Reprinted from

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 3 MARCH 1968

Structure of the Dirac Bracket in Classical Mechanics*

N. MUKUNDA† AND E. C. G. SUDARSHAN
Department of Physics, Syracuse University, Syracuse, New York

(Received 27 April 1967)

We discuss the structure of the Dirac bracket in classical mechanics. We consider a generalization of the usual Poisson bracket and show the close connection of this generalization to the Lagrange brackets of classical mechanics. We show how the Dirac bracket appears as a particular case of the generalized Poisson bracket, thus giving a simple reason why the Jacobi identity holds for the Dirac bracket. We also discuss the nature of the transformations generated via the Dirac bracket and the relation of these to canonical transformations.

INTRODUCTION

SEVERAL years ago, Dirac developed a canonical formalism for the Hamiltonian formulation of classical mechanical systems which are subject to constraints.[1] The usual Hamiltonian formulation of classical mechanics rests on the equivalence of the Lagrangian and the Hamiltonian equations of motion; and the passage from the Lagrangian variables of generalized position and velocity, q and $\dot{q}$, to the Hamiltonian variables of generalized position and momentum, q and p, is possible when and only when the velocities can be expressed in terms of the positions and the momenta. This requirement can be expressed in two equivalent ways: either (i) the Lagrangian equations of motion should specify *all* the accelerations as functions of positions and velocities; or (ii) the definitions of the momenta should not lead to any identities among the positions and momenta alone. The Dirac theory of constraints was intended to handle precisely those systems that do not fulfill this requirement, namely systems whose position and momentum variables obey certain identities and are therefore not independent. These identities are the constraints referred to earlier. In such cases the lack of complete specification of the accelerations by the Lagrangian equations of motion manifests itself also in ambiguities in the passage to an equivalent Hamiltonian formulation.

As a prelude to the quantization of such systems, Dirac proposed that the usual Poisson[2] brackets of classical mechanics be replaced by a new algebraic structure, now known as the Dirac bracket,[1] and that these new brackets be made to correspond to commutators in quantum theory. If $f(q, p)$ and $g(q, p)$ are two functions defined on a $2N$-dimensional phase space with coordinate variables $q_1 \cdots q_N$, $p_1 \cdots p_N$, then the Dirac bracket of f with g, $\{f, g\}^\star$ is defined by

$$\{f, g\}^\star = \{f, g\} - \{f, \theta^a\}C_{ab}\{\theta^b, g\}.$$

Here, the curly brackets without stars are ordinary Poisson brackets. The functions $\theta^a(q, p)$ are a certain subset of all those functions whose vanishing expresses the constraints. They have the important property that, if we form a matrix whose elements are the Poisson brackets of the θ^a with one another, then this matrix is nonsingular. (It follows that we have an even number of θ's.) The functions $C_{ab}(q, p)$ form the matrix inverse to the matrix of Poisson brackets:

$$C_{ab}\{\theta^b, \theta^c\} = \delta_a^c.$$

A summation over repeated indices is assumed in the equations above.

For systems involving constraints, the Hamiltonian equations of motion can be expressed in terms of Dirac brackets, in the same way in which the equations of motion of systems without constraints are expressible in terms of Poisson brackets. Before the Dirac bracket can be introduced, however, the set of all constraints has to be separated into two classes, known as first class and second class constraints. The functions θ^a are the second class constraints, and this class is characterized precisely by the existence of the matrix C_{ab}. The Dirac brackets share many of the standard properties of Poisson brackets, namely linearity, antisymmetry, and the Jacobi identity.[3] The main difference lies in the fact that with respect to them, the functions θ^a behave essentially like pure numbers. In other words, the Dirac bracket of θ^a with

* Work supported in part by the U.S. Atomic Energy Commission.

† Present address: Tata Institute of Fundamental Research, Bombay, India.

[1] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); "Lectures on Quantum Mechanics," Belfer Graduate School of Science Monograph Series No. 2 (Yeshiva University, New York, 1964).

[2] S. D. Poisson, J. de l'Ecole Polytech. **8**, 266 (1809).

[3] C. G. J. Jacobi, Compt. Rend. **11**, 529 (1841); *Vorlesungen über Dynamik*, A. Clebsch, Ed. (Reiner, Berlin, 1866, 2nd ed., 1884).

any other function is identically zero. It is only the set of second class constraints that can be eliminated in this way by the use of the Dirac bracket.

In this paper, we would like to study and clarify in an algebraic way the structure and properties of the Dirac bracket, by relating it to the other two algebraic structures of classical mechanics, namely Poisson and Lagrange brackets. We will also study the relationship between the transformations generated by the Dirac brackets on the one hand, and those generated by Poisson brackets on the other. The latter are, of course, the canonical transformations of classical mechanics. The motivation for this study is the following. The original proof of the Jacobi identity for the Dirac bracket consisted of a straightforward but rather lengthy *verification* of the identity,[4] without shedding much light on the structure of the bracket or suggesting any simple reason for suspecting that the identity might hold. Subsequently, it has been shown[5] by Bergmann and Goldberg that one can start from a certain continuous group of coordinate transformations in phase space having special properties with respect to the constraints; one then finds that the infinitesimal Lie brackets corresponding to this group are, in fact, Dirac brackets. The associativity of the group multiplication law then automatically guarantees the Jacobi identity for the Dirac bracket. Our interest is in exhibiting in a direct and algebraic way the reason why the Dirac bracket looks the way it does and the reason why it obeys the Jacobi identity, and after that examine the group of coordinate transformations generated by it.

In Sec. 1, we briefly review the properties of Poisson and Lagrange brackets and of canonical transformations in phase space. This material is completely standard and is included only for the sake of completeness. Section 2 consists of a straightforward extension of Poisson brackets to what we will call a generalized Poisson bracket. These brackets can be related in a direct way to Lagrange brackets. In Sec. 3, we show how the Dirac brackets arise as a special case of the generalized Poisson brackets. Finally, Sec. 4 contains a discussion of the coordinate transformations generated by the Dirac brackets and of the relation of these transformations to canonical transformations. In this paper, we will not be interested in any particular Lagrangians or Hamiltonians, and we will not need to make statements which are valid only when the constraint functions vanish.

1. POISSON AND LAGRANGE BRACKET: CANONICAL TRANSFORMATIONS[6]

In the $2N$-dimensional phase space of a classical mechanical system with canonical variables $q_1 \cdots q_N$, $p_1 \cdots p_N$ we define the Poisson bracket (PB) of any two functions $f(q, p)$ and $g(q, p)$ to be a third function given by

$$\{f, g\}(q, p) = \sum_{k=1}^{N} \left(\frac{\partial f}{\partial q_k} \cdot \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \cdot \frac{\partial g}{\partial q_k}\right). \quad (1.1)$$

Introducing the variables

$$\omega^\mu = \sum_{k=1}^{N} (\delta_{\mu k} q_k + \delta_{\mu, k+N} p_k) \quad (1.2)$$

and the constant matrix

$$\epsilon^{\mu\nu} = \delta_{\mu, \nu+N} - \delta_{\mu+N, \nu}, \quad (1.3)$$

we could rewrite Eq. (1.1), defining the PB in tensor notation[7]

$$\{f, g\}(\omega) = \epsilon^{\mu\nu} \frac{\partial f(\omega)}{\partial \omega^\mu} \cdot \frac{\partial g(\omega)}{\partial \omega^\nu}. \quad (1.4)$$

In either form [(1.1) or (1.4)], the PB satisfies the Jacobi identity

$$\{\{h, f\}, g\} + \{\{f, g\}, h\} + \{\{g, h\}, f\} = 0 \quad (1.5)$$

for any three functions f, g, h. Using the form (1.4), the only property of $\epsilon^{\mu\nu}$ used is its antisymmetry.

Canonical transformations can be characterized in the following way. The PB's of the basic variables ω^μ with one another have the standard values

$$\{\omega^\mu, \omega^\nu\} = \epsilon^{\mu\nu}.$$

Using the definiton (1.4), we see that the PB preserving property of canonical transformations can be transcribed as follows:

$$\frac{\partial \omega'^\mu}{\partial \omega^\sigma} \cdot \frac{\partial \omega'^\nu}{\partial \omega^\rho} \epsilon^{\sigma\rho} = \epsilon^{\mu\nu} \quad (1.6)$$

Thus canonical transformations are those transformations with respect to which $\epsilon^{\mu\nu}$ behaves as an invariant second rank antisymmetric tensor of contravariant type.

The covariant tensor $\epsilon_{\mu\nu}$ is defined as the inverse matrix to $\epsilon^{\mu\nu}$ and has the elements:

$$\epsilon_{\mu\nu} = -\delta_{\mu, \nu+N} + \delta_{\mu+N, \nu}. \quad (1.7)$$

Given any set $\phi^\alpha(\omega)$ of $2N$-independent functions we could express the ω^μ as functions of ϕ^α. Then we

[4] Compare the remarks by Dirac, "Lectures on Quantum Mechanics," Belfer Graduate School of Science Monograph Series No. 2 (Yeshiva University, New York, 1964), p. 42.

[5] P. G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955).

[6] See, for example, any of the standard texts, such as: H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1965); H. C. Corben and P. Stehle, *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1960).

[7] For an introduction to the tensor notation, see C. W. Kilmister, *Hamiltonian Dynamics* (John Wiley & Sons, Inc., New York, 1964).

define the Lagrange bracket (LB)[8] of ϕ^α and ϕ_β according to

$$L_{\alpha\beta}(\phi) = \epsilon_{\mu\nu} \frac{\partial\omega^\mu}{\partial\phi^\alpha} \cdot \frac{\partial\omega^\nu}{\partial\phi^\beta}. \tag{1.8}$$

It is then well known that

$$L_{\alpha\sigma}(\phi)\{\phi^\sigma, \phi^\beta\} = \delta_\alpha^\beta. \tag{1.9}$$

More relevant is the identity

$$\frac{\partial}{\partial\phi^\alpha} L_{\beta\gamma}(\phi) + \frac{\partial}{\partial\phi^\beta} L_{\gamma\alpha}(\phi) + \frac{\partial}{\partial\phi^\gamma} L_{\alpha\beta}(\phi) = 0 \tag{1.10}$$

which is the Lagrangian analogue of the Jacobi identity (1.5). It is seen that the left-hand side of (1.10) is totally antisymmetric in α, β, γ and that the identity holds as a consequence of the antisymmetry of $\epsilon_{\mu\nu}$ in its indices.

2. GENERALIZED POISSON BRACKETS

In this section we consider a generalization of the PB as given in (1.4).[9] Let there be given a set of functions $\eta^{\mu\nu}(\omega)$, antisymmetric in μ and ν, and obeying the identity (2.12) which will be derived later on. Define the generalized Poisson bracket (GPB) of any two functions $f(\omega)$, $g(\omega)$ to be a third function $h(\omega)$ given by

$$\{f, g\}^\star(\omega) \equiv h(\omega) = \eta^{\mu\nu}(\omega) \frac{\partial f(\omega)}{\partial\omega^\mu} \cdot \frac{\partial g(\omega)}{\partial\omega^\nu}. \tag{2.1}$$

We first consider the behavior of $\eta^{\mu\nu}(\omega)$ under coordinate transformations. Let $\omega^\mu \to \omega'^\mu$ be a general coordinate transformation, the ω'^μ being independent functions of the ω^μ. Given any function $f(\omega)$, we can define a new function f' by the equation

$$f'(\omega') = f(\omega). \tag{2.2a}$$

This is the transformation law characteristic of a "scalar field." By means of it we are led from a function f with a certain functional form to a function f' with a (generally) different functional form. The behavior of $\eta^{\mu\nu}(\omega)$ under a general transformation of coordinates is fixed by requiring that the GPB of two scalar fields be itself a scalar field. Thus, if in the variables ω^μ we have

$$\{f, g\}^\star(\omega) = h(\omega) \tag{2.1}$$

and $f(\omega)$, $g(\omega)$ go over into functions f' and g' according to (2.2a), then we also define $h(\omega)$ to go over into $h'(\omega')$ according to (2.2b):

$$h'(\omega') = h(\omega). \tag{2.2b}$$

Equation (2.1) expresses h in terms of f and g. Similarly, we can express h' in terms of f' and g':

$$\begin{aligned} h'(\omega') &= \eta^{\mu\nu}(\omega) \frac{\partial f(\omega)}{\partial\omega^\mu} \cdot \frac{\partial g(\omega)}{\partial\omega^\nu} \\ &= \eta^{\mu\nu}(\omega) \frac{\partial f'(\omega')}{\partial\omega'^\rho} \cdot \frac{\partial\omega'^\rho}{\partial\omega^\mu} \cdot \frac{\partial g'(\omega')}{\partial\omega'^\sigma} \frac{\partial\omega'^\sigma}{\partial\omega^\nu} \\ &= \eta'^{\rho\sigma}(\omega') \frac{\partial f'(\omega')}{\partial\omega'^\rho} \cdot \frac{\partial g'(\omega')}{\partial\omega'^\sigma}, \end{aligned} \tag{2.3}$$

where

$$\eta'^{\rho\sigma}(\omega') = \frac{\partial\omega'^\rho}{\partial\omega^\mu} \cdot \frac{\partial\omega'^\sigma}{\partial\omega^\nu} \eta^{\mu\nu}(\omega). \tag{2.4}$$

Thus $\eta^{\mu\nu}(\omega)$ transforms as a second-rank antisymmetric tensor of contravariant type.

We may state the content of (2.4) in the following form. *The* GPB *is an operation whereby, given two scalar fields f and g, a third one h is determined.* In each coordinate system, a given scalar field is represented by a specific function of those coordinates. The explicit expression of the function representing h in terms of those representing f and g, depends on the particular coordinate system. Equation (2.4) shows how this explicit expression changes when one goes from one set of coordinates to another.

Next we define the analogues of canonical transformations. For this purpose, we focus attention on the change in functional form, $f \to f'$, produced by a change of coordinates, $\omega \to \omega'$, when f' is defined in terms of f by (2.2a). Given two functions $f(\omega)$, $g(\omega)$, we consider the functions $f'(\omega)$, $g'(\omega)$ which are obtained by using the functional forms f', g' but with arguments ω instead of ω'. For an arbitrary change of coordinates, the GPB of $f(\omega)$ with $g(\omega)$

$$\{f, g\}^\star(\omega) \equiv h(\omega) = \eta^{\mu\nu}(\omega) \frac{\partial f(\omega)}{\partial\omega^\mu} \cdot \frac{\partial g(\omega)}{\partial\omega^\nu} \tag{2.5}$$

and that of $f'(\omega)$ with $g'(\omega)$

$$\{f', g'\}^\star(\omega) = k(\omega) = \eta^{\mu\nu}(\omega) \frac{\partial f'(\omega)}{\partial\omega^\mu} \frac{\partial g'(\omega)}{\partial\omega^\nu} \tag{2.6}$$

will not bear any special relationship to one another. However, if we demand that the transformation be such that

$$k(\omega) = h'(\omega), \tag{2.7}$$

[8] J. L. Lagrange, Memoires de l'Institut de France, (1808); reprinted in Oeuvres, Vol. VI, p. 713.

[9] See, for example, C. W. Kilmister, Ref. 7, Chap. 4.

where

$$h'(\omega') = h(\omega), \tag{2.8}$$

then we have the following consequence:

$$\{f', g'\}^\star(\omega) = (\{f, g\}^\star)'(\omega). \tag{2.9}$$

This is the statement that the operation of taking the GPB commutes with the operation of changing the functional form of a function according to the prescription given above. The requirement (2.7) is equivalent to the following one:

$$\eta^{\rho\sigma}(\omega') = \frac{\partial \omega'^\rho}{\partial \omega^\mu}\frac{\partial \omega'^\sigma}{\partial \omega^\nu}\eta^{\mu\nu}(\omega). \tag{2.10}$$

Notice that the same functions $\eta^{\mu\nu}$, but with different arguments, appear on the two sides of (2.10). Transformations $\omega \rightarrow \omega'$, which obey (2.10), may be called canonical with respect to the GPB defined by $\eta^{\mu\nu}(\omega)$. In the particular case when $\eta^{\mu\nu}$ happens to be $\epsilon^{\mu\nu}$, (2.10) coincides with (1.9), and we obtain the usual canonical transformations of classical mechanics.

We conclude this section with a discussion of the Jacobi identity.[10] We will demand that $\eta^{\mu\nu}(\omega)$ be such that for any three functions f, g, h, we have

$$\{\{f, g\}^\star, h\}^\star + \{\{g, h\}^\star, f\}^\star + \{\{h, f\}^\star, g\}^\star = 0. \tag{2.11}$$

If (2.1) is substituted in (2.11), two kinds of terms appear, those without derivatives of $\eta^{\mu\nu}$ and those with derivatives. The former vanish by themselves, due to the antisymmetry of $\eta^{\mu\nu}$. The vanishing of the latter leads to

$$\eta^{\lambda\mu}(\omega)\frac{\partial \eta^{\nu\rho}(\omega)}{\partial \omega^\mu} + \eta^{\nu\mu}(\omega)\frac{\partial \eta^{\rho\lambda}(\omega)}{\partial \omega^\mu} + \eta^{\rho\mu}(\omega)\frac{\partial \eta^{\lambda\nu}(\omega)}{\partial \omega^\mu} = 0. \tag{2.12}$$

Thus, the Jacobi identity for the GPB *is equivalent to* (2.12). We will assume that the GPB is nondegenerate in the sense that the functions $\eta^{\mu\nu}(\omega)$ form a nonsingular matrix, and we denote the matrix elements of the inverse matrix by $\eta_{\mu\nu}(\omega)$:

$$\eta_{\mu\nu}(\omega)\eta^{\nu\lambda}(\omega) = \delta_\mu^\lambda. \tag{2.13}$$

Then (2.12) can be written much more simply in terms of $\eta_{\mu\nu}(\omega)$:

$$\frac{\partial \eta_{\mu\nu}(\omega)}{\partial \omega^\lambda} + \frac{\partial \eta_{\nu\lambda}(\omega)}{\partial \omega^\mu} + \frac{\partial \eta_{\lambda\mu}(\omega)}{\partial \omega^\nu} = 0. \tag{2.14}$$

Notice the resemblance between (2.14) and (1.10). The only difference is that in place of the ω^μ variables in (2.14), we have the ϕ^α variables in (1.10), while $\eta_{\mu\nu}(\omega)$ is replaced by the LB $L_{\alpha\beta}(\phi)$. *This shows the close connection between the Jacobi identity for a* GPB, *on the one hand, and a standard property of* LB's *on the other*. We will make use of this connection in the next section to derive the Dirac bracket.

3. THE DIRAC BRACKET

Let there be given a set of $2(N - \gamma)$ functions

$$\theta^a(\omega), \quad a = 1, 2, \cdots, 2(N - \gamma), \tag{3.1}$$

which are independent of one another. Let us choose 2γ additional functions

$$\psi^m(\omega), \quad m = 1, 2, \cdots, 2\gamma, \tag{3.2}$$

so that the θ^a and ψ^m together form $2N$ independent functions. We can form two matrices, one made up of the PB's of the θ's and ψ's with each other, the other made up of their LB's. According to (1.9), these matrices are inverse to one another. This may be expressed as follows:

$$\{\theta^a, \theta^b\}L_{bc}(\theta, \psi) + \{\theta^a, \psi^m\}L_{mc}(\theta, \psi) = \delta_c^a, \tag{3.3a}$$

$$\{\theta^a, \theta^b\}L_{bn}(\theta, \psi) + \{\theta^a, \psi^m\}L_{mn}(\theta, \psi) = 0, \tag{3.3b}$$

$$\{\psi^m, \theta^a\}L_{ab}(\theta, \psi) + \{\psi^m, \psi^n\}L_{nb}(\theta, \psi) = 0, \tag{3.3c}$$

$$\{\psi^m, \theta^a\}L_{an}(\theta, \psi) + \{\psi^m, \psi^p\}L_{pn}(\theta, \psi) = \delta_n^m. \tag{3.3d}$$

A similar set of equations can be written down, corresponding to taking the matrix of LB's first, and that of PB's next. Here, the L's stand for the LB's:

$$L_{ab}(\theta, \psi) = \epsilon_{\mu\nu}\frac{\partial \omega^\mu}{\partial \theta^a}\cdot\frac{\partial \omega^\nu}{\partial \theta^b},$$

$$L_{am}(\theta, \psi) = -L_{ma}(\theta, \psi) = \epsilon_{\mu\nu}\frac{\partial \omega^\mu}{\partial \theta^a}\cdot\frac{\partial \omega^\nu}{\partial \psi^m}, \tag{3.4}$$

$$L_{mn}(\theta, \psi) = \epsilon_{\mu\nu}\frac{\partial \omega^\mu}{\partial \psi^m}\cdot\frac{\partial \omega^\nu}{\partial \psi^n}.$$

The LB's by themselves obey the identities (1.10). A subset of these involves differentiation with respect to ψ^m alone. These are

$$\frac{\partial L_{mn}(\theta, \psi)}{\partial \psi^l} + \frac{\partial L_{nl}(\theta, \psi)}{\partial \psi^m} + \frac{\partial L_{lm}(\theta, \psi)}{\partial \psi^n} = 0. \tag{3.5}$$

The considerations of the previous section show that if we set

$$\eta_{mn}(\theta, \psi) \equiv L_{mn}(\theta, \psi), \tag{3.6}$$

---

[10] J. M. Souriau, Commun. Math. Phys. 1, 374 (1966).

if $\eta_{mn}(\theta, \psi)$ possesses an inverse $\eta^{mn}(\theta, \psi)$

$$\eta_{mn}(\theta, \psi)\eta^{nl}(\theta, \psi) = \delta^l_m, \tag{3.7}$$

and if we define a bracket by

$$\{f, g\}^\star(\theta, \psi) \equiv \eta^{mn}(\theta, \psi)\frac{\partial f}{\partial \psi^m}\frac{\partial g}{\partial \psi^n}, \tag{3.8}$$

then this will obey the Jacobi identity and therefore be a GPB in 2γ variables. In this definition, we have used the fact that any function of ω^μ can be written as a function of θ^a, ψ^m. Partial differentiation with respect to a ψ^m is carried out keeping the θ^a and the other ψ's constant. Clearly,

$$\{f, \theta^a\}^\star = 0. \tag{3.9}$$

We now show that the bracket (3.8) *is just the Dirac bracket* (DB) *of f with g*. For this, we must first find the matrix η^{mn} inverse to η_{mn}. Let us assume that the submatrix of PB's of the θ^a with one another possesses an inverse:

$$C_{ab}(\theta, \psi)\cdot\{\theta^b, \theta^c\} = \delta^c_a. \tag{3.10}$$

From (3.3b), we then find

$$L_{bn}(\theta, \psi) = -C_{ba}(\theta, \psi)\{\theta^a, \psi^l\}\eta_{ln}(\theta, \psi). \tag{3.11}$$

Substituting this in (3.3d) gives

$$[\{\psi^m, \psi^l\} - \{\psi^m, \theta^b\}C_{ba}(\theta, \psi)\{\theta^a, \psi^l\}]\eta_{ln}(\theta, \psi) = \delta^m_n. \tag{3.12}$$

This shows that η^{mn} exists, and is given by

$$\eta^{mn}(\theta, \psi) = \{\psi^m, \psi^n\} - \{\psi^m, \theta^a\}C_{ab}(\theta, \psi)\{\theta^b, \psi^n\}. \tag{3.13}$$

Conversely, it may be easily shown that if η^{mn} exists, then so does C_{ab}.

We now use (3.13) in (3.8) to find

$$\{f, g\}^\star = \left[\frac{\partial f}{\partial \psi^m}\{\psi^m, \psi^n\} - \frac{\partial f}{\partial \psi^m}\{\psi^m, \theta^a\}C_{ab}\{\theta^b, \psi^n\}\right]\cdot\frac{\partial g}{\partial \psi^n}. \tag{3.14}$$

It is easy to see that by the addition of terms which in fact vanish, we can rewrite this in the form

$$\{f, g\}^\star = \{f, \psi^n\}\frac{\partial g}{\partial \psi^n} - \{f, \theta^a\}C_{ab}\{\theta^b, \psi^n\}\frac{\partial g}{\partial \psi^n}. \tag{3.15}$$

Once again, by the addition of vanishing terms, we write (3.15) in the final form

$$\{f, g\}^\star = \{f, g\} - \{f, \theta^a\}C_{ab}\{\theta^b, g\}. \tag{3.16}$$

We see at once that this GPB in 2γ variables is just the DB written down in the Introduction. We also see that it is an expression determined solely by the functions θ^a, and does not depend on the choice of the functions ψ^m which were originally used in (3.8) to define it.

4. DIRAC BRACKET TRANSFORMATIONS

In this last section, we consider the transformations generated via the DB,[11] or more generally, the coordinate transformations which are canonical with respect to the DB.

We first take up the question of the nature of these transformations *per se*. Written in the form (3.8), we see that the DB is a nondegenerate GPB in 2γ variables. For the moment, we ignore the presence of the variables θ^a. Now it is a well-known fact in the mathematical literature that all such symplectic structures in a given number of variables are locally isomorphic.[9] In other words, *by proper choice of the variables* ψ^m, *the coefficients in* (3.8) *can be made constants, so that the matrix* $\|\eta^{mn}\|$ *has exactly the same structure as the matrix* $\|\epsilon^{\mu\nu}\|$ *in* (1.3). Nevertheless, it may be useful to give here a simple proof of this statement, at least for the sake of completeness.

For this purpose, consider (3.5):

$$\frac{\partial \eta_{mn}}{\partial \psi^l} + \frac{\partial \eta_{nl}}{\partial \psi^m} + \frac{\partial \eta_{lm}}{\partial \psi^n} = 0. \tag{3.5}$$

This equation is analogous to one set of the Maxwell equations of electrodynamics, and exactly as in that case, one can show that η_{mn} may be expressed as a "curl" of a "vector"[12]:

$$\eta_{mn}(\psi) = \frac{\partial A_m(\psi)}{\partial \psi^n} - \frac{\partial A_n(\psi)}{\partial \psi^m}. \tag{4.1}$$

We have seen earlier that η^{mn} transforms as a contravariant tensor of the second rank, under an arbitrary change of coordinates $\psi \to \psi'$. Then, η_{mn} transforms as a covariant tensor, while $A_m(\psi)$ is a covariant vector field. The differentials of the coordinates,

[11] Given a GPB $\{f, g\}\star$, the transformation generated by a function $\phi(\omega)$ via this GPB is the transformation

$$f(\omega) \to f'(\omega) \equiv [(\exp\tilde{\phi})f](\omega) = f(\omega) + \{\phi, f\}\star(\omega) + (1/2!)\{\phi, \{\phi, f\}\star\}\star, (\omega) + \cdots.$$

Because of the Jacobi identity obeyed by the GPB, this transformation may be shown to be canonical with respect to this GPB.

[12] See, for instance, J. L. Synge, *Relativity: The Special Theory* (North-Holland Publishing Company, Amsterdam 1965), p. 344.

$d\psi^m$, form a contravariant vector, so that the expression

$$A_m(\psi)\, d\psi^m \tag{4.2}$$

is invariant. From the theory of the Pfaff problem,[13] it is known that one can always find a coordinate system, with coordinates ξ^m, in which (4.2) assumes the form

$$\xi^1\, d\xi^2 + \xi^3\, d\xi^4 + \cdots + \xi^{2r-1}\, d\xi^{2r}. \tag{4.3}$$

In this coordinate system, the components $A'_m(\xi)$ of the covariant vector field are linear in the coordinates, so that the $\eta'_{mn}(\xi)$ are constants. The same is then true of the η'^{mn} and without loss of generality we may assume that the matrix $\|\eta'^{mn}\|$ has the same structure as $\|\epsilon^{\mu\nu}\|$ in (1.3). Since the GPB is assumed to be nondegenerate, we conclude that $r = \gamma$.

Thus we can find functions $\xi^1, \cdots, \xi^{2r}$ of the ψ^m such that

$$\{f, g\}^\star(\psi) \equiv \eta^{mn}(\psi)\frac{\partial f}{\partial \psi^m}\frac{\partial g}{\partial \psi^n} = \epsilon^{mn}\frac{\partial f}{\partial \xi^m}\frac{\partial g}{\partial \xi^n}. \tag{4.4}$$

Written in this form, we immediately see that the coordinate transformations which are canonical with respect to the DB are none other than the usual canonical transformation *in the variables* ξ^m. (We are ignoring, of course, arbitrary transformations that may be performed among the variables θ^a by themselves.) The group of transformations generated via the DB (3.16) is isomorphic to the group of canonical transformations in $2r$ variables generated via the PB in $2r$ variables.[11] (Under these transformations, the variables θ^a do not change.)

We consider next the relation of these transformations to the usual canonical transformations in the basic variables ω^μ. We started with functions of the variables ω^μ, and given the set of functions $\theta^a(\omega)$, we defined the DB (3.16). We can then ask whether or not and under what circumstances transformations canonical with respect to the DB also belong to the group of usual canonical transformations on the ω^μ. We shall answer this question by first looking at a simple example.

Let us take the case where the $2(N-r)$ functions $\theta^a(\omega)$ are just a subset of the canonical q and p variables:

$$\theta^a(\omega) = q_{r+1}, \cdots, q_N; p_{r+1}, \cdots, p_N. \tag{4.5}$$

[13] E. Goursat, *Lecons sur le Probleme de Pfaff* (Hermann & Cie., Paris, 1922), p. 14.

Then the DB assumes the form

$$\{f, g\}^\star(q, p) = \sum_{k=1}^{r}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right). \tag{4.6}$$

In this case, the DB has the effect of "freezing" $(N-r)$ canonical pairs of degrees of freedom. If we now consider a transformation generated[11] via the DB by means of a function ϕ, depending only on $q_1, \cdots, q_r, p_1, \cdots, p_r$:

$$\begin{aligned} f(q,p) \to f'(q,p) &= f(q,p) + \{\phi, f\}^\star(q,p) \\ &\quad + \frac{1}{2!}\{\phi, \{\phi, f\}^\star\}^\star(q,p) + \cdots \\ &= ([\exp \tilde{\phi}]f)(q,p), \end{aligned} \tag{4.7}$$

such a transformation is also a canonical one in terms of the basic variables ω^μ, in which $q_1 \cdots q_r, p_1 \cdots p_r$ transform among themselves, while $q_{r+1} \cdots q_N$, $p_{r+1} \cdots p_N$ remain unchanged. [It should be noted that if the function ϕ used in (4.7) also depends on the "frozen" variables $q_{r+1} \cdots q_N$, $p_{r+1} \cdots p_N$, the resulting transformation is generally not canonical in the basic variables ω^μ.]

Returning to the general case of arbitrary functions θ^a, one checks easily that the DB is unaltered if in place of θ^a, one uses a set of independent functions of them, θ'^a. Thus in some cases, it may be possible to replace θ^a by θ'^a in such a way that the θ'^a are in fact a subset of $2(N-r)$ canonical q, p variables. Precisely when this can be done is shown by the following:

Theorem: The necessary and sufficient condition, that the DB determined by the $2(N-r)$ functions $\theta^a(\omega)$ corresponds to "freezing" $(N-r)$ pairs of canonical variables in the ordinary PB is that the functions θ^a form a function group of rank $2(N-r)$.[14]

To prove this theorem, we note that the necessity is obvious, since a set of $2(N-r)$ variables made up of $(N-r)$ q's and the corresponding p's does form a function group of rank $2(N-r)$. On the other hand, this condition is sufficient. For, given such a function group, one can replace the θ^a by functions of themselves, θ'^a, such that the PB's of the θ'^a with each other assume the constant values corresponding to $(N-r)$ pairs of canonical variables.[14]

We conclude by noting the distinguishing properties of the DB, which are suggested by (4.4). Given

[14] A set of functions $\theta^a(\omega)$ forms a function group if the PB's of the θ^a with each other can be expressed as functions of the θ^a alone. The rank of a function group is the number of independent functions in the function group. For further properties and details, see: L. P. Eisenhart, *Continuous Groups of Transformation* (Dover Publications, Inc., New York, 1961).

the functions $\theta^a(\omega)$, one could adjoin an arbitrary set of new functions $\zeta^m(\omega)$, making sure only that θ^a and ζ^m together form $2N$ independent functions; and, one could then define a bracket of two functions $f(\omega)$, $g(\omega)$ by

$$\{f, g\}^{\#} = \epsilon^{mn} \frac{\partial f}{\partial \zeta^m} \cdot \frac{\partial g}{\partial \zeta^n} . \tag{4.8}$$

This bracket certainly obeys the Jacobi identity, and treats the θ^a like pure numbers:

$$\{f, \theta^a\}^{\#} = 0. \tag{4.9}$$

One can then ask what is special about the DB. There are two features which are special to the DB. Firstly, while (4.4) shows that the DB is a particular case of the bracket (4.8), in general, for arbitrarily chosen functions $\zeta^m(\omega)$, (4.8) will not be a structure determined by the θ^a alone. The variables ξ^m appearing in (4.4) are, on the other hand, determined completely by the θ^a (up to a canonical transformation of the ξ^m among themselves). Secondly, the DB bears a special relationship to the PB in the following sense. If a function $f(\omega)$ is such that

$$\{f, \theta^a\} = 0, \quad \text{all} \quad a, \tag{4.10}$$

then for all functions $g(\omega)$, we have

$$\{f, g\}^{\star} = \{f, g\}. \tag{4.11}$$

Such a function $f(\omega)$ therefore generates the same transformation via the DB as it does via the PB.[11]

SUMMARY

We have shown that the Dirac bracket arises as a special case of the generalized Poisson bracket. We have traced the origin of the Jacobi identity for Dirac brackets to a standard property of Lagrange brackets. Finally, we have seen that when and only when the second class constraints θ^a form a function group does the Dirac bracket correspond to "freezing" a subset of canonical variables in the ordinary Poisson bracket.

ACKNOWLEDGMENTS

Part of this work was done while one of us (N. M.) was at the Palmer Physical Laboratory of Princeton University. He would like to thank Professor W. Bleakney and Professor M. L. Goldberger for the hospitality of that institution.