\frac{i}{k},\frac{i-1}{k}\right)k\right)V_k\left(\frac{i}{k},s\right)f\right.\right.\\&\qquad\left.\left.-V\left(t,\frac{i}{k}\right)C_j\left(\frac{i}{k}\right)V\left(\frac{i}{k},s\right)f\right\|\right\}\frac{1}{k}\\&\qquad\text{for all}\quad k>k_0(j).\end{aligned}\tag{2.9}$$

Now for any $u\in[s,t]$,

$$\begin{aligned}&\left\|V_k(t,u)\left(C_j\left(u,u-\frac{1}{k}\right)k\right)V_k(u,s)f-V(t,u)\,C_j(u)\,V(u,s)f\right\|\\&\quad\leqslant\|(V_k(t,u)-V(t,u))\,C_j(u)\,V(u,s)f\|\\&\qquad+\left\|V_k(t,u)\left(C_j\left(u,u-\frac{1}{k}\right)k-C_j(u)\right)V(u,s)f\right\|\\&\qquad+\left\|V_k(t,u)\,C_j\left(u,u-\frac{1}{k}\right)k(V_k(u,s)-V(u,s))f\right\|.\end{aligned}$$

Here, observe that in the first term $\operatorname{s-lim}_{k\to\infty}(V_k(t,u)-V(t,u))=0$ uniformly in u and $C_j(u)\,V(u,s)f$ is uniformly continuous in u. In the second term $V_k(t,u)$ is uniformly bounded, $\operatorname{s-lim}_{k\to\infty}(C_j(u,u-1/k)k-C_j(u))=0$ strongly, uniformly in u, and $V(u,s)f$ is uniformly continuous in u. Finally, in the last term $V_k(t,u)\,C_j(u,u-1/k)k$ is uniformly bounded and $\operatorname{s-lim}_{k\to\infty}(V_k(u,s)-V(u,s))=0$ strongly, uniformly in u. Hence using Lemma 2.3 we can choose $k_1(j)$ such that each of the three terms above are smaller than $\varepsilon/4$, uniformly in u, for $k>k_1(j)$. Hence from (2.9),

$$\begin{aligned}&\left\|W_k^{j,1}(t,s)f-\int_s^t V(t,r)\,C_j(r)\,V(r,s)\,dr\,f\right\|\\&\quad\leqslant\frac{\varepsilon}{4}+\sum_{i=[ks]+1}^{[kt]}\frac{3\varepsilon}{4}\cdot\frac{1}{k}\leqslant\frac{\varepsilon}{4}+\frac{3\varepsilon}{4}(t-s)\leqslant\varepsilon,\end{aligned}$$

i.e.,

$$\operatorname*{s-lim}_{k\to\infty} W_k^{j,1}(t,s)f = \int_s^t V(t,r)\, C_j(r)\, V(r,s)\, dr \equiv W^{j,1}(t,s),$$

the convergence being uniform in t and s. Since the integrand is strongly continuous, $W^{j,1}(t,s)$ is also strongly continuous in t and s. Inductively, assume that

$$\operatorname*{s-lim}_{k\to\infty} W_k^{j,m}(t,s) \equiv W^{j,m}(t,s) = \int_s^t V(t,r)\, C_j(r)\, W^{j,m-1}(r,s)\, dr$$

with properties as above, then arguing as above we obtain

$$\operatorname*{s-lim}_{k\to\infty} W_k^{j,m+1}(t,s) \equiv W^{j,m+1}(t,s) = \int_s^t V(t,r)\, C_j(r)\, W^{j,m}(r,s)\, dr$$

with continuity properties as before. We can rewrite (2.4) as

$$W_k^j(t,s) = \left(1 + C_j\left(t, \frac{[tk]}{k}\right)\right)\left[\sum_{m=0}^{[kt]-1} W_k^{j,m}(t,s)\right]\left(1 + C_j\left(\frac{[ks]}{k}, s\right)\right),$$

with $W_k^{j,0}(t,s) = V_k(t,s)$ for $j = 1, 2$. We have from assumption A2.1(ii) that

$$\lim_{k\to\infty} \left\| C_j\left(\frac{[ks]}{k}, s\right)\right\| = 0,$$

uniformly in s. Hence

$$\| W_k^{j,m}(t,s)\| \leqslant \frac{(t-s)^m}{m!} M_j^m \leqslant \frac{M_j^m}{m!} \qquad \text{for} \quad 0 \leqslant s, t \leqslant 1,$$

and as the series $\sum_m (M^m/m!)$ is summable independent of k, t, and s we get

$$\operatorname*{s-lim}_{k\to\infty} \sum_{m=0}^{[kt]-1} W_k^{j,m}(t,s) = \sum_{m=0}^{\infty} W^{j,m}(t,s) \equiv W^j(t,s),$$

uniformly in t and s which implies

$$\operatorname*{s-lim}_{k\to\infty} W_k^j(t,s) = W^j(t,s)$$

uniformly in t and s.

Let $f \in \mathscr{D}$, then for each fixed Brownian path w, $V_k(t,s)f \in D$ and

$$\operatorname*{s-lim}_{k\to\infty} (L(t)+i)^j V_k(t,s)f = \operatorname*{s-lim}_{k\to\infty} W_k^j(t,s)(L(s)+i)^j f = W^j(t,s)(L(s)+i)^j f.$$

Since $(L(t)+i)^j$ is closed and $\text{s-lim}_{k\to\infty} V_k(t,s)f = V(t,s)f$, we get

$$V(t,s)f \in D((L(t)+i)^j)$$

and (2.10)

$$(L(t)+i)^j V(t,s)f = W^j(t,s)(L(s)+i)^j f \qquad \forall j \in D.$$

Also for $u \in L_2(\mathbb{R}_+)$,

$$\begin{aligned}
&V(t,s)fe(u) - fe(u)\\
&\quad = \operatorname*{s-lim}_{k\to\infty} V_k(t,s)fe(u) - fe(u)\\
&\quad = \operatorname*{s-lim}_{k\to\infty} \int_s^t \left\{(-i)\, L\left(\frac{[kr]}{k}\right) V_k(r,s)\, dw(r)\, fe(u)\right.\\
&\qquad \left. - \frac{1}{2}\left(L\left(\frac{[kr]}{k}\right)\right)^2 V_k(r,s)\, dr\, fe(u)\right\}\\
&\quad = \operatorname*{s-lim}_{k\to\infty} \int_s^t \left\{(-i)\left[C_1\left(\frac{[kr]}{k}, r\right) + L(r)(L(r)+i)^{-1}\right]\right.\\
&\qquad \times (L(r)+i)\, V_k(r,s)\, dw(r)\\
&\qquad - \frac{1}{2}\left(C_2\left(\frac{[kr]}{k}, r\right) - 2iC_1\left(\frac{[kr]}{k}, r\right)(L(r)+i)^{-1} + L(r)^2(L(r)+i)^{-2}\right)\\
&\qquad \left. \times (L(r)+i)^2\, V_k(r,s)\, dr\right\} fe(u)\\
&\quad = \operatorname*{s-lim}_{k\to\infty} \int_s^t \left\{(-i)\left[C_1\left(\frac{[kr]}{k}, r\right) + L(r)(L(r)+i)^{-1}\right]\right.\\
&\qquad \times W_k^1(r,s)(L(s)+i)\, dw(r)\\
&\qquad - \frac{1}{2}\left(C_2\left(\frac{[kr]}{k}, r\right) - 2iC_1\left(\frac{[kr]}{k}, r\right)(L(r)+i)^{-1} + L(r)^2(L(r)+i)^{-2}\right)\\
&\qquad \left. \times W_k^2(r,s)(L(s)+i)^2\, dr\right\} fe(u).
\end{aligned}$$

Note that $\|C_j([kr]/k, r)\| \leqslant M_j|r - [kr]/k| \leqslant M_j/k \to 0$, $L(r)(L(r)+i)^{-1}$,

$(L(r)+i)^{-1}$ and $L(r)^2(L(r)+i)^{-2}$ are uniformly bounded, s-$\lim_{k\to\infty} W_k^j(r,s) = W^j(r,s)$ uniformly in r, s. Thus we get by (2.10)

$$\begin{aligned} V(t,s)\,fe(u)-fe(u) &= \int_s^t \{(-i)\,L(r)(L(r)+i)^{-1}\,W^1(r,s)(L(s)+i)\,dw(r) \\ &\quad -\tfrac{1}{2}L(r)^2(L(r)+i)^{-2}\,W^2(r,s)(L(s)+i)^2\,dr\}\,fe(u) \\ &= \int_s^t \{(-i)\,L(r)\,V(r,s)\,dw(r)-\tfrac{1}{2}L(r)^2\,V(r,s)\,dr\}\,fe(u), \end{aligned}$$

i.e.,

$$d_t V(t,s) = [-iL(t)\,dw(t)-\tfrac{1}{2}L(t)^2\,dt]\,V(t,s). \quad ∎$$

As mentioned in the Introduction, we are interested in the unitarity of $V(t,s)$ and for this we prove the convergence of $V_k^*(t,s)$. We introduce the dual cocycle following Journé [8] and Mohari [9]. Define

$$\tilde{w}(t)=\begin{cases} w(1)-w(1-t), & 0\leqslant t\leqslant 1 \\ w(t), & t\geqslant 1 \end{cases}$$

and

$$\tilde{t}=1-t \qquad \text{for} \quad 0\leqslant t\leqslant 1.$$

Then we note that

$$w(t)=\begin{cases} \tilde{w}(1)-\tilde{w}(1-t), & 0\leqslant t\leqslant 1 \\ \tilde{w}(t) & t\geqslant 1, \end{cases}$$

and note that $\tilde{w}$ is again a standard Brownian motion and we can apply Ito's formula with respect to $\tilde{w}$.

From Eq. (2.5) by taking the adjoint we have

$$V_k^*(t,s)=\overrightarrow{\prod_{m=[ks]}^{[kt]}} e^{i[w(t\wedge(m+1)/k)-w(s\vee m/k)]\,L(m/k)},$$

i.e.,

$$V_k^*(t,s)=\overleftarrow{\prod_{\tilde{m}=\{k\tilde{t}\}}^{\{k\tilde{s}\}}} e^{i[\tilde{w}(\tilde{s}\wedge\tilde{m}/k)-\tilde{w}(\tilde{t}\vee(\tilde{m}-1)/k)]\,\tilde{L}(\tilde{m}/k)} \equiv \tilde{V}_k(\tilde{s},\tilde{t}),$$

where we have defined $\sim$ by $\tilde{m}=k-m$, $\tilde{L}(t)=L(\tilde{t})$, and $\{x\}$ is the smallest integer greater than or equal to x. We have also noted that the transformation $t \to \tilde{t}$ changes the order $0 \leqslant s \leqslant t \leqslant 1$ to $0 \leqslant \tilde{t} \leqslant \tilde{s} \leqslant 1$ and that

$$\left(s \vee \frac{m}{k}\right)^{\sim} = \tilde{s} \wedge \frac{\tilde{m}}{k} \qquad \text{and} \qquad \left(t \wedge \frac{m+1}{k}\right)^{\sim} = \tilde{t} \vee \frac{\tilde{m}-1}{k}.$$

Equivalently defining

$$\tilde{V}_k(t,s) \equiv \overleftarrow{\prod_{m=\{ks\}}^{\{kt\}}} e^{i[\tilde{w}(t \wedge m/k) - \tilde{w}(s \vee (m-1)/k)]} \tilde{L}\left(\frac{m}{k}\right) \qquad \text{for} \quad 0 \leqslant s \leqslant t \leqslant 1$$

just as for $V_k(t,s)$ except for a sign change (of $-i$ to $+i$) and $\{\ \}$, $\tilde{w}$, and $\tilde{L}$ replacing $[\ ]$, w, and L, respectively, we make the estimates as we did for $V_k(t,s)$. The change from $[\ ]$ to $\{\ \}$ is minor and does not affect the estimates as $k \to \infty$, and we obtain the convergence of $\tilde{V}_k(t,s)$. Hence $\tilde{V}_k(\tilde{s}, \tilde{t}) = V_k^*(t,s)$ will converge strongly, in which case the limit must be $V^*(t,s)$ and therefore $V(t,s)$ must be unitary. Also, the propagator $V^*(t,s)$ satisfies the differential equation

$$\begin{aligned} d_t V(t,s)^* &= V(t,s)^* \left[iL(t)\, dw(t) - \tfrac{1}{2} L(t)^2\, dt\right], \qquad 0 \leqslant s \leqslant t \leqslant 1 \\ V(s,s) &= I. \end{aligned} \tag{2.11}$$

Combining the above results with Propositions 2.2 and 2.4, we have

THEOREM 2.5. *Defining* $V_k(t,s)$ *as in* (2.2), $V(t,s) = \text{s-lim}_{k \to \infty} V_k(t,s)$ *exists in* $h \otimes \Gamma(L^2(\mathbb{R}_+))$ *uniformly in* t, s *for* t, s *in* $[0,1]$. *Furthermore,* $V(t,s)$ *is a strongly continuous* (*both in* s *and* t) *family of unitary operators satisfying the quantum stochastic differential equation*

$$d_t V(t,s) = \left[-iL(t)\, dw(t) - \tfrac{1}{2} L(t)^2\, dt\right] V(t,s) \qquad \textit{for} \quad 0 \leqslant s \leqslant t < \infty$$

with $V(s,s) = I$.

We also have

THEOREM 2.6. *There exists a unique unitary solution of the quantum stochastic differential equation* (2.1),

$$dV(t) = \left[-iL(t)\, dw(t) - \tfrac{1}{2} L(t)^2\, dt\right] V(t), \qquad V(0) = I, \quad 0 \leqslant t < \infty,$$

with $V(t)(\mathscr{D} \otimes \Gamma(L^2(\mathbb{R}_+))) \subset D(L(t)^2) \otimes \Gamma(L^2(\mathbb{R}_+))$. *The solution is also strongly continuous in* t.

Proof. The existence of a strongly continuous unitary solution is clear, once we put

$$V(t) = V(t, 0) \qquad \text{for} \quad 0 \leqslant t \leqslant 1$$

and use Theorem 2.5.

Now suppose $V_1(t)$ is another unitary solution of the above differential equation, with $V_1(t)(\mathscr{D} \otimes \Gamma(L^2(\mathbb{R}_+))) \subset D(L(t)^2) \otimes \Gamma(L^2(\mathbb{R}_+))$, then by quantum Ito's formula [2],

$$\begin{aligned}
&\langle V(t) fe(u), V_1(t)\, ge(v)\rangle - \langle fe(u), ge(v)\rangle \\
&= \int_0^t \langle -iL(s)\, V(s) fe(u), V_1(s)\, ge(v)\rangle (dv(s) + d\bar{u}(s)) \\
&\quad + \int_0^t \langle -\tfrac{1}{2} L(s)^2\, V(s) fe(u), V_1(s)\, ge(v)\rangle\, ds \\
&\quad + \int_0^t \langle V(s) fe(u), -iL(s)\, V_1(s)\, ge(v)\rangle (d\bar{u}(s) + dv(s)) \\
&\quad + \int_0^t \langle V(s) fe(u), -\tfrac{1}{2} L(s)^2\, V_1(s)\, ge(v)\rangle\, ds \\
&\quad + \int_0^t \langle (-i)\, L(s)\, V(s) fe(u), (-i)\, L(s)\, V_1(s)\, ge(v)\rangle\, ds \\
&\qquad \text{for} \quad f, g \in \mathscr{D}, u, v \in L^2(\mathbb{R}_+), 0 \leqslant t \leqslant 1 \\
&= 0.
\end{aligned}$$

Hence $V^*(t)\, V_1(t)\, ge(v)$ for $g \in \mathscr{D}$ and $v \in L^2(\mathbb{R}_+)$, i.e., $V^*(t)\, V_1(t) = I$, but $V(t)$ is unitary, so $V_1(t) = V(t)$, proving uniqueness.

3. A Special Case

In this section, we apply the results of the last section to obtain unitary solutions of the quantum stochastic differential equation,

$$dU(t) = U(t)[iL\, dw(t) + (-\tfrac{1}{2} L^2 + iH)\, dt], \qquad V(0) = I \tag{3.1}$$

on $h \otimes \Gamma(L^2(\mathbb{R}_+))$, where L and H are, possibly unbounded, self-adjoint operators on h.

In this section we assume

A3.1. (i) $\exists \mathscr{D} \subset D(L^2)$ such that $\mathscr{D}$ is a core for L^j, $j = 1, 2$, and $\mathscr{D}$ is left invariant by e^{iHt} and e^{iLt} for every $t \in \mathbb{R}$.

(ii) H, $[L, H]$ and $[L, [L, H]]$ are L-bounded, where the second and third expressions have to be interpreted as the L-bounded extensions of the associated forms on $D(L)$.

We set

$$L(t) = e^{iHt} L e^{-iHt}. \tag{3.2}$$

LEMMA 3.1. *Let L and H satisfy* A3.1. *Then* $L(t) = e^{iHt} L e^{-iHt}$ *satisfies assumption* A2.1.

Proof. It follows trivially that $L(t)$ satisfies assumption A2.1(i) by assumption (i) above.

The proof of the second part is by direct computation and all the domains are just right by virtue of assumptions.

It is clear that $L(t)$ is essentially self-adjoint on $\mathscr{D}$. Hence the $(L(t)+i)^{-j}$ are bounded operators on h. We also have

$$(L(t)+i)^{-j} = e^{iHt}(L+i)^{-j} e^{-iHt} \qquad \text{for} \quad j = 1, 2.$$

Now for $s < t$,

$$C_1(t, s) = \int_s^t e^{iHr} i[H, L+i](L+i)^{-1} e^{-iHr} e^{iHr}(L+i) e^{-iHr}(L(s)+i)^{-1} \, dr$$

and writing $\beta = \|i[H, L+i](L+i)^{-1}\|$, and $\alpha(t, s) = \|C_1(t, s)\|$ we get

$$\begin{aligned}
\alpha(t, s) &\leqslant \beta \int_s^t \|(L(r)+i)(L(s)+i)^{-1}\| \, dr \\
&\leqslant \beta \int_s^t \|1 + C_1(r, s)\| \, dr \\
&\leqslant \beta \left\{ (t-s) + \int_s^t \alpha(r, s) \, dr \right\}.
\end{aligned}$$

Fix s and take $k(t) = (t-s) + \int_s^t \alpha(r, s) \, dr$ for $t > s$, to obtain

$$k'(r) = 1 + \alpha(r, s) \leqslant 1 + \beta k(r).$$

Solving the above differential inequality with initial value $k(s) = 0$, we have

$$k(t) \leqslant \int_s^t e^{\beta(t-r)} \, dr.$$

Hence $\alpha(t,s)\leqslant\beta k(t)\leqslant\beta e^{\beta}(t-s)$ (as $0\leqslant s<t\leqslant 1$), i.e., $\|C_1 t,s)\|\leqslant M_1(t-s)$ for some constant M_1. We rewrite $C_2(t,s)$ as

$$C_2(t,s)=\int_s^t e^{iHr}\, i[H,(L+i)^2](L+i)^{-2}e^{-iHr}e^{iHr}(L+i)^2e^{-iHr}(L(s)+i)^{-2}\,dr,$$

and note that

$$\begin{aligned}&[H,(L+i)^2](L+i)^{-2}\\&\quad=[(L+i),[H,(L+i)]](L+i)^{-2}+2[H,(L+i)](L+i)^{-1},\end{aligned}$$

so that $[H,(L+i)^2](L+i)^{-2}$ is a bounded operator by assumption A3.1(ii). Let $\delta=\|[H,(L+i)^2](L+i)^{-2}\|$ and $\gamma(t,s)=\|C_2(t,s)\|$. Then

$$\begin{aligned}\gamma(t,s)&\leqslant\delta\int_s^t\|(L(r)+i)^2(L(s)+i)^{-2}\|\,dr\\&\leqslant\delta(t-s)+\int_s^t\gamma(r,s)\,dr.\end{aligned}$$

As before this leads to

$$\|C_2(t,s)\|\leqslant M_2(t-s),$$

for some constant M_2.

To prove that the second part of the assumption A2.1(ii) holds, observe that

$$i[H,(L(t)+i)^j](L(t)+i)^{-j}=e^{iHt}\{i[H,(L+i)^j](L+i)^{-j}\}\,e^{-iHt},$$

which is bounded uniformly in t in norm and is strongly (uniformly) continuous in t. Also note that

$$\begin{aligned}\|(L(s)+i)^{-j}-(L(t)+i)^{-j}\|&=\|(L(t)+i)^{-j}C_j(t,s)\|\\&\leqslant\|(L+i)^{-j}\|\,M_j(t-s)\to 0,\qquad\text{as}\quad s\uparrow t.\end{aligned}$$

Now,

$$\begin{aligned}&(t-s)^{-1}C_j(t,s)\\&\quad=e^{iHs}\left\{\frac{e^{iH(t-s)}(L+i)^je^{-iH(t-s)}-(L+i)^j}{(t-s)}\right\}(L+i)^{-j}e^{-iHs}\\&\quad=e^{iHs}\left[\frac{1}{(t-s)}\int_0^{t-s}dr\,e^{iHr}(i[H,(L+i)^j](L+i)^{-j})\right.\\&\qquad\left.\times e^{-iHr}\{1+C_j(r,0)\}\right]e^{-iHs}.\end{aligned}$$

Since $\|C_j(r, 0)\| \to 0$ as $r \to 0$ and since $i[H, (L+i)^j](L+i)^{-j}$ is a bounded operator by hypothesis, the

$$\operatorname*{s-lim}_{s \uparrow t} \left\{ \frac{1}{(t-s)} \int_0^{t-s} dr\, e^{iHr}(i[H, (L+i)^j](L+i)^{-j})\, e^{-iHr}\{1 + C_j(r, 0)\} \right\}$$

exists and equals $i[H, (L+i)^j](L+i)^{-j}$. Hence the result. ∎

Immediately we have the theorem.

THEOREM 3.2. *Assume* A3.1. *Then there exists a unique unitary solution of the quantum stochastic differential equation* (3.1),

$$dU(t) = U(t)[iL\, dw(t) + (-\tfrac{1}{2}L^2 + iH)\, dt], \qquad U(0) = I,$$

with $U(t)(\mathscr{D} \otimes \Gamma(L^2(\mathbb{R}_+))) \subset D(L^2) \otimes \Gamma(L^2(\mathbb{R}_+))$. *The solution is also strongly continuous in* t.

Proof. Lemma 3.1 ensures that we can apply Theorem 2.6 to obtain a strongly continuous unitary $V(t)$ satisfying

$$dV(t) = [-iL(t)\, dw(t) - \tfrac{1}{2}L(t)^2\, dt]\, V(t), \qquad V(0) = I.$$

Clearly $V^*(t)$ is strongly continuous, unitary and since by Theorem 2.6, $V(t)(\mathscr{D} \otimes \Gamma(L^2(\mathbb{R}_+))) \subseteq D(L(t)^2) \otimes \Gamma(L^2(\mathbb{R}_+))$ it satisfies the adjoint equation

$$dV^*(t) = V^*(t)[iL(t)\, dw(t) - \tfrac{1}{2}L(t)^2\, dt], \qquad V^*(0) = I.$$

Now set $U(t) = V^*(t)\, e^{iHt}$ and verify that $U(t)$ satisfies the required differential equation. Uniqueness follows as before. ∎

4. APPLICATION TO ONE DIMENSIONAL DIFFUSION

In this section we demonstrate a simple application of our results to one dimensional diffusion. Let a be a real valued C^∞ function on $\mathbb{R}$ with a' bounded. Consider the autonomous system

$$\frac{d\psi_\alpha(x)}{d\alpha} = a(\psi_\alpha(x)), \qquad \psi_0(x) = x, \quad \alpha \in \mathbb{R}. \tag{4.1}$$

The properties of solutions of such systems are well known and we have

PROPOSITION 4.1. *Let* a *and* ψ_α *be as above. Then,*

(i) ψ_α *defines a group of* C^∞ *diffeomorphisms of* $\mathbb{R}$.

(ii) *For every* $\alpha \in \mathbb{R}$, *define* U_α *on* $L^2(\mathbb{R})$ *by*

$$U_\alpha f(x) = \sqrt{\left|\frac{d\psi_\alpha(x)}{dx}\right|}\, f(\psi_\alpha(x)), \qquad V_0 = I \tag{4.2}$$

then U_α *is a group of unitary operators in* $L^2(\mathbb{R})$.

(iii) U_α *leaves* $C_0^\infty(\mathbb{R})$ *invariant for every* α.

(iv) *Let* L *be the self-adjoint generator of* U_α. *Then* $C_0^\infty(\mathbb{R})$ *is a core for* L *and* $Lf = \frac{1}{2}(aP + Pa)f$ *for* $f \in C_0^\infty(\mathbb{R})$, P *being the momentum operator* $(-i(d/dx))$.

Proof. (i) As a' is bounded a is globally Lipschitz. Now apply Theorem 4.21 of [10, pp. 44–50] to obtain the group of C^∞ diffeomorphisms ψ_α.

(ii) This is clear from (i) by direct computation.

(iii) Let $f \in C_0^\infty(\mathbb{R})$ with support$(f) \subset K$, where K is a compact subset of $\mathbb{R}$. Then from (4.2) it is clear that support $(U_\alpha(f)) \subset \psi_\alpha^{-1}(K)$ and as ψ_α is a diffeomorphism $\psi_\alpha^{-1}(K)$ is compact.

Also observe that $d\psi_\alpha(x)/dx = \exp(\int_0^\alpha a'(\psi_\beta(x))\, d\beta) > 0$, and hence $\sqrt{|d\psi_\alpha(x)/dx|}$ is C^∞ and consequently $U_\alpha f$ is in $C_0^\infty(\mathbb{R})$.

(iv) $C_0^\infty(\mathbb{R}) \subset \mathscr{D}(L)$ and $Lf = \frac{1}{2}(aP + Pa)f$ for $f \in C_0^\infty(\mathbb{R})$ is clear from direct computation. From (iii), $e^{i\alpha L}$ leaves $C_0^\infty(\mathbb{R})$ invariant for every real α and hence $C_0^\infty(\mathbb{R})$ is a core for L (see Theorem VIII.11 of [11]). ∎

We also need essential self-adjointness of L^2 on $C_0^\infty(\mathbb{R})$.

LEMMA 4.2. *Let* $a \in C^\infty(\mathbb{R})$ *with* $a > 0$, a' *and* aa'' *bounded then* $C_0^\infty(\mathbb{R})$ *is a core for* L^2 (L *is defined as in Proposition* 4.1).

Proof. An easy computation shows that for $f \in C_0^\infty(\mathbb{R})$,

$$L^2 f(x) = -\frac{d}{dx}\, p(x) \frac{d}{dx} f(x) + q(x) f(x),$$

where $p = 4a^2$ and $q = -(2aa'' + a'^2)$. It is clear from assumptions on a that p is strictly positive and q is bounded below. Now Theorem XIII.6.15 of [12] gives the required result. ∎

By replacing the function a by a real valued C_0^∞ function b in (4.1) we obtain another group of diffeomorphisms π_α. Let V_α be the corresponding group of unitaries from Proposition 4.1 and H be its self-adjoint generator. In order that L and H satisfy assumptions A3.1 we make the following assumptions on a and b.

A4.1. a, b are real valued, $a \in C^\infty(\mathbb{R})$, $a(x) > \lambda > 0$ for some λ, a', and aa'' bounded, and $b \in C_0^\infty(\mathbb{R})$.

LEMMA 4.3. *On $C_0^\infty(\mathbb{R})$, H, $[L, H]$, and $[L, [L, H]]$ are L bounded.*

Proof. For $f \in C_0^\infty(\mathbb{R})$, observe that

$$Hf = \left(\frac{b}{a}\right) Lf + (-i)\frac{(ab' - ba')}{2a} f.$$

Now as a is bounded away from zero and as $b \in C_0^\infty$, L boundedness of H is clear. In a similar way $[L, H]$ and $[L, [L, H]]$ are also seen to be L bounded on $C_0^\infty(\mathbb{R})$ by looking at their actions on C_0^∞ functions. ∎

THEOREM 4.4. *Let L and H be the self-adjoint operators on $L^2(\mathbb{R})$ defined above with a and b satisfying the assumptions* A4.1. *Then there exists a unique unitary process $U(t)$ satisfying the quantum stochastic differential equation*

$$dU(t) = U(t)[iL\, dw(t) + (-\tfrac{1}{2}L^2 + iH)\, dt], \qquad U(0) = I \tag{4.3}$$

on the domain $C_0^\infty(\mathbb{R}) \otimes \Gamma(L^2(\mathbb{R}_+))$.

Proof. From Proposition 4.1, Lemmas 4.2 and 4.3 we know that L and H satisfy assumptions A3.1. Hence by Theorem 3.2 we have the required result. ∎

Let $\mathscr{A}$ be the algebra of bounded twice continuously differentiable functions $C_b^2(\mathbb{R})$. Define $j_t \colon \mathscr{A} \to \mathscr{B}(\Gamma(L^2(\mathbb{R}_+)))$ by

$$j_t(\phi) = U(t)\, \phi U(t)^*, \tag{4.4}$$

where we denote by the symbol ϕ also the operator of multiplication by the function $\phi \in C_b^2(\mathbb{R})$.

THEOREM 4.5. *The homomorphism j_t defined above is a quantum stochastic flow satisfying*

$$dj_t(\phi) = j_t(a\phi')\, dw(t) + j_t(\tfrac{1}{2}a^2\phi'' + c\phi')\, dt, \tag{4.5}$$

where $c = b + \frac{1}{2}aa'$. Moreover the family $\{j_t(\phi)\}$ is commutative.

Proof. That j_t satisfies the required quantum stochastic differential equation is seen from computing $dj_t(\phi)$ using quantum Ito's formula [2].

The commutativity of the family $\{j_t(\phi)\}$ is clear as

$$\begin{aligned} j_t(\phi) &= \operatorname*{s-lim}_{k\to\infty} V_k(t,0)^*\, e^{-iHt}\phi(x)\, e^{iHt} V_k(t,0) \\ &= \operatorname*{s-lim}_{k\to\infty} V_k(t,0)^*\, \phi(\pi_t(x))\, V_k(t,0) \\ &= \operatorname*{s-lim}_{k\to\infty} \phi(\psi_{w(t)-w([kt]/k)} \circ \pi_{[kt]/k} \circ \psi \circ \cdots \circ \pi_{[kt]/k} \circ \pi_t(x)). \end{aligned} \tag{4.6}$$

Then $j_t(\phi)$ commutes with any multiplication operator and hence is itself a multiplication operator. ∎

It should be remarked that (4.6) and also the very construction of $V(t, s)$ in Section 2 is in a sense, a random Trotter–Kato product formula. One may see [13] for some results in this direction.

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References

1. R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, *Comm. Math. Phys.* **93** (1984), 301–323.
2. K. R. Parthasarathy, An introduction to quantum stochastic calculus, *in* "Monographs in Mathematics," Birkhäuser-Verlag, Basel, 1992.
3. F. Fagnola, On quantum stochastic differential equation with unbounded coefficients, *Probab. Theory Related Fields* **86** (1990), 501–516.
4. F. Fagnola and K. B. Sinha, Quantum flows with unbounded structure maps and finite degrees of freedom, *J. London Math. Soc.*, in press.
5. M. A. Pinsky, Multiplicative operator functionals, *in* "Advances in Probability and Related Topics," Vol. 3, Dekker, New York, 1974.
6. G. Da Prato, M. Iannelli, and L. Tubaro, Some results on linear stochastic differential equations in Hilbert spaces, *Stochastics* **6** (1982), 105–116.
7. K. Yosida, "Functional Analysis," Springer-Verlag, Berlin, 1965.
8. J. L. Journé, Structure des cocycles markoviens sur l'espace de Fock, *Probab. Theory Related Fields* **75** (1987), 291–316.
9. A. Mohari, "Quantum Stochastic Calculus with Infinite Degrees of Freedom and Its Applications," Ph.D. Thesis, Indian Statistical Institute, New Delhi, 1991.
10. W. D. Curtis and F. R. Miller, "Differential Manifolds and Theoretical Physics," Academic Press, London, 1985.
11. M. Reed and B. Simon, "Methods of Modern Mathematical Physics," Vol. I, Academic Press, London, 1972.
12. N. Dunford and J. T. Schwartz, "Linear Operators," Part II, Interscience, New York, 1963.
13. K. R. Parthasarathy and K. B. Sinha, A random Trotter–Kato product formula, *in* "Statistics and Probability: Essays in honour of C. R. Rao," pp. 553–565, North-Holland, Amsterdam, 1982.