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Quantum Theory of the Infinite-Component Majorana Field and the Relation of Spin and Statistics*

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The quantum theory of the infinite-component Majorana field is formulated. The present paper discusses the three classes (slower than light, lightlike, and faster than light) of solutions to this equation and their Wigner classification. Particular attention is paid to the question of the normalization of the faster-than-light solutions. The current operator is shown to be timelike even for the spacelike solutions, and it is shown to lead to a finite process of emission of light by charged Majorana particles. The quantum theory of the Majorana field is formulated in accordance with the substitution law, and the usual connection between spin and statistics is recovered.

I. INTRODUCTION

THE usefulness of the local covariant field description of particle phenomena has been amply demonstrated in the successes of quantum electrodynamics and of the chiral $V-A$ weak interactions. It has been conventional in such treatments to use local fields, each of which describes only one kind of particle, with a definite mass and a definite spin. Many years ago the late H. J. Bhabha systematically investigated the possibility of describing a family of particles with varying masses and spins by a single irreducible equation.[1] As a special class of such equations, Bhabha studied relativistic wave equations of the form

$$(i\Gamma^\mu\partial/\partial x^\mu-\kappa)\psi=0\,, \qquad (1.1)$$

where the matrices Γ^μ together with the spin matrices $S^{\mu\nu}$ satisfied the commutation relations of the de Sitter group. Of course, the $S^{\mu\nu}$ themselves satisfy the commutation relations of the homogeneous Lorentz group, and the matrices Γ^μ constitute a four-vector operator with respect to this group. Bhabha's additional assumption was that the matrices Γ^μ among themselves satisfied the commutation relations of the form

$$[\Gamma^\mu,\Gamma^\nu]=i\lambda S^{\mu\nu}.$$

From this de Sitter structure, Bhabha was able to obtain a mass-spin spectrum in which the mass decreased as the spin increased. These equations include the spin-$\frac{1}{2}$ Dirac equation and the spin-0 and -1 Duffin-Kemmer-Petiau equations. But except for these special cases, the Bhabha equations lead to the necessity of introducing an indefinite metric of an unsatisfactory kind. This difficulty can be traced to the unfortunate restriction to finite-component fields, which necessarily correspond to nonunitary representations of the homogeneous Lorentz group. We should therefore relax this restriction

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[1] H. J. Bhabha, Rev. Mod. Phys. 17, 200 (1945).

and search for Bhabha equations for infinite-component fields.

It is curious that the simplest of such equations were discovered by Majorana[2] several decades ago, and were rediscovered by Gel'fand and Yaglom. There are two equations, one of which describes a family of particles with integral spins, and the other a family with half-integral spins. Both of them can be written in the standard form of Eq. (1.1), with Γ^μ and $S^{\mu\nu}$ together furnishing a Hermitian representation of the Lie algebra of the de Sitter group $O(3,2)$.

In spite of renewed interest in infinite-component wave equations and their group-theoretic basis, there are several problems connected with the quantum theory of such a system. The first concerns the apparent possibility of restriction of the field to only one sign of the frequency and the consequent danger of violating basic features of conventional field theory like *CPT* invariance and the substitution law. The second difficulty concerns the existence of spacelike solutions to these equations. It was also found that the usual proofs of the connection between spin and statistics were not valid in these cases.

It is the purpose of this paper to resolve these difficulties and to present a quantum field theory of the infinite-component Majorana field. Necessary for the accomplishment of this task are two theoretical formulations. One concerns the quantum theory of faster-than-light particles.[3] The other is a theorem on the connection between spin and statistics.[4]

The present paper deals only with the Majorana field. The treatment of systems more complicated but physically interesting is reserved for a forthcoming publication.

The material of this paper is arranged as follows. In Sec. II, we examine in detail the algebraic properties of the Majorana representations of $O(3,2)$ and the reduction of these representations under various subgroups of interest.[5] Section III discusses the different kinds of solutions of the Majorana equations, with special attention to finding the representation of the "little group" that goes with each class of solutions. The solutions are separated subsequently into Poincaré-invariant and Poincaré-irreducible sets. In Sec. IV, we consider normalization and completeness of the set of spinor solutions of the wave equation for a fixed value of the space momentum at various energies. While no new features are expected in the case of the timelike solutions, we justify the careful treatment of the spacelike ones and provide such a treatment. In Sec. IV we also discuss the behavior of the "current" operator Γ_μ in various kinds of states and its relevance to the spontaneous emission of radiation by spacelike particles. Section V discusses the quantum-field theory of the Majorana field, and the spin-statistics connection. In the Appendix we summarize a few properties of the discrete representations of the group $O(2,1)$.

[2] E. Majorana, Nuovo Cimento **9**, 335 (1932); D. M. Fradkin, Am. J. Phys. **34**, 314 (1966); I. M. Gel'fand and A. M. Yaglom, Zh. Eksperim. i Teor. Fiz. **18**, 703 (1948); **18**, 109b (1948); **18**, 1105 (1948); V. Bargmann, Math. Rev. **10**, 583 (1949); **10**, 584 (1949); E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967).

[3] M. E. Arons and E. C. G. Sudarshan, Phys. Rev. **173**, 1622 (1968); J. Dhar and E. C. G. Sudarshan, *ibid.* **174**, 1808 (1968).

[4] E. C. G. Sudarshan, Proc. Indian Acad. Sci. **A67**, 284 (1968).

[5] A discussion of the Majorana representations is also given by A. Böhm, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. Brittin (Gordon and Breach, Science Publishers, Inc., New York, 1968), Vol. X B; and D. Stoyanov and I. Todorov, J. Math. Phys. **9**, 2146 (1968).

II. PROPERTIES OF MAJORANA REPRESENTATIONS

We describe here the mathematical properties of the two "Majorana" representations of the homogeneous Lorentz group.

The de Sitter algebra $O(3,2)$ is generated by the elements S_{AB} $(=-S_{BA})$ satisfying the commutation relations

$$-i[S_{AB},S_{CD}]=g_{BC}S_{AD}-g_{AC}S_{BD}+g_{BD}S_{CA}-g_{AD}S_{CB}\,, \tag{2.1}$$

with

$$g_{00}=g_{55}=+1\,,\quad g_{11}=g_{22}=g_{33}=-1\,, \tag{2.2}$$

and all other g_{AB} vanishing. Letting $\mu, \nu, \ldots,$ denote the space-time dimensions 0, 1, 2, 3, we identify $S_{\mu\nu}$ with the (spin part of the) generators of the homogeneous Lorentz group $O(3,1)$, and

$$\Gamma_\mu=S_{\mu 5} \tag{2.3}$$

as the four-vector operator in the wave equation. We write

$$\begin{aligned}S_{jk}&=\tfrac{1}{2}\epsilon_{jkl}\eta^\dagger\sigma_l\eta\,,\\ S_{0k}&=\tfrac{1}{4}(\eta^\dagger\sigma_k C\eta^{\dagger T}-\eta^T C\sigma_k\eta)\,,\\ \Gamma_0&=\tfrac{1}{2}(\eta^\dagger\eta+1)\,,\\ \Gamma_k&=-\tfrac{1}{4}i(\eta^\dagger\sigma_k C\eta^{\dagger T}+\eta^T C\sigma_k\eta)\,,\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}[][\eta_1,\eta_1^\dagger]&=[\eta_2,\eta_2^\dagger]=1\,,\\ [\eta_1,\eta_2]&=[\eta_1,\eta_2^\dagger]=0\,,\\ C&=i\sigma_2\,.\end{aligned} \tag{2.5}$$

It is easily verified that the construction (2.4) satisfies the de Sitter commutation relations (2.1), and, by inspection, we verify that it is Hermitian. It is, however, reducible since the nontrivial operator

$$\exp(i\pi\eta^\dagger\eta)=\exp[i\pi(2\Gamma^0+1)] \tag{2.6}$$

commutes with all the generators. According as $\frac{1}{2}\eta^\dagger\eta$ is an integer or a half-integer, we have representations of the de Sitter algebra which, on restriction to its $O(3)$ subgroup, describes integral or half-integral spin representations. In what follows we will find it advantageous to deal with the generators (2.4) directly without explicitly mentioning the restriction to one or the other of the two Majorana representations. When needed, we may use the projection operators $\frac{1}{2}[1\pm\exp(i\pi\eta^\dagger\eta)]$ to these two different representations.

The two representations of $O(3,2)$ provided by the construction (2.4) are remarkable in that they remain irreducible when consideration is restricted to the $O(3,1)$ subgroup generated by $S_{\mu\nu}$. The two representations of $O(3,1)$ involved here are, in the conventional notation, $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$; these are the only irreducible representations of $O(3,1)$ permitting the construction of a four-vector operator. The first one belongs to the supplementary series, and the second one to the principal series, of unitary irreducible representations of $O(3,1)$. The irreducibility under $O(3,1)$ leads to many relations among the operators $S_{\mu\nu}$ and Γ_μ, which we now describe.

There are two Casimir operators for the $O(3,1)$ algebra:

$$\mathcal{C}_1=\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu}\,,\quad \mathcal{C}_2=\tfrac{1}{8}\epsilon^{\mu\nu\lambda\sigma}S_{\mu\nu}S_{\lambda\sigma}\,, \tag{2.7}$$

and in the realization (2.4) they have the following values:

$$\mathcal{C}_1=\tfrac{3}{4}\,,\quad \mathcal{C}_2=0\,. \tag{2.8}$$

At the $O(3,2)$ level, one again has two Casimir invariants. To define the first of them we first define the five-vector R^A:

$$R^A=\tfrac{1}{8}\epsilon^{ABCDE}S_{BC}S_{DE}\,, \tag{2.9}$$

and then contract R^A with itself. However, the component R^5 is nothing but the $O(3,1)$ invariant $\mathcal{C}_2$, which happens to vanish. It then follows, since R^A is an irreducible tensor operator under $O(3,2)$, that R^A vanishes identically, for all values of A. The extra relations we obtain in this way are

$$\begin{aligned} R_0&=J_k\Gamma_k=0\,,\\ R_j&=\Gamma_0J_j-\epsilon_{jkl}K_k\Gamma_l=0\,. \end{aligned} \tag{2.10}$$

We have used the customary notation $K_k=S_{0k}$ and $J_k=\frac{1}{2}\epsilon_{klm}S_{lm}$. The other Casimir invariant of $O(3,2)$ is the quadratic expression

$$\tfrac{1}{2}S^{AB}S_{AB}=\Gamma^\mu\Gamma_\mu-\mathcal{C}_1\,. \tag{2.11}$$

Either from the constancy of this expression, or more simply from the irreducibility of the representations under $O(3,1)$ and from the fact that $\Gamma^\mu\Gamma_\mu$ is a Lorentz scalar, we deduce that $\Gamma^\mu\Gamma_\mu$ must be a pure number. Using (2.4), we easily deduce that

$$\Gamma^\mu\Gamma_\mu=-\tfrac{1}{2}\,. \tag{2.12}$$

We now examine this at the $O(3,2)$ level. Taking the commutator of Γ_λ with (2.12) we find that

$$\{S_{\lambda\mu},\Gamma^\mu\}=0\,. \tag{2.13}$$

It is clear that if we define an irreducible tensor $T_B{}^C$ under $O(3,2)$ as

$$\begin{aligned} T_B{}^C&=\{S_{BA},S^{AC}\}-\tfrac{1}{5}\delta_B{}^C\{S_{DA},S^{AD}\}\\ &=\{S_{BA},S^{AC}\}-\delta_B{}^C\,, \end{aligned} \tag{2.14}$$

then (2.13) expresses the vanishing of the components T_λ.[5] It follows that in fact $T_B{}^C$ vanishes identically,[6] i.e.,

$$\{S_{BA},S^{AC}\}=\delta_B{}^C\,. \tag{2.15}$$

In addition to (2.12) and (2.13), we obtain from (2.15)

$$\{\Gamma_\lambda,\Gamma_\mu\}=\{S_{\lambda\nu},S^\nu{}_\mu\}-g_{\lambda\mu}\,. \tag{2.16}$$

Since the commutator $[\Gamma_\lambda,\Gamma_\mu]$ is equal to $-iS_{\lambda\mu}$, this yields

$$\Gamma_\lambda\Gamma_\mu=-\tfrac{1}{2}iS_{\lambda\mu}+\tfrac{1}{2}\{S_{\lambda\nu},S^\nu{}_\mu\}-\tfrac{1}{2}g_{\lambda\mu}\,. \tag{2.17}$$

Using these identities, which are characteristic of the Majorana representations of $O(3,2)$, we can obtain a great deal of useful information about how the representations reduce under various subgroups of interest. First we consider the reduction under the maximal compact subgroup $O(3)\otimes O(2)$ of $O(3,2)$. Here, $O(3)$ acts on the indices 1, 2, 3 and is generated by the operators J_j, while $O(2)$ acts on the indices 0 and 5 and is generated by $S_{05}=\Gamma_0$. Thus the reduction would be accomplished by simultaneously diagonalizing the three operators: J^2, J_3, Γ_0, the first being the Casimir invariant of $O(3)$. However, by using (2.17) for the case $\mu=\nu=0$, we see that the eigenvalue of Γ_0 determines that of J^2:

$$(\Gamma_0)^2=J^2+\tfrac{1}{4}\,. \tag{2.18}$$

Thus in the reduction under $O(3)\otimes O(2)$, each representation of $O(3)$ goes with one representation of $O(2)$. The corresponding basis is called the "canonical basis" since representations of $O(3,1)$ are usually expressed in it; we shall write its elements in the form

$$\psi_{jm}{}^{(1)}:\quad (\psi_{j'm'}{}^{(1)},\psi_{jm}{}^{(1)})=\delta_{j'j}\delta_{m'm}\,. \tag{2.19}$$

The range of j values is the set of all integers in the representation $(0,\frac{1}{2})$, and all half-odd integers in the representation $(\frac{1}{2},0)$. On this basis we have

$$\begin{aligned} J^2\psi_{jm}{}^{(1)}&=j(j+1)\psi_{jm}{}^{(1)},\quad J_3\psi_{jm}{}^{(1)}=m\psi_{jm}{}^{(1)}\,,\\ \Gamma_0\psi_{jm}{}^{(1)}&=(j+\tfrac{1}{2})\psi_{jm}{}^{(1)}\,. \end{aligned} \tag{2.20}$$

[Notice that the vectors $\psi_{jm}{}^{(1)}$ form a basis in spin space alone, and that the scalar product used in (2.19) also refers to this space.] In terms of the construction given in (2.4), we can write

$$\psi_{jm}{}^{(1)}=[(j+m)!(j-m)!]^{-1/2}\times(\eta_1{}^\dagger)^{j+m}(\eta_2{}^\dagger)^{j-m}|0\rangle\,. \tag{2.21}$$

The matrix elements of all the components of K can be determined from those of K_3, for which we have

$$K_3\psi_{jm}{}^{(1)}=\tfrac{1}{2}[(j+1)^2-m^2]^{1/2}\psi_{j+1,m}{}^{(1)}+\tfrac{1}{2}(j^2-m^2)^{1/2}\psi_{j-1,m}{}^{(1)}\,. \tag{2.22}$$

The matrix elements of Γ can then be obtained by

[6] A. Böhm (Ref. 5) has shown that if one starts with the de Sitter commutation relations and imposes the identities (2.10) and (2.15) one ends up with the Majorana representations.

using the relation

$$\Gamma = e^{-(i\pi/2)\Gamma_0} \mathbf{K} e^{(i\pi/2)\Gamma_0}. \quad (2.23)$$

This same basis is suited to the reduction of the $O(3,2)$ representation under an "unphysical" $O(2,1)\otimes O(2)$ subgroup defined as follows: The $O(2,1)$ acts on the indices 0, 3, and 5 and is generated by Γ_0, Γ_3, and K_3; the $O(2)$ acts on the indices 1 and 2 and is generated by J_3. We shall refer to this $O(2,1)$ subgroup as $\tilde{O}(2,1)$; it is not contained in the physical $O(3,1)$ group. The Casimir invariant of $\tilde{O}(2,1)$ is

$$\tilde{Q} = (\Gamma_3)^2 + (K_3)^2 - (\Gamma_0)^2; \quad (2.24)$$

by using (2.17) for $\mu=\nu=0$ and $\mu=\nu=3$, we find that

$$\tilde{Q} = (J_3+\tfrac{1}{2})(\tfrac{1}{2}-J_3). \quad (2.25)$$

Since J_3 is quantized, only the discrete-class unitary irreducible representations (UIR) of $\tilde{O}(2,1)$ appear, and since Γ_0 is positive definite, only those of the type $D_k^{(+)}$ appear in the conventional notation. Thus we find that for a fixed value of m (eigenvalue of J_3) the states $\psi_{jm}^{(1)}$ for $j \geqslant |m|$ form the basis for the UIR $D_{|m|+\frac{1}{2}}^{(+)}$ of the $\tilde{O}(2,1)$ group generated by Γ_0, Γ_3, and K_3. Each UIR of $\tilde{O}(2,1)$ that appears, in fact, appears twice; once for $J_3=m$ and once for $J_3=-m$.

Next we consider the $O(2,1)\otimes O(1,1)$ subgroup of $O(3,2)$; this $O(2,1)$ is a subgroup of the physical $O(3,1)$ group[7] and acts on the indices 0, 1, and 2, being generated by the operators J_3, K_1, and K_2, while the $O(1,1)$ is generated by Γ_3 and is associated with the indices 3 and 5. The Casimir invariant of $O(2,1)$ is

$$Q = (K_1)^2 + (K_2)^2 - (J_3)^2; \quad (2.26)$$

by considering (2.17) for $\mu=\nu=3$, we establish easily that

$$Q = \tfrac{1}{4} + (\Gamma_3)^2. \quad (2.27)$$

The reduction under the group $O(2,1)\otimes O(1,1)$ would be accomplished by simultaneously diagonalizing the operators Q, J_3, and Γ_3. However, by virtue of (2.27), this is equivalent to diagonalizing J_3 and Γ_3. Already (2.27) shows that we will encounter only the continuous nonexceptional series of UIR's of $O(2,1)$. From the earlier discussion of the group $\tilde{O}(2,1)$, we see that the problem of diagonalizing Γ_3 is a problem at the level of the $O(2,1)$ group structure. In the canonical basis, we have Γ_0 and J_3 diagonal and at a fixed value of J_3 we wish to pass from this basis to one in which Γ_3 is diagonal. This is equivalent to diagonalizing a noncompact $O(1,1)$ generator in a $D^{(+)}$-type UIR of $O(2,1)$ starting with the basis in which the compact $O(2)$ generator is diagonal. The way to do this is explained in the Appendix; here we quote the results. We shall use the variable σ to denote the eigenvalues of Γ_3, and we shall write $\psi_{\sigma,m}^{(2)}$ for the eigenvectors of Γ_3 and J_3:

$$\Gamma_3\psi_{\sigma,m}^{(2)} = \sigma\psi_{\sigma,m}^{(2)},$$
$$J_3\psi_{\sigma,m}^{(2)} = m\psi_{\sigma,m}^{(2)} \quad (-\infty<\sigma<\infty), \quad (2.28)$$
$$(\psi_{\sigma'm'}^{(2)},\psi_{\sigma m}^{(2)}) = \delta_{m'm}\delta(\sigma'-\sigma).$$

Then this basis is related to the canonical one:

$$\psi_{\sigma m}^{(2)} = \sum_j \mathcal{C}_{|m|+\frac{1}{2}}(j+\tfrac{1}{2}, -\sigma)\psi_{j,m}^{(1)},$$

$$\mathcal{C}_{|m|+\frac{1}{2}}(j+\tfrac{1}{2}, -\sigma) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\Gamma(2|m|+1)} \left[\frac{\Gamma(|m|+j+1)}{\Gamma(j-|m|+1)}\right]^{1/2} \times 2^{|m|+\frac{1}{2}}$$
$$\times e^{-(i\pi/2)(|m|-j)} F(|m|-j, |m|+\tfrac{1}{2}-i\sigma, 2|m|+1, 2)$$
$$\times |\Gamma(|m|+\tfrac{1}{2}+i\sigma)|. \quad (2.29)$$

The new basis states $\psi_{\sigma m}^{(2)}$ in the spin space have two alternative interpretations, as did the canonical basis $\psi_{jm}^{(1)}$. For fixed m, the states $\psi_{\sigma m}^{(2)}$ span the space for the UIR $D_{|m|+\frac{1}{2}}^{(+)}$ of the subgroup $\tilde{O}(2,1)$ as σ varies continuously from $-\infty$ to $+\infty$. On the other hand, according to (2.27) we have

$$Q\psi_{\sigma m}^{(2)} = (\tfrac{1}{4}+\sigma^2)\psi_{\sigma m}^{(2)}. \quad (2.30)$$

Thus for a fixed value of σ, as m varies in discrete steps from $-\infty$ to $+\infty$, the states $\psi_{\sigma m}^{(2)}$ span the space of an UIR of the physical $O(2,1)$ group. The states $\psi_{\sigma m}^{(2)}$ and $\psi_{-\sigma m}^{(2)}$ give rise to one and the same UIR of $O(2,1)$. Thus in the reduction of the Majorana representations of $O(3,1)$ we have found every continuous nonexceptional representation of the subgroup $O(2,1)$ twice, these two occurrences being distinguished by the value of Γ_3.

Last of all we consider how the Majorana representations reduce under the Euclidean subgroup[8] $E(2)$ contained in the physical Lorentz group $O(3,1)$. $E(2)$ is generated by the three operators J_3, J_1-K_2, and J_2+K_1, and its Casimir invariant is

$$\epsilon^2 = (J_1-K_2)^2 + (J_2+K_1)^2. \quad (2.31)$$

The combination of the Γ_μ that commutes with the $E(2)$ group is $\Gamma_0+\Gamma_3$, and as in the previous cases, this operator can be related to ϵ^2. The easiest way to do this is to start with Eq. (2.18), unitarily transform both sides by means of the operator $e^{i\alpha K_3}$, and then take the limit as $\alpha\to\infty$. In this way one establishes that

$$\epsilon^2 = (\Gamma_0+\Gamma_3)^2. \quad (2.32)$$

The reduction of the representation space under the direct product of $E(2)$ and the one parameter group generated by $\Gamma_0+\Gamma_3$ is accomplished by simultaneously

[7] Reduction of $O(3,1)$ under $O(2,1)$ has been done by: principal series—S. Strom, Arkiv Fysik **34**, 215 (1967); A. Sciarrino and M. Toller, J. Math. Phys. **8**, 1252 (1967); N. Mukunda, *ibid.* **9**, 50 (1968); supplementary series—N. Mukunda, *ibid.* **9**, 417 (1968).

[8] Reduction of $O(3,1)$ under $E(2)$ has been done by S. Strom (Ref. 7).

diagonalizing J_3 and $\Gamma_0+\Gamma_3$, since then ϵ^2 will be automatically diagonal. As in the reduction with respect to $O(2,1)\otimes O(1,1)$, this reduces to the diagonalization of a parabolic generator in a $D^{(+)}$ UIR of $O(2,1)$ starting with the compact $O(2)$ generator diagonal. We shall write $\psi_{\epsilon,m}{}^{(3)}$ for the corresponding basis:

$$(\Gamma_0+\Gamma_3)\psi_{\epsilon,m}{}^{(3)}=\epsilon\psi_{\epsilon,m}{}^{(3)}, \quad J_3\psi_{\epsilon,m}{}^{(3)}=m\psi_{\epsilon,m}{}^{(3)},$$
$$(\psi_{\epsilon'm'}{}^{(3)},\psi_{\epsilon m}{}^{(3)})=\delta_{m'm}\delta(\epsilon'-\epsilon) \quad (0<\epsilon<\infty), \tag{2.33}$$
$$\epsilon^2\psi_{\epsilon,m}{}^{(3)}=\epsilon^2\psi_{\epsilon,m}{}^{(3)}.$$

From the analysis given in the Appendix, we have

$$\psi_{\epsilon,m}{}^{(3)}=\sum_j \mathcal{B}_{|m|+\frac{1}{2}}(j+\tfrac{1}{2},\,\epsilon)\psi_{j,m}{}^{(1)},$$
$$\mathcal{B}_{|m|+\frac{1}{2}}(j+\tfrac{1}{2},\,\epsilon)=\frac{1}{\Gamma(2|m|+1)}\left(\frac{\Gamma(j+|m|+1)}{\Gamma(j-|m|+1)}\right)^{1/2}$$
$$\times e^{-\epsilon}2^{|m|+\frac{1}{2}}\epsilon^{|m|}\Phi(|m|-j,\,2|m|+1,\,2\epsilon). \tag{2.34}$$

Thus we find that the Majorana representations contain each infinite-dimensional representation of $E(2)$ exactly once, corresponding to ϵ^2 spanning the range 0–∞.

To summarize, we have introduced three different orthonormal bases in spin space, namely, $\psi_{jm}{}^{(1)}$, $\psi_{\sigma m}{}^{(2)}$, and $\psi_{\epsilon m}{}^{(3)}$. The first one exhibits the reduction of the Majorana representations under $O(3)\otimes O(2)$, the second under $O(2,1)\otimes O(1,1)$, and the third under $E(2)\otimes(\Gamma_0+\Gamma_3)$. Further, each of them exhibits the reduction under the "unphysical" $\tilde{O}(2,1)\otimes O(2)$ subgroup, with an elliptic (compact), hyperbolic (noncompact), and parabolic (noncompact) generator of $\tilde{O}(2,1)$ being diagonal in $\psi^{(1)}$, $\psi^{(2)}$, and $\psi^{(3)}$, respectively.

III. CLASSIFICATION OF PLANE-WAVE SOLUTIONS

Having examined the algebraic properties of the Majorana representations in some detail, we are now in a position to classify the solutions of the wave equations according to the kinds of unitary irreducible representations of the Poincaré group that they generate.[9] The important question of normalization of these solutions will be taken up in the Sec. IV.

Let us consider the Majorana equations

$$(i\Gamma^\mu\partial/\partial x^\mu-\kappa)\psi(x)=0, \tag{3.1}$$

where κ is a real positive constant and Γ^μ is given by (2.4). Here $\psi(x)$ is at the same time a function of x and an infinite-component vector in spin space transforming according to the unitary representation of $O(3,1)$ generated by $\mathbf{J}$ and $\mathbf{K}$. The plane-wave solutions of this equation are of the form

$$\psi(x)=\varphi(p)e^{-ip\cdot x},$$

with $\varphi(p)$ satisfying the equation

$$(\Gamma^\mu p_\mu-\kappa)\varphi(p)=0. \tag{3.2}$$

Given such a solution $\varphi(p_\mu)$, we can get another solution for a different value p' (lying on the same hyperboloid as p_μ so that $p^2=p'^2$) by Lorentz transforming the amplitude $\varphi(p_\mu)$. If $U(\Lambda)$ is the unitary operator for a Lorentz transformation Λ, then

$$\varphi'(p_\mu)=U(\Lambda)\varphi(p_\mu)$$

satisfies the equation

$$(U\Gamma^\mu U^{-1}p_\mu-\kappa)\varphi'(p_\mu)=0.$$

Remembering that

$$U\Gamma^\mu U^{-1}=\Lambda^\mu{}_\nu\Gamma^\nu,$$

we see that $\varphi'(p_\mu)$ satisfies the same equation as $\varphi(p_\mu)$, but with p_μ replaced by $\Lambda^\nu{}_\mu P_\nu$. As a consequence of this observation it is sufficient to consider certain standard configurations and obtain all other solutions from them. As long as κ is nonzero, not all the momenta can vanish at the same time. The solutions may then be divided into three classes according as the momentum vector is timelike, lightlike, or spacelike. In each case, it is not sufficient to find the "mass spectrum," but it is also necessary to determine the "spin," i.e., the behavior of the solution with respect to the "little group." This we shall do for each of these three classes.

Returning to (3.2) we can parametrize p^μ in general as

$$p^\mu=(E,p\cos\phi\sin\theta,p\sin\phi\sin\theta,p\cos\theta), \tag{3.3}$$

where p ($\geqslant 0$) is the magnitude of the space part of p^μ. We can also demand that $\varphi(p_\mu)$ be an eigenstate of helicity with eigenvalue λ, so that we shall write $\varphi_\lambda(p_\mu)$, and have

$$(\Gamma^0E-\Gamma^1p\sin\theta\sin\phi-\Gamma^2p\sin\theta\cos\phi-\Gamma^3p\cos\theta-\kappa)\times\varphi_\lambda(p_\mu)=0, \tag{3.4}$$
$$\mathbf{J}\cdot\mathbf{p}\varphi_\lambda(p_\mu)=\lambda p\varphi_\lambda(p_\mu).$$

Such a $\varphi_\lambda(p_\mu)$ can be obtained from the solution $\varphi_\lambda(E,p)$ of the equations

$$(\Gamma^0E-\Gamma^3p-\kappa)\varphi_\lambda(E,p)=0,$$
$$J_3\varphi_\lambda(E,p)=\lambda\varphi_\lambda(E,p), \tag{3.5}$$

by means of a spatial rotation

$$\varphi_\lambda(p_\mu)=e^{-i\phi J_3}e^{-i\theta J_2}\varphi_\lambda(E,p). \tag{3.6}$$

We can therefore work with (3.5). We consider now the three classes of solutions in turn.

Solutions of Class I: Slower-than-Light Particles

In this case we parametrize E and p:

$$E=M\cosh\zeta, \quad p=M\sinh\zeta \quad (M>0,\,\zeta\geqslant 0). \tag{3.7}$$

[9] A general discussion of solutions of infinite-component field equations is given in W. Ruhl, Commun. Math. Phys. 6, 312 (1967).

The standard configuration for slower-than-light particles is that in which $\mathbf{p}=0$. The little group corresponding to this configuration is generated by J_1, J_2, J_3 with the Casimir invariant J^2. The amplitude ψ_λ in the standard configuration is obtained from $\varphi_\lambda(E,p)$ by a Lorentz transformation:

$$\psi_\lambda = e^{-i\zeta K_3}\varphi_\lambda(E,p)\,, \tag{3.8}$$

and obeys

$$(\Gamma^0 M-\kappa)\psi_\lambda=0\,,\quad J_3\psi_\lambda=\lambda\psi_\lambda\,. \tag{3.9}$$

In addition, we demand that ψ_λ be an eigenfunction of J^2 with eigenvalue $s(s+1)$, this being the square of the spin. On the other hand, the eigenvalues of Γ^0 are $(s+\frac{1}{2})$. The solutions of these equations are easily written down in terms of the canonical basis $\psi^{(1)}$. The structure of the Majorana representations and the wave equation give a relation between the mass and the spin:

$$M=M(s)=\kappa/(s+\tfrac{1}{2})\,. \tag{3.10}$$

As the spin increases, the mass decreases. Labeling the solutions by the mass, spin, and helicity values, the amplitudes in the standard configuration are given, up to a normalization constant, by

$$\psi_\lambda{}^{M(s),s}=\psi_{s,\lambda}{}^{(1)}\,. \tag{3.11}$$

To obtain $\varphi_\lambda(E,p)$, we have to apply the transformation $e^{i\zeta K_3}$ to $\psi_{s,\lambda}{}^{(1)}$.

We note that the energy for a particle at rest is always of the same sign (positive). This is in marked contrast to the nature of the solutions of finite-dimensional wave equations.

Solutions of Class II: Lightlike Particles

We now parametrize E and p appearing in (3.5) in the following way:

$$E=p=\kappa e^{\zeta}\quad(-\infty<\zeta<\infty)\,. \tag{3.12}$$

(Negative values of E are absent; see below.) The standard configuration for lightlike particles can be chosen so that the spatial momentum is of magnitude κ and lies in the positive third direction. The little group corresponding to this configuration is $E(2)$, generated by J_3, J_1-K_2, and J_2+K_1, with Casimir invariant ϵ. The amplitude ψ_λ in the standard configuration is defined once again by (3.8) and obeys

$$(\Gamma_0+\Gamma_3-1)\psi_\lambda=0\,,\quad J_3\psi_\lambda=\lambda\psi_\lambda\,. \tag{3.13}$$

In addition we demand that ψ_λ be an eigenfunction of ϵ. The solutions to these equations are now given in terms of the basis $\psi^{(3)}$ in spin space. We see that the wave equation and the structure of the Majorana representations single out a unique value for the invariant ϵ, namely, $\epsilon=+1$. Labeling the amplitude by the mass (which vanishes), the value of ϵ, and the helicity, in the standard configuration we have

$$\psi_\lambda{}^{0,\epsilon=+1}=\psi_{1,\lambda}{}^{(3)}\,, \tag{3.14}$$

again up to a normalization factor.

The representation of $E(2)$ involved is infinite-dimensional, and the lightlike particle belongs to the so-called "continuous-spin" type. Note that, in contrast to the previous class of solutions, here only one irreducible representation of the Poincaré group has been generated. Stated in another way, in the standard configuration only one member of the basis $\psi_{\epsilon,m}{}^{(3)}$ corresponding to $\epsilon=+1$ is involved in constructing solutions of the wave equation, whereas in the case of solutions of class I, in the rest frame each element of the basis $\psi_{jm}{}^{(1)}$ gives rise to a different physical solution. Notice also that we do not have lightlike solutions in which the energy is negative; this is due to the fact that both $\Gamma_0+\Gamma_3$ and $\Gamma_0-\Gamma_3$ are positive-definite operators. As before, the solutions to (3.5) are obtained by applying the Lorentz transformation $e^{i\zeta K_3}$ to $\psi_{1,\lambda}{}^{(3)}$.

Solutions of Class III: Faster-than-Light Particles

This, the most novel case, is not normally encountered in finite-component wave equations.[10] We can parametrize E and p appearing in (3.5) in the present case as

$$E=l\sinh\zeta\,,\quad p=l\cosh\zeta\,. \tag{3.15}$$

The quantity l is the positive square root of the Lorentz-invariant expression $-p^\mu p_\mu$; we shall continue to refer to it as the mass of the particle. The standard configuration is now one in which the particle has infinite velocity and zero energy; and the momentum is oriented along the positive third axis. The amplitude in the standard configuration, ψ_λ, is again related to $\varphi_\lambda(E,p)$ via (3.8), and obeys

$$(\Gamma_3 l-\kappa)\psi_\lambda=0\,,\quad J_3\psi_\lambda=\lambda\psi_\lambda\,. \tag{3.16}$$

The little group $O(2,1)$ leaving this configuration invariant is generated by J_3, K_1, and K_2. The solutions to (3.16) belonging to definite representations of $O(2,1)$ are the vectors $\psi_{\sigma,\lambda}{}^{(2)}$. The relation between the "square of the spin" and the "mass" is given by (2.27):

$$l=\kappa/\sigma\,,\quad Q=\tfrac{1}{4}+\sigma^2\,. \tag{3.17}$$

Thus we find a continuous set of (positive) values for the mass l, varying from zero to infinity. For each value of the mass, a unique infinite-dimensional continuous nonexceptional type UIR of the little group $O(1,2)$ is determined by (3.17). This relation looks more familiar if we write

$$s'=-\tfrac{1}{2}+i\sigma\,,$$

so that

$$-Q=s'(s'+1)\,.$$

[10] For a systematic discussion of class-III particles, see O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962); their quantum field theory of spinless class-III particles is formulated in the papers cited in Ref. 3.

Then the mass formula (3.17) can be put in a form very similar to the slower-than-light mass formula (3.10) by rewriting it in the form

$$l=i\kappa/(s'+\tfrac{1}{2}). \tag{3.18}$$

Since the faster-than-light particles have an imaginary rest mass, this is the analytic continuation of the mass formula for the slower-than-light particles. The amplitude in the standard configuration can be labeled by the mass l, the spin σ, and the helicity λ, and is given up to a normalization by

$$\psi_\lambda{}^{l(\sigma),\sigma}\approx\psi_{\sigma,\lambda}{}^{(2)}. \tag{3.19}$$

Notice that in the standard configuration, only those members of the basis $\psi_{\sigma,m}{}^{(2)}$ that correspond to positive eigenvalues σ for Γ_3 give rise to solutions of the Majorana equation. Thus in the respective standard configurations, each member of the basis $\psi_{jm}{}^{(1)}$, half the members of the basis $\psi_{\sigma,m}{}^{(2)}$ (with $\sigma>0$), and one member of the basis $\psi_{\epsilon m}{}^{(3)}$ (with $\epsilon=+1$) give us solutions of the wave equation.

For the faster-than-light particles, energy can take either sign. There is no invariant distinction between the solutions of class III with positive energies and those with negative energies, since a Lorentz transformation can carry one kind of solution into another.

Thus, we have found three classes of solutions $\psi_\lambda{}^{M(s),s}$, $\psi_\lambda{}^{0,\epsilon=+1}$, and $\psi_\lambda{}^{l(\sigma),\sigma}$ which correspond, respectively, to slower-than-light, lightlike, and faster-than-light particles. The first and third families of particles contain an infinite number of Poincaré irreducible solutions, while there is essentially only one lightlike solution. The class-I particles constitute a denumerable infinity, each labeled by a mass M and spin s with the mass formula (3.10) relating them. The class-III particles, on the other hand, constitute a nondenumerable infinity labeled by an imaginary rest mass il and a complex spin $s'=-\frac{1}{2}+i\sigma$, which denotes a member of the continuous nonexceptional family of representations of $O(2,1)$. All three classes have all single-valued representation or all double-valued representations according as the operator (2.6) has the eigenvalue $+1$ or -1.

IV. NORMALIZATION, COMPLETENESS, AND BEHAVIOR OF CURRENT

The covariant wave functions $\psi(\mathbf{x},t)$ in (3.1) furnish several irreducible representations of the Poincaré group. We can construct a four-vector of charge current

$$\mathcal{J}_\mu(\mathbf{x},t)=\psi^\dagger(\mathbf{x},t)\Gamma_\mu\psi(\mathbf{x},t). \tag{4.1}$$

Under Lorentz transformations these quantities transform as the components of a four-vector density. It is also conserved, by virtue of the equations of motion:

$$\partial^\mu\mathcal{J}_\mu(\mathbf{x},t)=0. \tag{4.2}$$

As a consequence, the space integral of the time component is Lorentz-invariant and time-dependent:

$$Q(t)=Q=\int\psi^\dagger(\mathbf{x},t)\Gamma_0\psi(\mathbf{x},t)d^3x. \tag{4.3}$$

This quantity is positive definite and serves to define a norm for the wave functions. Thus in spin space, *the physically relevant norm is not the one with respect to which the Majorana representations of O(3,1) are unitary, but the one that uses the positive-definite matrix* Γ_0 *as a metric operator.*

We wish now to verify that the solutions of the Majorana equation for fixed $\mathbf{p}$ and various values of E are orthogonal to one another with respect to the metric Γ_0, and can be normalized appropriately. For the timelike solutions (class I) we could have expected this to emerge, since these solutions are generated from the elements of the canonical basis $\psi_{jm}{}^{(1)}$ in spin space, and between these vectors all the operators Γ_μ have finite matrix elements. However, the situation is completely different for the solutions of class III, which are obtained by Lorentz transformations from the elements of the basis $\psi_{\sigma,m}{}^{(2)}$ in spin space. It is a known fact that the notion of "matrix elements" of the operators Γ_μ between such vectors is mathematically meaningless; in other words, the vectors $\psi_{\sigma,m}{}^{(2)}$ are not in the domain of the operators Γ_μ for $\mu\neq3$. Thus, it is necessary to verify in detail that these solutions do submit to a delta-function normalization with respect to Γ_0 (since there is a continuum of such solutions).

Since we are concerned with the solutions of the wave equation for a fixed spatial momentum $\mathbf{p}$ and various values of E, we can assume that the momentum $\mathbf{p}$ is directed along the positive third axis. Thus we are dealing essentially with Eqs. (3.5). Further, since Γ_0 commutes with the helicity operator, we can work with one fixed value of the helicity; in other words, we can restrict ourselves to Eq. (3.5), in which it is understood that Γ_0, Γ_3, and the Lorentz generator K_3 generate a single discrete UIR of the unphysical $\tilde{O}(2,1)$ group.[11,12] If the fixed value of the helicity is λ, the UIR of $\tilde{O}(2,1)$ involved is $D_{|\lambda|+\frac{1}{2}}{}^{(+)}$. For brevity we shall write $k=|\lambda|+\frac{1}{2}$.

Let us first consider the normalization of the timelike solutions. For a given value of the spin s, and corresponding mass $M=\kappa/(s+\frac{1}{2})$, the solution of Eq. (3.5),

$$(\Gamma^0E-\Gamma^3p-\kappa)\varphi_\lambda(E,p)=0,$$

where

$$E=[p^2+\kappa^2/(s+\tfrac{1}{2})^2]^{1/2},$$

is given, according to (3.8) and (3.11), by

$$\varphi_\lambda(E,p)=e^{i\zeta(s,p)K_3}\psi_{s,\lambda}{}^{(1)},\quad \tanh\zeta(s,p)=p/[p^2+\kappa^2/(s+\tfrac{1}{2})^2]^{1/2}. \tag{4.4}$$

[11] On the $O(2,1)$ representations in an $O(1,1)$ basis: E. C. G. Sudarshan, in *Proceedings of the Coral Gables Conference* (W. J. Freeman and Co., San Francisco, 1966); and N. Mukunda, J. Math. Phys. **8**, 2210 (1967).

[12] J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, J. Math. Phys. **9**, 2100 (1968).

Since

$$e^{-i\zeta K_3}\Gamma^0 e^{i\zeta K_3}=\Gamma^0\cosh\zeta+\Gamma^3\sinh\zeta\,, \qquad (4.5)$$

and since Γ^3 has no matrix elements connecting $\psi_{s,\lambda}^{(1)}$ to itself [see (2.22) and (2.23)], we get

$$\varphi_\lambda^\dagger(E,P)\Gamma^0\varphi_\lambda(E,p)=\cosh\zeta(s,p)\ (s+\tfrac{1}{2}) = [\kappa/M^2(s)][p^2+M^2(s)]^{1/2}. \qquad (4.6)$$

On the other hand, from (3.5) we can show easily that two solutions corresponding to different values of mass and spin are orthogonal with respect to Γ^0. The matrix elements that enter in this demonstration are all finite, since any matrix Γ_μ acting on one of the states $\psi_{s,m}^{(1)}$ gives back a finite linear combination of these states. We therefore choose the timelike solution corresponding to mass $\kappa/(s+\frac{1}{2})$, spin s, helicity λ, and momentum p in the third direction as

$$\psi_{\lambda,p}^{M(s),s}=M(s)\kappa^{-1/2}[p^2+M^2(s)]^{-1/4}\times e^{i\zeta(s,p)K_3}\psi_{s,\lambda}^{(1)}, \qquad (4.7)$$

so that

$$\psi_{\lambda',p}^{M(s'),s'}\Gamma^0\psi_{\lambda,p}^{M(s),s}=\delta_{s's}\delta_{\lambda'\lambda}. \qquad (4.8)$$

For the discussion of the continuum of spacelike solution, we use the realization of the UIR's $D_k^{(+)}$ of $\tilde{O}(2,1)$ described in the Appendix. In this realization, the operators Γ^0, Γ^3, and K^3 are linear differential operators in a real variable z, $0\leq z<\infty$. The infinite number of components of ψ is replaced by a dependence on the continuous variable z. The effect of a Lorentz transformation generated by K^3 is given by

$$e^{-i\zeta K_3}\psi(z)=e^{\dagger\zeta}\psi(ze^\zeta). \qquad (4.9)$$

For a fixed value of p, the values of the mass l that appear, and the possible values of E, are both parametrized by the variable ζ varying from $-\infty$ to $+\infty$ in (3.15). For the present, let us denote the solution of (3.5) for a given value of ζ by ζ itself. This solution is obtained by applying a Lorentz transformation to an appropriate eigenfunction of Γ^3. (These eigenfunctions have been given in detail in the Appendix.) Therefore, the solution to the equation

$$(\Gamma^0 l\sinh\zeta-\Gamma^3 l\cosh\zeta-\kappa)\varphi_\zeta(z)=0$$

is given by

$$\varphi_\zeta(z)=e^{-i\zeta K^3}\psi_{-(\kappa\cosh\zeta)/p}(z) = e^\zeta\frac{e^{-\pi\kappa/2l}}{(2\pi)^{1/2}}\frac{|\Gamma(k+i\kappa/l)|}{\Gamma(2k)}e^{(\kappa-1)\zeta} \times\Phi(k-i\kappa/l,\,2k;\,ize^\zeta)z^{k-1}\exp(-\tfrac{1}{2}ize^\zeta). \qquad (4.10)$$

[$\psi_{-\kappa/l}(z)$ is an eigenfunction of Γ^3 with eigenvalue $-\kappa/l$, and $l=p/\cosh\zeta$.] Let $\varphi_{\zeta'}(z)$ denote another such amplitude. In the space of the UIR $D_k^{(+)}$, the product $(\cdots,\cdots)$ involves an integration with respect to z from 0 to ∞. We denote this integral—with, however, the upper limit being a finite quantity Λ rather than ∞—by $(\cdots,\cdots)_\Lambda$. Then, using the differential equations obeyed by $\varphi_\zeta(z)$ and $\varphi_{\zeta'}(z)$ and the relation $\Gamma^3=\Gamma^0-\frac{1}{2}z$, we get

$$(\tanh\zeta-\tanh\zeta')(\varphi_{\zeta'},\Gamma^0\varphi_\zeta)_\Lambda = (1-\tanh\zeta')[(\varphi_{\zeta'},\Gamma^0\varphi_\zeta)_\Lambda-(\Gamma^0\varphi_{\zeta'},\varphi_\zeta)_\Lambda]. \qquad (4.11)$$

In each term here it is understood that Γ^0 acts as a differential operator on the function of z standing immediately to its right. Now the quantity in square brackets on the right-hand side of (4.11) can be evaluated as $\Lambda\to\infty$. First, since the integrand involved is a perfect differential, we have

$$(\varphi_{\zeta'},\Gamma^0\varphi_\zeta)_\Lambda-(\Gamma^0\varphi_{\zeta'},\varphi_\zeta)_\Lambda=z\varphi_\zeta(z)(d/dz)[z\varphi_{\zeta'}^*(z)] - z\varphi_{\zeta'}^*(z)(d/dz)[z\varphi_\zeta(z)].$$

Using next the asymptotic form of the confluent hypergeometric function,[13] we write, for large Λ,

$$\Lambda\varphi_\zeta(\Lambda)\approx(2/\pi)^{1/2} \times\cos[\tfrac{1}{2}\Lambda e^\zeta-(\kappa/l)\ln\Lambda-(\kappa/l)\zeta-\tfrac{1}{2}\pi k-\eta(\zeta)], \qquad (4.12)$$
$$\eta(\zeta)=\arg\Gamma(k-i\kappa/l).$$

Putting all this together we find that

$$(\tanh\zeta-\tanh\zeta')(\varphi_{\zeta'},\Gamma^0\varphi_\zeta)_\Lambda\simeq(e^\zeta/\pi)(1-\tanh\zeta') \times\sin\left[\frac{e^\zeta-e^{\zeta'}}{2}\Lambda+\kappa\left(\frac{1}{l'}-\frac{1}{l}\right)\times\ln\Lambda+\kappa\left(\frac{\zeta'}{l'}-\frac{\zeta}{l}\right)+\eta(\zeta')-\eta(\zeta)\right]$$
$$+(\text{oscillating terms which vanish at }\zeta'=\zeta). \qquad (4.13)$$

We can now divide both sides by the factor $\tanh\zeta-\tanh\zeta'$ and go to the limit $\Lambda\to\infty$. In doing so, we use the rule that

$$\lim_{\Lambda\to\infty}\frac{\sin(x'-x)\Lambda}{(x'-x)}=\pi\delta(x'-x), \qquad (4.14)$$

and also the rule that an oscillating quantity without a singular denominator is to be counted as vanishing in the limit. In this way we obtain

$$(\varphi_{\zeta'},\Gamma^0\varphi_\zeta)=\cosh\zeta\ \delta(\zeta'-\zeta). \qquad (4.15)$$

This detailed analysis shows that even though the spacelike states are generated by Lorentz transformations acting on eigenfunctions of the operator Γ^3, and that these eigenfunctions do not lie in the domain of Γ^0, *nevertheless the spacelike states can be normalized in the delta-function sense in the* Γ^0 *metric.* This happens because we are not required to compute directly the (nonexistent) matrix elements of Γ^0 between eigen-

[13] For the behavior of confluent hypergeometric functions, see *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I.

vectors of Γ^3. Between Γ^0 and these eigenvectors stand the finite Lorentz transformations generated by K_3.

It is more convenient to convert the delta function appearing in (4.15) to one involving Lorentz-invariant arguments. We define the normalized amplitude corresponding to mass $l(\sigma)$, spin-$(-\frac{1}{2}+i\sigma)$, helicity λ, space momentum (in the positive 3 direction) p, and sign of the energy ϵ, by a formula as close to (4.7) as possible:

$$\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma}=\frac{l(\sigma)}{\kappa^{1/2}[p^2-l^2(\sigma)]^{1/4}}e^{i\epsilon\zeta(\sigma,p)K_3}\psi_{\sigma,\lambda}{}^{(2)},$$
$$\tanh\zeta(\sigma,p)=[p^2-l^2(\sigma)]^{1/2}/p\,,\quad l(\sigma)=\kappa/\sigma\,,\qquad(4.16)$$
$$\epsilon=\pm\,,\quad(\kappa/p)\leq\sigma<\infty\,.$$

With this definition, we have

$$\psi_{\lambda',p,\epsilon'}{}^{l(\sigma'),\sigma'\dagger}\Gamma^0\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma}=\delta_{\epsilon'\epsilon}\delta_{\lambda'\lambda}\delta(\sigma'-\sigma)\,.\quad(4.17)$$

Thus one can think of (4.16) as an "analytic continuation" of (4.7), and of (4.17) as a continuation of (4.8). It is worth emphasizing that here we have a complete and unambiguous definition of the spacelike solutions, since the basic ingredient, namely, the eigenfunctions of Γ^3, are also unambiguously normalized by their scalar products with one another [see (2.28) and the Appendix]. From (4.16) follows (4.17). Note that ϵ is not a Lorentz-invariant quantity; also that in the definition of $\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma}$ appears the square root of the magnitude of the energy $\sqrt{|E|}$.

The last solution left to be discussed is the lightlike one. It is easy enough to write down the function of z that this solution corresponds to: It is obtained by applying the operator $e^{i\zeta K^3}$ to the eigenfunction of $\Gamma^0+\Gamma^3$ with eigenvalue $+1$ [E and p being parametrized as in (3.12)]. Thus the lightlike solution is given by

$$e^{\zeta}\delta(ze^{\zeta}-2)$$

up to a normalizing factor. Now the main question is whether this has a finite Γ^0 norm (in which case it would appear as a discrete contribution in the completeness relation involving all solutions for a fixed momentum p), or whether it has an infinite Γ^0 norm (in which case it is to be treated just as a limit of the spacelike solutions). It is not hard to convince oneself that the *lightlike solution has infinite* Γ_0 *norm, so that it does not have to be taken into account in a discrete form.*

We are now in a position to write down the basic completeness relation involving all the solutions of the wave equation for various energies and at a fixed momentum p. (That a timelike solution is orthogonal to a spacelike one in the Γ^0 metric is shown by a method similar to the one we used to normalize the spacelike ones.) Using (4.8) and (4.17), we have

$$\sum_{j,\lambda}\psi_{\lambda,p}{}^{M(s),s}\psi_{\lambda,p}{}^{M(s),s\dagger}\Gamma^0+\sum_{\epsilon,\lambda}\int_{\kappa/p}^{\infty}d\sigma\,\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma}\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma\dagger}\Gamma^0=1.\quad(4.18)$$

Since this is a relation at a fixed value of spatial momentum, all the timelike solutions, but only a subset of the spacelike ones, are present. The generalization to an arbitrary direction for p is achieved via a space rotation, and amounts to replacing p everywhere by $\mathbf{p}$. For $p=0$, the set of solutions corresponding to slower-than-light particles at rest by itself forms a complete set.

At this point, it is worthwhile clarifying certain aspects of the completeness relation (4.18) which are peculiar to our dealing with an infinite-dimensional spin space. The originally defined space of the unitary Majorana representations of $O(3,2)$ is a Hilbert space H in which a vector is assigned the norm $(\psi,\psi)^{1/2}=(\psi^\dagger\psi)^{1/2}$. With respect to the corresponding scalar product, Γ_μ is a Hermitian operator. Now the set of vectors ψ in H which have the added property that

$$(\psi,\Gamma^0\psi)=\psi^\dagger\Gamma^0\psi<\infty$$

forms a linear set D which is a proper subset of H because Γ^0 is unbounded. D is clearly dense in H; if completed with respect to the scalar product (ψ,ψ), then D yields H. However, D forms a Hilbert space in its own right if we define the norm in D to be the quantity $(\psi,\Gamma^0\psi)^{1/2}$. The completeness relation (4.18) actually refers to D viewed as a Hilbert space in this sense. In other words, if ψ is a vector in D such that $(\psi,\Gamma^0\psi)<\infty$, then there follows an expansion

$$\psi=\sum_{s,\lambda}a(s,\lambda)\psi_{\lambda,p}{}^{M(s),s}+\int_{\kappa/p}^{\infty}d\sigma\sum_{\epsilon,\lambda}b(\sigma,\epsilon,\lambda)\psi_{\lambda,p,\epsilon}{}^{l(\sigma),\sigma},\quad(4.19)$$

where $a(s,\lambda)$ is a sequence of complex numbers and $b(\sigma,\epsilon,\lambda)$ is a sequence of Lebesgue-measurable functions of σ, such that

$$\psi^\dagger\Gamma^0\psi=\sum_{s,\lambda}|a(s,\lambda)|^2+\sum_{\epsilon,\lambda}\int_{\kappa/p}^{\infty}d\sigma|b(\sigma,\epsilon,\lambda)|^2<\infty\,.\quad(4.20)$$

Since the space D is dense in H, one may expect that an expansion of the form (4.19) is valid also for vectors in H that are not in D. However, for such vectors, the right-hand side of (4.20) will diverge. [It is to be noted that if we wish to express the quantity $\psi^\dagger\psi$ in terms of the representatives $a(s,\lambda)$ and $b(\sigma,\epsilon,\lambda)$, one will obtain a nonlocal expression in these coefficients.]

We conclude this section with some comments on the behavior of the "current" $\mathcal{J}_\mu$ in the spacelike states. As we have noted previously, since Γ_0 is positive definite, the UIR's of $\tilde{O}(2,1)$ generated by Γ_0, Γ_3, and K_3 are of the discrete (positive) class. In such a case, the quantities $\Gamma_0+\Gamma_3$ and $\Gamma_0-\Gamma_3$ are both positive-definite operators. Since without loss of generality we could transform the charge current density at any point to have components only along the 0 and 3 axes, this leads us to assert that *the charge-current four-vector is always positive timelike.* We could deduce this alternatively by noting that $\mathcal{J}_\mu(x,t)$ transforms as a four-vector density whose time component is always positive definite.

In the case of a timelike solution, this is easy enough to understand; in fact, by routine manipulations one can show that the expectation value of Γ_μ in a given timelike state with four-momentum p_μ is directly proportional to p_μ apart from normalization factors involving the energy:

$$\psi_{\lambda',\mathbf{p}}{}^{M(s),s\dagger}\Gamma_\mu\psi_{\lambda,\mathbf{p}}{}^{M(s),s}=(p_\mu/E)\delta_{\lambda'\lambda}. \tag{4.21}$$

What seems somewhat puzzling is that the expectation value of Γ_μ in a state with spacelike four-momentum p_μ, assuming such an expectation value exists, should turn out to be timelike.

In order to examine this point further, for the moment we only consider solutions of the wave equation in one space and one time dimension, namely, those based on the operators Γ_0, Γ_3, K_3 and involving a single UIR of $\tilde{O}(2,1)$. In this case, K_3 is the (sole) generator of physical Lorentz transformations. Let φ and φ' be two solutions corresponding, respectively, to momentum two-vectors (E,p) and (E',p') both of which are spacelike, i.e., $p^2>E^2$ and $p'^2>E'^2$, so φ and φ' obey

$$\begin{aligned}(\Gamma^0E-\Gamma^3p-\kappa)\varphi&=0,\\(\Gamma^0E'-\Gamma^3p'-\kappa)\varphi'&=0.\end{aligned} \tag{4.22}$$

If the "rapidities" corresponding to φ and φ' are ζ and ζ', and the masses are l and l' ($\tanh\zeta=E/p$, etc.), then φ is obtained from an eigenfunction of Γ_3 with eigenvalue $+\kappa/l$:

$$\varphi=e^{i\zeta K_3}\psi_{+\kappa/l}, \tag{4.23}$$

and similarly for φ'. Now from (4.22) we deduce that

$$\begin{aligned}\varphi'^\dagger\Gamma^0\varphi&=\frac{p'-p}{p'E-pE'}\kappa, & \varphi'^\dagger\varphi&=\frac{(p'-p)\kappa}{l'l\sinh(\zeta-\zeta')}\varphi'^\dagger\varphi,\\ \varphi'^\dagger\Gamma^3\varphi&=\frac{E'-E}{p'E-pE'}\kappa, & \varphi'^\dagger\varphi&=\frac{(E'-E)\kappa}{l'l\sinh(\zeta-\zeta')}\varphi'^\dagger\varphi.\end{aligned} \tag{4.24}$$

As long as ζ and ζ' are different, both $\sinh(\zeta-\zeta')$ and $\varphi'^\dagger\varphi$ are finite and nonzero quantities. The latter is in fact the matrix element between eigenvectors of Γ_3, with eigenvalues κ/l' and κ/l of the finite Lorentz transformation $e^{-i(\zeta'-\zeta)K_3}$ generated by K_3, and as long as $\zeta'-\zeta\neq0$, this is a finite quantity. Further, both $\zeta'-\zeta$ and $\varphi'^\dagger\varphi$ are invariant when one and the same Lorentz transformation is applied to both φ and φ'. From this we see that the matrix elements of Γ^0 and Γ^3 do behave as a two-vector, since in the degenerate case of only one space dimension (p,E) forms a two-vector if (E,p) does. However, if we now try to take the limit $\varphi'\to\varphi$ in (4.26), all the expressions diverge. As for the matrix element of Γ^0, we know that we get $\delta(E'-E)$ on taking the limit $p'\to p$ and that this would become infinite when we set $E'=E$. The situation is similar for Γ^3. Thus, in a spacelike state which is an eigenstate of energy momentum, the operators Γ_μ have no well-defined expectation values; they either diverge or are nonexistent. The trouble arises essentially because an attempt to evaluate such an expectation value amounts to evaluating "matrix elements" of Γ_μ between eigenvectors of the "noncompact" operator Γ_3, and such things do not exist.

In the realistic case of three-dimensional space it is not possible to rewrite the matrix elements of Γ_μ between two arbitrary spacelike states in as simple a form as the above. However, one can express them in terms of the "kinematic" operators $S_{\mu\nu}$ and the four-momenta p and p' corresponding to solutions φ and φ' of the Majorana equation. For this, one can make use of the identity (2.17). The expression is

$$\begin{aligned}4\kappa(\varphi',\Gamma_\mu\varphi)=&-P_\mu(\varphi',\varphi)\\&+P^\lambda(\varphi',\{S_{\mu\nu},S_\lambda{}^\nu\}\varphi)+iQ^\lambda(\varphi',S_{\mu\lambda}\varphi),\\&P=p'+p,\quad Q=p'-p.\end{aligned} \tag{4.25}$$

From these remarks one sees that the current operator Γ_μ has finite matrix elements between two spacelike states if these four-momenta do not coincide; but these diverge if we allow the four-momenta to approach one another. This fact prevents us from a direct verification of the statement that even in a spacelike state, the expectation value of the current operator is a timelike four-vector.

These properties are of interest in connection with the emission of radiation by spacelike particles. Assuming that the coupling to the electromagnetic field is via the operator Γ_μ, we see that there is a finite amplitude for the spontaneous emission of a photon by a spacelike particle. For, in such a process the photon would carry away a nonvanishing four-momentum k_μ, so that the initial and final momenta, p_μ and p_μ', of the spacelike particle would differ by the amount k_μ. This transition would generally result in a change in the mass l of the spacelike particle. Thus, at least in lowest order of perturbation theory, there is a *finite process* that could lead to emission of radiation by such particles, if they exist, a process quite distinct from the Čerenkov effect.

V. THEORY OF QUANTIZED MAJORANA FIELD

While Eqs. (3.1) and (3.2) admit solutions for slower-than-light and lightlike particles, in a proper quantum theory we expect to be able to have both positive- and negative-frequency parts of the field operator so as to be able to describe both absorptions and emissions of the particles described by Eqs. (3.1) and (3.2). In the more familiar case of finite-component relativistic wave fields, the positive- and negative-frequency parts are irreducibly contained in the same local operator. The decomposition into positive- and negative-frequency parts is, in these cases, a nonlocal operation. In the present case, however, the negative-frequency class-I or -II solutions do not appear. Instead, the irreducible parts contain the infinity of solutions which belong to the three classes of particles that we have described. It therefore follows that the construction of a quantum theory of this field would involve the introduction of another field with negative-frequency solutions of classes I and II. The necessity of doing this for class III, namely, of assigning the complete set of solutions of the primitive field to only annihilation operators and the need to introduce a conjugate field with only creation operators of class III was elsewhere demonstrated for a simple scalar field of faster-than-light particles.[3] This suggests that the proper method of quantization is one in which the entire Majorana field $\psi(x)$ obeying (3.1) is assigned to absorption operators, and in which a conjugate local field obeying a different equation and consisting entirely of emission operators is to be introduced. We must further demand that the absorption part of the field and the emission part of the (conjugate) field enter the dynamics of the quantized field on the same basis; and they can be interchanged without altering the dynamical law. This property is automatically present in usual finite-component field theories and leads to the so-called substitution law. We would have to involve the symmetry between emission and absorption as a constraint on the dynamical law; in this wider context we shall refer to the requirement of symmetry between emission and absorption as the S principle.[4]

Let us now return to the problem of quantization of the Majorana field. We could deduce (3.1) as an Euler-Lagrange equation from an action principle where the Lagrangian density is

$$\mathfrak{L}=\psi^\dagger\tfrac{1}{2}i(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)\Gamma^\mu\psi-\kappa\psi^\dagger\psi\,, \tag{5.1}$$

where $\psi^\dagger$ is the Hermitian conjugate of ψ. If (5.1) is considered as a *classical* Lagrangian density, the momentum-density conjugate to ψ is given by

$$\pi_r=\partial\mathfrak{L}/\partial\dot{\psi}_r=i(\psi^\dagger\Gamma^0)_r. \tag{5.2}$$

The Weiss-Schwinger action principle also leads to the Poisson-bracket relation

$$[\psi(\mathbf{x}),i\psi^\dagger(\mathbf{y})\Gamma^0]_{\mathrm{P.b.}}=\mathbf{1}\times\delta(\mathbf{x}-\mathbf{y}). \tag{5.3}$$

The four-vector

$$\mathcal{J}_\mu(x)=\psi^\dagger(x)\Gamma_\mu\psi(x) \tag{5.4}$$

is conserved by virture of the field equations, and the Lorentz-invariant quantity

$$Q=\int\psi^\dagger(\mathbf{x})\Gamma^0\psi(\mathbf{x})d^3x \tag{5.5}$$

is time-independent. The quantity Q could be seen to be the generator of gauge transformations in the field $\psi(x)$:

$$[Q,\psi(x)]_{\mathrm{P.b.}}=i\psi(x). \tag{5.6}$$

In a similar manner we can construct the dynamical variables of energy and momentum:

$$H=\int\psi^\dagger(x)(\mathbf{\Gamma}\cdot\mathbf{p}+\kappa)\psi(x)d^3x,$$
$$\mathbf{P}=\int\psi^\dagger(x)\Gamma_0\mathbf{p}\psi(x)d^3x, \tag{5.7}$$

which act as the generators of time and space translations.

In constructing this Hamiltonian version of the classical theory it was necessary to introduce the quantities $\psi^\dagger(x)$, which obey a different wave equation. Hence the theory requires the use of two fields, ψ and $\psi^\dagger$.

We can now proceed to construct a quantum field theory for the Majorana field along parallel lines. But the theory so constructed would not necessarily possess any symmetry between the positive- and negative-frequency solutions. If this symmetry is required, we should do two things: First, we should consider both $\psi(x)$ and $\psi^{\dagger T}(x)$ as field variables to be treated on an equal footing. Second, we should demand a certain symmetry under the exchange of the positive- and negative-frequency parts. This is characteristic of all finite-component relativistic fields, but we impose it in the form of the demand that the action be invariant under the interchange of $\psi(\mathbf{x},t)$ and $V\psi^{\dagger T}(+\mathbf{x},-t)$. Here V is the real matrix $V=\exp(i\pi J_2)=\exp(\frac{1}{2}i\pi\eta^\dagger\sigma_2\eta)$ chosen to ensure that the two quantities ψ and $V\psi^{\dagger T}$ transform in the same manner, so that the interchange is explicitly Lorentz-invariant.

We now use the action principle in the form

$$[\psi(x),\delta A]=+i\delta\psi(x). \tag{5.8}$$

We choose, in place of (5.1), the amended action

$$A=\tfrac{1}{4}i\int d^4x[\psi^\dagger(\mathbf{x},t)(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)\Gamma^\mu\psi(\mathbf{x},t)$$
$$+\psi^T(\mathbf{x},-t)V^{-1T}(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)\Gamma^\mu V\psi^{\dagger T}(\mathbf{x},-t)]. \tag{5.9}$$

But we know that V is antisymmetric for the two-valued representation (half-integral spins) and symmetric for the one-valued representation (integral spins). In either case Γ^0 commutes with V. Hence, at $t=0$, we have

$$\int d^3y[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})\Gamma^0\delta\psi(\mathbf{y})+\delta\psi^T(\mathbf{y})V^{-1T}\Gamma^0V\psi^{\dagger T}(\mathbf{y})]$$
$$=\int d^3y[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})\Gamma^0\delta\psi(\mathbf{y})\mp\delta\psi^T(\mathbf{y})\Gamma^0\psi^{\dagger T}(\mathbf{y})]=\delta\psi(\mathbf{x}), \tag{5.10}$$

with the $\mp$ signs chosen according as V is antisymmetric (half-integral spin) or symmetric (integral spins). We still have the freedom to choose the field variation $\delta\psi(\mathbf{y})$ to commute or anticommute with all the field quantities. Consistency of (5.10) demands that anticommuting variations be chosen for the half-integral-spin case (with V antisymmetric) and commuting variations be chosen for the integral spin case. Accordingly, we obtain the fundamental relations

$$[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})]_\pm=(\Gamma^0)^{-1}\delta(\mathbf{x}-\mathbf{y}) \tag{5.11}$$

and the standard connection between spin and statistics.

We now expand the fields ψ and $\psi^\dagger$ in terms of annihilation and creation operators, respectively. For a given three-momentum $\mathbf{p}$, we can for brevity label the set of spinors appearing in (4.18) by the energy E $(-|\mathbf{p}|\leq E<\infty)$, and indicate the process of summing over the timelike ones and integrating over the spacelike ones by a formal integration with respect to E. Omitting also the helicity labels, we write

$$\begin{aligned}\psi(x)&=\int d^3p\int dE\; e^{-ip\cdot x}\psi_{\mathbf{p},E}a(\mathbf{p},E)\,,\\ \psi^\dagger(x)&=\int d^3p\int dE\; e^{ip\cdot x}\psi_{\mathbf{p},E}{}^\dagger a(\mathbf{p},E)\,.\end{aligned} \tag{5.12}$$

Then the orthonormality and completeness properties of $\psi_{\mathbf{p},E}$ at fixed $\mathbf{p}$ (and with respect to Γ_0) together with (5.11) imply

$$[a(\mathbf{p},E),a^\dagger(\mathbf{p}',E')]_\pm=\delta(\mathbf{p}-\mathbf{p}')\delta(E-E')\,. \tag{5.13}$$

In terms of these operators we can rewrite the particle number, energy, and momentum operators:

$$\begin{aligned}N&=\int d^3p\int dE\; a^\dagger(\mathbf{p},E)a(\mathbf{p},E)\,,\\ H&=\int d^3p\int dE\; Ea^\dagger(\mathbf{p},E)a(\mathbf{p},E)\,,\\ \mathbf{P}&=\int d^3p\int dE\; \mathbf{p}a^\dagger(\mathbf{p},E)a(\mathbf{p},E)\,.\end{aligned} \tag{5.14}$$

These expressions enable us to interpret the second-quantized Majorana fields as assemblies of an infinite number of types of quantum-mechanical particles obeying Bose or Fermi statistics according as the spin is integral or is half-integral. This association extends also to the spacelike particles, the relevant thing then being whether the representation of the little group is single or double valued.

APPENDIX: THE DISCRETE UIR'S OF $O(2,1)$

Here we describe a few of the properties of the positive discrete class UIR's of the group $O(2,1)$. There is one such UIR for each value of a parameter k, where k goes over the range $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$. These are denoted as $D_k{}^{(+)}$. The three generators J_0, J_1, and J_2 obey

$$[J_0,J_1]=iJ_2\,,\quad [J_0,J_2]=-iJ_1\,,\quad [J_1,J_2]=-iJ_0 \tag{A1}$$

and are Hermitian. J_0 generates the compact $O(2)$ groups, and its eigenvalues m in $D_k{}^{(+)}$ consist of $m=k$, $k+1$, $\cdots$, ∞. J_1 and J_2 both generate the noncompact $O(1,1)$ group, and each of them has every real number, both positive and negative, as eigenvalues, with exactly one eigenvector per eigenvalue (in the UIR's $D_k{}^{(+)}$). The operators $J_0\pm J_1$ are of parabolic type, are positive definite, and have every positive real number as eigenvalues.

Elsewhere we have described a construction of these UIR's in a Hilbert space H of functions on the positive real line. Elements of H are functions $f(z)$ of a real variable z in the range $0\leq z<\infty$, and the scalar product of two functions f and g is defined by

$$(g,f)=\int_0^\infty zdz\; g^*(z)f(z)\,. \tag{A2}$$

In this space, the generators are given as linear differential operators as follows:

$$\begin{aligned}J_0&=-\frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz}+\frac{k(k-1)}{z}+\tfrac{1}{4}z\,,\\ J_1&=-\frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz}+\frac{k(k-1)}{z}-\tfrac{1}{4}z\,,\\ J_2&=-i\left(z\frac{d}{dz}+1\right).\end{aligned} \tag{A3}$$

One may verify explicitly that the Casimir operator $Q=(J_1)^2+(J_2)^2-(J_0)^2$ has the value $k(1-k)$.

We turn now to the determination of the eigenvectors of the generators and the scalar products between them. Let us write φ_m, ψ_λ, and χ_μ for the eigenvectors of J_0, J_1, and J_2, respectively:

$$\begin{aligned}&J_0\varphi_m=m\varphi_m\,,\quad J_1\psi_\lambda=\lambda\psi_\lambda\,,\quad J_2\chi_\mu=\mu\chi_\mu\,,\\ &m=k,\,k+1,\cdots,\infty\,;\quad -\infty<\lambda,\,\mu<\infty\,.\end{aligned} \tag{A4}$$

In properly normalized form, these are found to be

$$\begin{aligned}\varphi_m(z)&=\frac{1}{\Gamma(2k)}\left[\frac{\Gamma(m+k)}{\Gamma(m+1-k)}\right]^{1/2}z^{k-1}e^{-z/2}\Phi(k-m,\,2k;\,z)\,,\\ \psi_\lambda(z)&=\frac{e^{\pi\lambda/2}}{(2\pi)^{1/2}}\frac{|\Gamma(k-i\lambda)|}{\Gamma(2k)}\times z^{k-1}e^{-iz/2}\Phi(k+i\lambda,\,2k;\,iz)\,,\\ \chi_\mu(z)&=\frac{1}{(2\pi)^{1/2}}z^{i\mu-1}\,.\end{aligned} \tag{A5}$$

(Φ is the confluent hypergeometric function.) Each set of eigenvectors obeys the following orthormality and completeness relationships:

$$(\varphi_{m'},\varphi_m)=\delta_{m'm}\,,\quad (\psi_{\lambda'},\psi_\lambda)=\delta(\lambda-\lambda')\,,$$
$$(\chi_{\mu'},\chi_\mu)=\delta(\mu'-\mu)\,,$$
$$\sum_{m=k}^{\infty}\varphi_m(z)\varphi_m(z')=\int_{-\infty}^{\infty}d\lambda\,\psi_\lambda(z)\psi_\lambda{}^*(z') \tag{A6}$$
$$=\int_{-\infty}^{\infty}d\mu\,\chi_\mu(z)\chi_\mu{}^*(z')=\frac{1}{z}\delta(z-z')\,.$$

It is worth noting that with our choice of the eigenvectors φ_m of J_0, the generator J_1 as given in (A3) has the standard form

$$J_1\varphi_m=\tfrac{1}{2}[(m-1)(m+k-1)]^{1/2}\varphi_{m-1}+\tfrac{1}{2}[(m+k)(m-k+1)]^{1/2}\varphi_{m+1}. \tag{A7}$$

Next we turn to the scalar products between these eigenvectors. Between eigenvectors of J_0 and J_1, we find

$$\mathcal{A}_k(m,\lambda)=(\varphi_m,\psi_\lambda)=\frac{1}{(2\pi)^{1/2}}\frac{1}{\Gamma(2k)}\left[\frac{\Gamma(m+k)}{\Gamma(m+1-k)}\right]^{1/2}$$
$$\times 2^k|\Gamma(k-i\lambda)|e^{\frac{1}{2}i\pi(m-k)}F(k-m,\,k+i\lambda;\,2k;\,2)\,. \tag{A8}$$

Using this expression, one can write each eigenvector of J_1 in terms of those of J_0:

$$\psi_\lambda(z)=\sum_{m=k}^{\infty}\mathcal{A}_k(m,\lambda)\varphi_m(z) \tag{A9}$$

and vice versa. The scalar product of a φ with a χ, and of a χ with a ψ, can also be worked out (though we did not use these in the text). They are

$$(\varphi_m,\chi_\mu)=\frac{1}{(2\pi)^{1/2}}\frac{1}{\Gamma(2k)}\left[\frac{\Gamma(m+k)}{\Gamma(m+1-k)}\right]^{1/2}2^{k+i\mu}\Gamma(k+i\mu)F(k-m,\,k+i\mu;\,2k;\,2)\,,$$
$$(\chi_\mu,\psi_\lambda)=(1/2\pi)e^{i[\eta_k(\mu)-k\pi/2]}2^{-i\mu}[e^{\pi(\lambda-\mu)/2}e^{i[\eta_k(\mu)-\eta_k(\lambda)]}2^{-i\lambda}\Gamma(i\mu-i\lambda)F(k+i\lambda,\,1-k+i\lambda,\,1+i\lambda-i\mu;\,\tfrac{1}{2}i)$$
$$+e^{\pi(\mu-\lambda)/2}e^{i[\eta_k(\lambda)-\eta_k(\mu)]}2^{-i\mu}\Gamma(i\lambda-i\mu)F(k+i\mu,\,1-k+i\mu,\,1+i\mu-i\lambda;\,\tfrac{1}{2}i)]. \tag{A10}$$

Lastly, we consider the parabolic generators $J_0\pm J_1$. The combination J_0-J_1 is particularly simple, since it is just equal to $\frac{1}{2}z$. Therefore, the solutions to the equations

$$(J_0-J_1)\xi_\nu(z)=\nu\xi_\nu(z)\quad(0\le\nu<\infty) \tag{A11}$$

are

$$\xi_\nu(z)=\nu^{-1/2}\delta(z-2\nu) \tag{A12}$$

and obey

$$(\xi_{\nu'},\xi_\nu)=\delta(\nu'-\nu)\,. \tag{A13}$$

From this one sees that

$$\mathcal{B}_k(m,\nu)=(\varphi_m,\xi_\nu)=\frac{1}{\Gamma(2k)}\left[\frac{\Gamma(m+k)}{\Gamma(m+1-k)}\right]^{1/2}2^k\nu^{k-1/2}e^{-\nu}$$
$$\times\Phi(k-m,\,2k;\,2\nu)\,, \tag{A14}$$

and there follows an expansion of the eigenvectors of J_0-J_1 in terms of those of J_0:

$$\xi_\nu(z)=\sum_{m=k}^{\infty}\mathcal{B}_k(m,\nu)\varphi_m(z)\,. \tag{A15}$$

The eigenvectors of the other parabolic generator J_0+J_1, normalized in the same way as ξ_ν, turn out to be

$$\zeta_\nu(z)=\frac{1}{(2z)^{1/2}}J_{2k-1}(\sqrt{(2\nu z)})\,.$$

In the use we have made of these results, we have identified the generators Γ_0, Γ_3, and K_3 of $\tilde{O}(2,1)$ with the J's in the following way:

$$\Gamma_0\to J_0\,,\quad \Gamma_3\to -J_1\,,\quad K_3\to J_2 \tag{A16}$$

and of course, k corresponds to $|m|+\frac{1}{2}$.