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Generalized Shmushkevich Method: Proof of Basic Results*

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We here derive certain orthogonality properties of the Clebsch-Gordan (CG) coefficients of an arbitrary compact group G. Our discussion recognizes the fact that the irreducible representations (IR's) of G need not be equivalent to their complex conjugates and that the same IR can appear more than once in the reduction of the direct product of two IR's of G. The properties obtained allow the development of a generalized Shmushkevich method for directly writing down consequences of the invariance of particle interactions under G. The discussion given is sufficiently general to apply to the currently interesting cases of SU_3 and G_2.

1. INTRODUCTION

THE aim of this work is to exhibit the proofs of certain facts concerning compact groups and their Clebsch-Gordan (CG) coefficients which are used in the development of the generalized Shmushkevich method for writing down consequences of the invariance of the strong interactions with respect to a given compact group. We commence with an explanation of this method.

Shmushkevich[1] originally described the method now known as Shmushkevich's method in connection with the charge-independent theory of the strong interactions. It is a technique for writing down linear relationships among the cross sections for various elementary-particle reactions that exist as a result of the assumption of charge independence or invariance with respect to the isospin rotation group, R_3. Simple expositions of it with examples may be found in recent books on elementary-particle physics.[2] Its notable characteristics are its economy, particularly in complicated physical contexts, and the fact that it proceeds without the use of (and therefore without the need for knowledge of) numerical values of CG coefficients of R_3. Formal proof[3] of the results which underlie the method depends only on certain general properties of CG coefficients of R_3, to be noted below.

In view of the great interest currently surrounding the theories[4,5] which use SU_3 and G_2 as invariance groups of the strong interactions, it is desirable to possess a generalization of Shmushkevich's method for these theories. Provided we assume that those properties of CG coefficients of R_3 which were used in the justification of the method for R_3 generalize to SU_3 and G_2, we can proceed directly to the development of the generalized Shmushkevich methods for SU_3 and G_2. Several illustrations have already been given of how consequences of SU_3 and G_2 invariance may be written down by the method: relationships between decay weights for the decays of certain resonances that exist as a result of SU_3 invariance[6] or G_2 invariance,[7] relationships between meson–baryon scattering cross sections that exist as a result of SU_3 invariance.[8]

The two properties of CG coefficients of R_3[9] used in the justification of Shmushkevich's method for R_3 are the following[3]:

(1) Orthogonality

$$\sum_{m_1m_2} C(j_1j_2j;\ m_1m_2m)C(j_1j_2j';\ m_1m_2m') = \delta(jj')\delta(mm'); \quad (1.1)$$

(2) Modified orthogonality[10]

$$\sum_{m_2m} C(j_1j_2j;\ m_1m_2m)C(j_1'j_2j;\ m_1'm_2m) = [(2j+1)/(2j_1+1)]\delta(j_1j_1')\delta(m_1m_1'). \quad (1.2)$$

The modified orthogonality rule arises from the

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[1] I. M. Shmushkevich, Dokl. Akad. Nauk SSSR **103**, 235 (1955).
[2] R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience Publishers, Inc., New York, 1961), p. 185; P. Roman, *Theory of Elementary Particles* (North-Holland Publishing Company, Amsterdam, 1960), p. 443.
[3] Such a proof is given in a forthcoming paper, A. J. Macfarlane and E. C. G. Sudarshan, "Shmushkevich's Method for a Charge Independent Theory: Nucleon–Anti-Nucleon Annihilation" (to be published).
[4] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961); M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 944, **28**, 24 (1962).
[5] R. E. Behrends and L. F. Landowitz, Phys. Rev. Letters **11**, 296 (1963).
[6] C. Dullemond, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. Letters **10**, 423 (1963).
[7] A. J. Macfarlane, N. Mukunda, and E. C. G. Sudarshan, Phys. Rev. **133**, B. 475, (1964).
[8] E. C. G. Sudarshan, *Proceedings of the Athens Conference on Newly Discovered Resonant Particles*, edited by B. A. Munir and L. J. Gallaher (Ohio University, Athens, Ohio, 1963).
[9] We use the notation of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).
[10] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 791.

ordinary orthogonality rule by means of the symmetry property

$$C(j_1j_2j;\ m_1m_2m) = (-)^{j_2+m_2}[(2j+1)/(2j_1+1)]^{\frac{1}{2}} \times C(jj_2j_1;\ -m\ m_2\ -m_1). \quad (1.3)$$

In this paper, we define CG coefficients for an arbitrary compact group, and establish that results analogous to Eqs. (1.1) and (1.2) obtain, so that a generalized Shmushkevich method may indeed justifiably be used to write down consequences of invariance with respect to any compact group, in particular with respect to SU_3 or G_2. That an orthogonality rule like (1.1) obtains is to be expected. It is not obvious although true, however, that a natural generalization of (1.2) exists. Indeed the opposite might be expected since a natural generalization of (1.3) does not exist—e.g., (1.3) for R_3 has to be replaced by a complicated crossing relation for an arbitrary compact group. The reason for this stems from the fact that the representation theory of R_3 is simpler than that of an arbitrary compact group in two important respects. These are as follows:

(A) An irreducible representation of an arbitrary compact group need not be equivalent to its complex conjugate.

(B) The direct product of two irreducible representations of an arbitrary compact group may contain the same irreducible representation more than once in its reduction.

For R_3 neither (A) nor (B) can occur. Since for SU_3 both (A) and (B) can occur[11] and do in practically interesting cases, the relevance of the present discussion becomes clear.

Among previous literature on the subject, we note that Wigner[14] has discussed CG coefficients of finite groups which do not allow either (A) or (B), and that Sharp[15] has discussed the same problem for compact groups and also for compact groups for which (A) but not (B) can occur. On the other hand, Hamermesh[16] has discussed groups which allow (B) but not (A). We here use the notation used by Hamermesh and refer to his book for many of the general properties of irreducible representations of groups used in the ensuing sections.

The material of the paper is presented as follows. In Sec. 2, we mention various facts regarding the representation theory of an arbitrary compact group and define its CG coefficients. In Sec. 3, we derive the desired generalizations of Eqs. (1.1) and (1.2). Sec. 4 contains an illustrative example.

2. SIMPLE PROPERTIES OF IR's AND CG COEFFICIENTS

We consider an arbitrary compact group G with general element R. Since G is compact, each of the IR's is of finite dimension and equivalent to a unitary IR. Thus we may confine attention to unitary IR's of G, i.e., to $D^\mu(R)$ which satisfy

$$D^\mu(R)^\dagger = D^\mu(R)^{-1}. \quad (2.1)$$

Here we use as a labeling of the IR's of G a single lower-case Greek letter $\mu, \nu, \rho \cdots$, which may in fact stand for several labels. For example, we may write

$$\mu = (\mu_1, \mu_2, \cdots \mu_l)$$

in the case of the IR of highest weight μ of l components if G is of rank l.

We do not assume that the IR $D^\mu(R)$ is equivalent to its complex conjugate $D^\mu(R)^*$, which is still however an IR of G, but set

$$D^\mu(R)^* = J(\mu', \mu)^{-1}D^{\mu'}(R)J(\mu', \mu), \quad (2.2)$$

where J is unitary and independent of R.[17] We shall apply primes to lower-case Greek letters always exactly in this sense and never at all to other letters. It is obvious that passage from $D^\mu(R)$ to $D^{\mu'}(R)$ is an involution, so that $D^{\mu''}(R) = D^\mu(R)$. Also,

$$J(\mu, \mu') = \tilde{J}(\mu', \mu), \quad (2.3)$$

where the tilde denotes transposition. We may summarize the situation by saying that the set $\{\cdots \mu, \nu, \rho \cdots\}$ of all IR's of G is the same as the set $\{\cdots \mu', \nu', \rho' \cdots\}$, possibly in a different order.

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[11] In the currently popular form of unitary symmetry theory[4,12] the baryons, pseudoscalar and vector mesons, are classified according to the octet or IR (1, 1) of SU_3, and the spin-$\frac{3}{2}$ baryon resonances are classified according to the decuplet or IR (3,0). The direct product of two octets contains an octet twice—a fact which reflects the possibility of constructing two independent Yukawa-type meson–baryon interactions. The IR (3,0) is not equivalent to its complex conjugate, but to the complex conjugate of the IR (0,3). The notation used here for IR's of SU_3 is explained in Ref. 13.

[12] S. L. Glashow and J. J. Sakurai, Nuovo Cimento **26**, 622 (1962).

[13] A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento **30**, 845 (1963).

[14] E. P. Wigner, Am. J. Math **63**, 57 (1941), and "On the Matrices Which Reduce the Kronecker Product of Representations of Simply Reducible Groups" (unpublished).

[15] W. T. Sharp, "Racah Algebra and the Contraction of Groups," CRT–935, Chalk River, Canada, 1960.

[16] M. Hamermesh, *Group Theory* (Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1962). See especially Chap. 5 and Sec. 7–14.

[17] The matrix J plays a role for the general compact group analogous to that played by the "$1 - j$ symbol" for R_3.

We go on to consider the direct product

$$D^\mu(R) \otimes D^\nu(R) \tag{2.4}$$

of a pair of IR's of G. Since G is compact, we know that this is equivalent to a direct sum of IR's of G, possibly containing several IR's more than once. We may use

$$D^\mu(R) \otimes D^\nu(R) \cong \textstyle\sum_\rho (\mu\nu\rho) D^\rho(R) \tag{2.5}$$

to define $(\mu\nu\rho)$ as the number of times $D^\rho(R)$ occurs as a direct summand in the reduction of (2.4). Allowed values of $(\mu\nu\rho)$ are 0, 1, 2 $\cdots$. We have

$$(\mu\nu\rho) = (\nu\mu\rho). \tag{2.6}$$

If $D^\mu(R)$ has character $\chi^\mu(R)$, Eq. (2.5) implies

$$\chi^\mu(R)\chi^\nu(R) = \textstyle\sum_\rho (\mu\nu\rho)\chi^\rho(R). \tag{2.7}$$

We may use the orthogonality relation for characters

$$\int \chi^\mu(R)\chi^\nu(R)^*\, dR = A\,\delta(\mu\nu), \tag{2.8}$$

where A is a normalization constant and the integration is the usual left- and right-invariant integration over the group manifold of G, to give

$$(\mu\nu\rho) = A^{-1}\int \chi^\mu(R)\chi^\nu(R)\chi^\rho(R)^*\, dR. \tag{2.9}$$

Hence using the consequence

$$\chi^\mu(R)^* = \chi^{\mu'}(R) \tag{2.10}$$

of Eq. (2.2), we may deduce the important result

$$(\mu\nu\rho) = (\rho'\nu\mu'). \tag{2.11}$$

We now define the CG coefficients of G for the direct product (2.4) and show that they are the elements of the unitary matrix which generates the similarity transformation that brings the direct sum on the left side of (2.5) into equivalence with the right side. If n_μ is the dimension of $D^\mu(R)$, suppose ψ^μ_j with j standing for a set of labels which take on n_μ distinct sets of values is an orthonormal basis in the representation space of the IR μ. Under R, we have

$$\psi^\mu_j \underset{R}{\to} O_R\psi^\mu_j = \psi^\mu_k D^\mu(R)_{kj}. \tag{2.12}$$

Here we are using summation convention for Latin indices but not Greek ones. Similarly, ψ^ν_k is an orthonormal basis for the representation space of the IR ν, so that the products $\psi^\mu_j\psi^\nu_k$ are the basis functions for the product (2.4). Reduction of this product into a direct sum of IR's $D^\rho(R)$, various ρ, involves a unitary change of basis wherein we replace the products $\psi^\mu_j\psi^\nu_k$ by sets of basis functions ψ^ρ_l which transform according to $D^\rho(R)$,

$$\psi^\rho_l \underset{R}{\to} O_R\psi^\rho_l = \psi^\rho_m D^\rho(R)_{ml}. \tag{2.13}$$

Since $(\mu\nu\rho) > 1$ is possible for a given ρ appearing in the reduction, there can be more than one independent set of n_ρ basis functions ψ^ρ_l. To distinguish these we add a Latin capital label (to which summation convention does not apply) to the basis functions, e.g., $\psi^{\rho A}_l$, there being $(\mu\nu\rho)$ allowed (sets of) values for the (perhaps composite) label A. We demand that basis functions $\psi^{\rho A}_l$ for different A be orthogonal. Also we arrange[18] to have

$$\psi^{\rho A}_l \underset{R}{\to} O_R\psi^{\rho A}_l = \psi^{\rho A}_m D^\rho(R)_{ml}, \tag{2.14}$$

with $D^\rho(R)_{ml}$ independent of A. We may define $\psi^{\rho A}_l$ explicitly by setting

$$\psi^{\rho A}_l = \psi^\mu_j\psi^\nu_k(\mu j, \nu k \mid \rho A l), \tag{2.15}$$

where

$$(\mu j, \nu k \mid \rho A l) \tag{2.16}$$

is the generalized CG coefficient of G for the product (2.4). We may also give an inverse to Eq. (2.15) in the form

$$\psi^\mu_j\psi^\nu_k = \sum_{\rho A} \psi^{\rho A}_l(\rho A l \mid \mu j, \nu k), \tag{2.17}$$

where the quantities

$$(\rho A l \mid \mu j, \nu k)$$

are elements of the matrix inverse to that with the CG coefficients (2.16) as elements, i.e.,[19]

$$(\mu j, \nu k \mid \rho A l)(\sigma B m \mid \mu j, \nu k) = \delta(\rho\sigma)\,\delta(AB)\,\delta(lm), \tag{2.18}$$

$$\sum_{\rho A} (\mu j, \nu k \mid \rho A l)(\rho A l \mid \mu p, \nu q) = \delta(jp)\,\delta(kq). \tag{2.19}$$

We can now exhibit that the CG coefficients (2.16) are the elements of the

$$n_\mu n_\nu = \textstyle\sum_\rho (\mu\nu\rho) n_\rho \tag{2.20}$$

-dimensional matrix of the similarity transformation which brings the direct sum on the right of Eq. (2.5) into equivalence with the direct product on the left. We apply O_R to (2.17), use (2.15) and cancel the product basis functions, as they are linearly independent, obtaining

$$D^\mu(R)_{pj}D^\nu(R)_{qk} = \sum_{\rho A} (\mu p, \nu q \mid \rho A l) D^\rho(R)_{lm}(\rho A m \mid \mu j, \nu k), \tag{2.21}$$

which demonstrates the equivalence.

[18] Ref. 16, p. 150.
[19] Ref. 16, p. 149.

We should stress the fact that there is a great deal of arbitrariness[20] in the definition of the CG coefficient of G for the product (2.4) if for given ρ, $(\mu\nu\rho) > 1$, so that there are several orthogonal sets of basis functions $\psi^{\rho A}{}_l$. This is because we can make unitary transformations with respect to A for fixed ρ without disturbing the explicit reduction of the product (2.4) or the orthogonality of the sets $\psi^{\rho A}{}_l$ with different A. Fortunately, we do not have to dwell on this arbitrariness in our work.

We now proceed to obtain analogs of Eqs. (1.1)–(1.3) in terms of the quantities (2.16).

3. GENERALIZATION OF EQS. (1.1)–(1.3).

The generalization of (1.1) is immediate. As the matrix of CG coefficients is unitary, we have

$$(\mu j, \nu k \mid \rho A l)^* = (\rho A l \mid \mu j, \nu k), \tag{3.1}$$

so that (2.18) becomes

$$(\mu j, \nu k \mid \rho A l)(\mu j, \nu k \mid \sigma B m)^* = \delta(\rho\sigma)\delta(AB)\delta(lm), \tag{3.2}$$

which is the required generalization of (1.1).

To generalize (1.3), we need a lemma.

Lemma. The direct-product representation $D^\mu(R) \otimes D^\nu(R)$ contains the identity representation O only if $\nu = \mu'$, and then only once.

The corresponding normalized wavefunction is

$$(n_\mu)^{\frac{1}{2}}\psi^{(\rho=0)} = J(\mu, \mu')_{ik}\psi^\mu{}_i\psi^{\mu'}{}_k. \tag{3.3}$$

The first part of the lemma follows (2.9) and (2.8) on setting $\rho = 0$ and $\chi^0(R) = 1$. To verify the statement that the $\psi^{(0)}$ as given by (3.3) is the correct invariant basis function, we use Eqs. (2.1), (2.2), and (2.14) as well as the fact that $J(\mu, \mu')$ is unitary, in the following way:

$$\begin{aligned}\psi^{(0)} &= J(\mu, \mu')_{ik}\psi^\mu{}_i\psi^{\mu'}{}_k \\ &\underset{R}{\rightarrow} J(\mu, \mu')_{ik}D^\mu(R)_{mi}D^{\mu'}(R)_{lk}\psi^\mu{}_m\psi^{\mu'}{}_l \\ &= J(\mu', \mu)_{ki}D^\mu(R)_{mi}J(\mu', \mu)_{ln} \\ &\quad\times D^\mu(R)^*{}_{n\nu}J(\mu', \mu)^\dagger{}_{\nu k}\psi^\mu{}_m\psi^{\mu'}{}_l \\ &= D^\mu(R)_{m\nu}D^\mu(R)^*{}_{n\nu}J(\mu, \mu')_{nl}\psi^\mu{}_m\psi^{\mu'}{}_l \\ &= J(\mu, \mu')_{nl}\psi^\mu{}_n\psi^{\mu'}{}_l = \psi^{(0)}.\end{aligned}$$

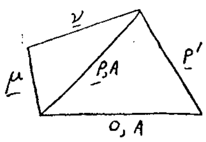

FIG. 1. The $(\mu\nu)\,\rho' \rightarrow \rho\,\rho' \rightarrow 0$ coupling.

[20] Ref. 16, p. 261.

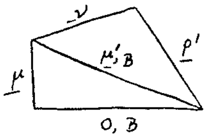

FIG. 2. The $\mu\,(\nu\rho') \rightarrow \mu\,\mu' \rightarrow 0$ coupling.

The normalization of $\psi^{(0)}$ follows from the unitarity of $J(\mu, \mu')$. Thus the proof of the lemma is complete.

Suppose now we have three sets of basis functions $\psi^\mu{}_i$, $\psi^\nu{}_k$ and $\psi^{\rho'}{}_l$ together with their associated matrices $D^\mu(R)$, $D^\nu(R)$, $D^{\rho'}(R)$. We seek combinations of the product functions $\psi^\mu{}_i\psi^\nu{}_k\psi^{\rho'}{}_l$ that are invariant under G. In general, a whole subspace of such combinations exists. We can build a basis for this subspace in the following manner. From the lemma it is clear that in an invariant linear combination of products, whatever multiplies the $\psi^{\rho'}{}_l$, must be a quantity of the type $\psi^\rho{}_m$. That is, we must first combine the products $\psi^\mu{}_i\psi^\nu{}_k$ to a wavefunction of type ρ, and then combine that with the $\psi^{\rho'}{}_l$ to get an invariant. Thus we arrive at a set of $(\mu\nu\rho)$ orthonormal invariant states labeled by a letter A,

$$(n_\rho)^{\frac{1}{2}}\psi^{0A} = J(\rho, \rho')_{ml}(\mu j, \nu k \mid \rho A m)\psi^\mu{}_i\psi^\nu{}_k\psi^{\rho'}{}_l. \tag{3.4}$$

These states form an orthonormal basis for the manifold of invariant states in the triple-product space. We may represent Eq. (3.4) schematically as in Fig. 1.

It is clear however that we must obtain the same manifold if we start by coupling the $\psi^{\rho'}{}_l$ and $\psi^\nu{}_k$ to form states of type $\psi^{\mu' B}{}_m$, and then combine these with $\psi^\mu{}_i$ to form invariant states ϕ^{0B}. In this way, we get

$$(n_\mu)^{\frac{1}{2}}\phi^{0B} = J(\mu, \mu')_{im}\psi^\mu{}_i(\rho' l, \nu k \mid \mu' B m)\psi^{\rho'}{}_l\psi^\nu{}_k. \tag{3.5}$$

This may be represented as in Fig. 2. Since ψ^{0A} and ϕ^{0B} span the same subspace of the triple-product space, there is a unitary transformation connecting them, so that we have

$$\psi^{0A} = \textstyle\sum_B M(\mu\nu\rho)_{AB}\phi^{0B}. \tag{3.6}$$

We now insert (3.4) and (3.5) into (3.6) and obtain, after dropping the linearly independent product functions, the relationship[21]

$$\begin{aligned}&(n_\rho)^{-\frac{1}{2}}J(\rho, \rho')_{ml}(\mu j, \nu k \mid \rho A m) \\ &= \textstyle\sum_B M(\mu\nu\rho)_{AB}(n_\mu)^{-\frac{1}{2}}J(\mu', \mu)_{nj}(\rho' l, \nu k \mid \mu' B n).\end{aligned} \tag{3.7}$$

In the absence of multiplicities, i.e., when $(\mu\nu\rho) =$

[21] Following Hamermesh, (Ref. 16, Sec. 7–14), we may show that the arbitrariness in the definition of CG coefficients of G may be disposed of in such a way that $M(\mu\nu\rho) = 1$. We can however proceed without effecting this.

$(\rho'\nu\mu') = 1$, Eq. (3.7) reduces to

$$(n_\rho)^{-\frac{1}{2}}J(\rho,\rho')_{ml}(\mu j,\nu k \mid \rho m) = (n_\mu)^{-\frac{1}{2}}J(\mu',\mu)_{nj}(\rho' l,\nu k \mid \mu' n). \quad (3.8)$$

Equations (3.8) and (3.7)[21] exhibit how a crossing relation replaces the simple symmetry property (1.3) when IR's are no longer all equivalent to their complex conjugates and when $(\mu\nu\rho) > 1$ is allowed.

It is now a simple matter to derive from (3.7) the required generalization of Eq. (1.2). To do this, we first obtain from Eq. (3.7)

$$(n_\rho)^{-\frac{1}{2}}J(\rho,\rho')^{\dagger}_{lq}(\lambda p,\nu k \mid \rho Cq)^* = \sum_D M(\lambda\nu\rho)^{\dagger}_{DC}(n_\lambda)^{-\frac{1}{2}}J(\lambda',\lambda)^{\dagger}_{ps}(\rho' l,\nu k \mid \lambda' Ds)^* \quad (3.9)$$

by complex conjugation and relabeling. Then we multiply corresponding sides of Eqs. (3.7) and (3.9) and sum over k and l. First, we note that the unitarity of $J(\rho,\rho')$ simplifies the left side to

$$(n_\rho)^{-1}(\mu j,\nu k \mid \rho Am)(\lambda p,\nu k \mid \rho Cm)^*;$$

summation over k and m implied here as always. On the right side, we first use (3.2) to obtain a factor

$$\delta(\mu\lambda)\,\delta(BD)\,\delta(ns).$$

Now the M's on the right refer to the same triples of IR's, and the J's to the same pair of complex conjugates of IR's, so that their unitarity reduces the right side to

$$(n_\lambda)^{-1}\delta(\mu\lambda)\,\delta(jp)\,\delta(AC).$$

We thus obtain

$$(\mu j,\nu k \mid \rho Am)(\lambda p,\nu k \mid \rho Cm)^* = (n_\rho/n_\mu)\,\delta(\mu\lambda)\,\delta(jp)\,\delta(AC) \quad (3.10)$$

as the required generalization of (1.2).

4. ILLUSTRATIVE EXAMPLE

Justification of the application of Shmushkevich's method to SU_3 is here provided in a simple case on the basis of the work of the previous sections.

We use the notation (μ_1,μ_2) for an IR of SU_3, the integers μ_1 and μ_2 being the components of its highest weight. Basis states for any IR $\mu = (\mu_1,\mu_2)$ are obtained as simultaneous eigenstates of operators which may be identified with $\mathbf{I}^2$, total isospin; I_z its z component; and Y, hypercharge. Thus in place of $\psi^\mu{}_j$ we have $|\mu_1\mu_2; II_zY\rangle$. It is customary[4,12] to associate sets of particles of the same spins and parities, "approximately" the same masses, and appropriate I and Y with such IR's of SU_3 in order to set up a unitary symmetry theory. For the purpose of illustration let us consider all the allowed decays of a particle belonging to the IR $\rho = (\rho_1,\rho_2)$ into two particles belonging to μ and ν. Such decays have matrix elements

$$\langle \mu I_1I_{1z}Y_1; \nu I_2I_{2z}Y_2 |\, T\, | \rho II_zY\rangle = \sum_A (\mu I_1I_{1z}Y_1, \nu I_2I_{2z}Y_2 \mid \rho AII_zY)\langle \rho A \| T \| \rho\rangle. \quad (4.1)$$

In order to derive the results which correspond to the Shmushkevich tables used in Refs. 6 and 8, we must consider the following sums over the squared moduli of matrix elements (4.1). These are $Q(II_zY)$, the sum over I_1, I_{1z}, Y_1, and I_2, I_{2z}, Y_2 at fixed I, I_z, Y; and $R(I_1, I_{1z}, Y_1)$, the sum over I_2, I_{2z}, Y_2 and I, I_z, Y at fixed I_1, I_{1z}, Y_1.

We get

$$\begin{aligned} Q(II_zY) &= \sum\sum_A (\mu I_1I_{1z}Y_1, \nu I_2I_{2z}Y_2 \mid \rho AII_zY)\langle\rho A\| T \|\rho\rangle \\ &\quad\times \sum_B (\mu I_1I_{1z}Y_1, \nu I_2I_{2z}Y_2 \mid \rho BII_zY)^*\langle\rho B\| T\|\rho\rangle^* \\ &= \sum_{AB} \delta(AB)\langle\rho A\| T\|\rho\rangle\langle\rho B\| T\|\rho\rangle^* \\ &= \sum_A |\langle\rho A\| T\|\rho\rangle|^2. \end{aligned} \quad (4.2)$$

The sum set first in the first line is by definition over I_1, I_{1z}, Y_1, I_2, I_{2z}, Y_2 at fixed I, I_z, Y—just that required to allow the use of Eq. (3.2). The result (4.2)—$Q(II_zY)$ independent of I, I_z and Y—states the equality of the total widths for all decays for different members of the unitary multiplet ρ.

Likewise, we get

$$R(I_1, I_{1z}, Y_1) = (n_\rho/n_\mu)\sum_A |\langle\rho A\| T\|\rho\rangle|^2, \quad (4.3)$$

the summation involved in the derivation being, by definition, just that required to allow the use of Eq. (3.10). Thus $R(I_1, I_{1z}, Y_1)$ is independent of I_1, I_{1z}, and Y_1, which is just what has been called[6] a Shmushkevich theorem for the decay situation.

No further information than is provided by Eqs. (3.2), (3.10) is required for the writing down of Shmushkevich theorems for more complex situations.