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Constraint dynamics of particle world lines

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The Dirac generator fomulations for relativistic Hamiltonian dynamics is extended by explicitly separating the question of dynamical evolution in an inertial frame from that of changes of frame. We obtain a new eleven-generator formalism: ten to realize the Poincaré Lie algebra, and one to provide equations of motion. For point-particle systems, a new form of the world-line conditions is developed. It is demonstrated that with these extensions one can consistently describe classical point particles with interaction in a relativistically invariant way while maintaining invariant world lines. This paper uses the approach based on independent particle variables to set up such theories.

I. INTRODUCTION

The requirement of special-relativistic invariance for a classical dynamical system encompasses two distinct notions: one is the identity of dynamical laws in all inertial frames, and the other is manifest covariance. The independence of these two notions is seen most clearly within the canonical formalism, to which one is led automatically if one has a Lagrangian starting point. With a Lagrangian one has a definite parameter of evolution with respect to which one has differential equations of motion. From the Lagrangian description one can pass to the equivalent Hamiltonian one based on a phase space and the idea of Poisson brackets; some definite set of phase-space variables, referring to physical conditions at a common value of the evolution parameter, is identified as forming a complete set of variables, and suitable bracket relations are postulated among them. Dynamical evolution within an inertial frame, as well as the passage from one frame to another, are both represented by canonical transformations on the phase space. Such a Hamiltonian description can be set up directly even in the absence of a Lagrangian. The identity of physical laws in all frames is then guaranteed by having a realization of the Poincaré group by canonical transformations on phase space. The added requirement that for selected dynamical variables the canonical transformation laws under the elements of the Poincaré group be compatible with an independently stated geometrical transformation law expresses manifest covariance.

To set up a classical relativistic Hamiltonian theory one must therefore start with a suitable phase space and then do two things: a definite parameter of evolution must be specified, and eleven distinguished functions on phase space must be given. The first of these is the generator of the canonical transformations describing dynamical evolution in any inertial frame; the remaining ten generate a canonical realization of the Poincaré group and so describe changes of frame. It is the latter ten generators that must provide, via their brackets, a realization of the Lie algebra of the Poincaré group. Conditions of manifest covariance provide additional restrictions.

Many years ago Dirac[1] proposed three natural forms of Hamiltonian relativistic dynamics, namely the instant, point, and front forms. These correspond essentially to three different ways in which one might pick a parameter of evolution out of the four space-time coordinates that are assigned to an event in special relativity. Dirac's list of possibilities is not exhaustive in the following crucial sense: It was assumed that the parameter of evolution is chosen kinematically, i.e., in the same way for all the states of motion. The instant form, for example, uses the "laboratory time", one of the four coordinates assigned by an inertial observer to each space-time event, as the evolution parameter. Thus in this form the generator of dynamical evolution in one frame coincides with that element in the Poincaré algebra that generates time displacements between inertial frames. More generally in each of Dirac's forms of dynamics only ten fundamental quantities need to be specified, fulfilling the bracket relations of the Poincaré algebra; the eleventh generator is always one of the ten, or a suitable linear combination of them. The fundamental phase-space variables, out of which the ten generators are built, vary of course from one frame to another.

The attempt to describe a collection of classical relativistic point particles within the Dirac program, with a condition of manifest covariance included, led to the astonishing and deep result that the particles must necessarily be free and no interactions are possible. To be specific, this important no-interaction theorem[2] was proved within Dirac's instant form of dynamics with the following detailed assumptions: (i) The set of three-dimensional position coordinates of all the particles at a common laboratory time forms one half of a system of canonical variables in the phase space of the entire system; (ii) Under the Euclidean subgroup of the Poincaré group (characteristic of the instant form) the canonical and geometrical transformation laws for these coordinates coincide; (iii) If in any state of motion, as seen in one inertial frame, the world lines of the particles are imagined drawn in space-time, then the canonical rules of transformation that transfer the description to another inertial frame preserve the objective reality of these world lines. Assumptions (ii) and (iii) express the idea of manifest covariance in the present context.

The fact that the objective reality of world lines is a definite condition not implied by the structure relations of the Lie algebra of the Poincaré group was recognized long ago by Pryce.[3] The explicit expression of this condition in the language of Poisson brackets and generators on phase space was given by Currie *et al.*,[2] and is called the world-line condition (WLC). This condition was given by them within Dirac's instant form of dynamics, but as we shall see the idea itself is much more general.[4] As we said above, Dirac's program uses only ten independent generators and not eleven. In retrospect one can see that it is the fact that the ten generators have to do double duty, namely, obey the Lie relations of the Poincaré group on the one hand, and obey the WLC on the other, that is the fundamental origin of the no-interaction theorem.

Recently there have been several attempts[4,5] to set up theories of classical relativistic interacting point particles which are designed to avoid the no-interaction theorem, even though they are described in a generalized Hamiltonian framework. All of them are constructed in the constrained Hamiltonian formalism. This formalism was invented by Dirac[6] to express a theory based on a singular Lagrangian, in a generalized phase-space form; and introduced the ideas of constraints and Dirac brackets in that context. However, as Dirac himself pointed out, one can use these ideas directly even in the absence of a Lagrangian. One starts out with a suitable phase space, adopts some number of algebraically independent constraints, and some Hamiltonian compatible with the constraints; one can then write a generalized Hamiltonian equation of motion with respect to some (specified or unspecified) parameter of evolution. Such a formalism has two characteristic features[7]: The initial phase space invariably has more independent coordinates than are needed for the physical system one intends to describe ultimately; (ii) The physical identification of the variables in terms of particles, and the final system of brackets, must both be delayed until all necessary constraints have been set down.

It is important to understand exactly how one has succeeded in avoiding the no-interaction theorem, i.e., precisely which assumption or assumptions underlying the theorem have been given up. One gets the impression from the literature that what has been given up is the existence of objective world lines; i.e., the WLC. If this is so, it is hard to admit that recent work constitutes a definite advance over what had been known for a long time. Indeed, soon after Dirac's paper on the forms of relativistic dynamics, but well before the no-interaction theorem was proved, Thomas[8] explicitly suggested that the objective reality of world lines be given up, and interacting theories of relativistic point particles be set up in Dirac's instant form by constructing ten generators for the Poincaré group. Such theories were presented by Bakamjian and Thomas,[9] but were soon shown by Foldy[10] to have a serious physical defect: they did not possess the cluster decomposition property. This defect shows in the recently presented models that attempt to escape the no-interaction theorem.

It appears to us that there is no physically well-founded reason at the classical level to give up the objective reality of world lines for point particles, unless one points to the no-interaction theorem itself. We intend to show that the way out of this impasse lies in an altogether different direction[11]: One must go beyond the boundaries of Dirac's program for relativistic dynamics, and envisage choices of evolution parameter that are dynamically, not kinematically, determined. In such a framework, all the eleven generators for a relativistic Hamiltonian theory enter with independent and equal status. The impossibly strong conditions that the ten generators of the Poincaré group obey the bracket relations of the Lie algebra of that group, contain interaction, and obey the WLC, get weakened: the WLC now need to be obeyed by, or are a condition on, the generators of the Poincaré group and the eleventh generator of dynamical evolution. It is in this way that the existence of interaction and of objectively real world lines become compatible with one another: the implicit assumption of a clear-cut separation of kinematics and dynamics underlying all of Dirac's forms of relativistic dynamics, must be given up. The formalism of constrained Hamiltonian dynamics is then to be viewed as a very convenient means for the construction of such theories.

The problem of cluster decomposition[10] remains, however, unresolved.

Since the possibility of describing a relativistic Hamiltonian theory with all eleven generators is novel and unfamiliar, we start in Sec. II with a description of a free single relativistic particle in such a formalism. The case of two interacting particles is taken up in Sec. III. It is explicitly shown here that one recovers the no-interaction theorem if one adopts a system of constraints that reduce the framework to Dirac's instant form; but that with use of a different set of constraints and the eleven-generator formalism, both interactions and invariant world lines can coexist. Basic to this demonstration are a careful analysis of what exactly constitutes a state of motion for the two-particle system, and a new form of the WLC. The extension to a system of N particles occupies Sec. IV. In both Secs. III and IV, we use independent particle variables rather than, say, "center of mass" and relative ones, and the evolution parameter is essentially the time in the center-of-momentum frame. Other ways of choosing the evolution parameter are described in Sec. V. The paper ends with concluding remarks in Sec. VI.

II. SINGLE FREE PARTICLE

Our objective is to describe a free relativistic point particle with mass m in a formalism flexible enough to allow for different choices of evolution parameter. We will also develop the WLC in this formalism, and show how and when it reduces to the form in which it was used in the proof of the no-interaction theorem.

To this end we begin with an eight-dimensional phase space Γ with basic independent variables x^μ, p^μ, and postulate the Poisson bracket relations

$$\{x^\mu, x^\nu\}=0, \quad \{x^\mu, p^\nu\}=g^{\mu\nu}, \quad \{p^\mu, p^\nu\}=0 . \tag{1}$$

Denote a general element of the Poincaré group by (Λ, a). Then the mappings of Γ onto itself given by

$$R(\Lambda, a): \ x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{2}$$

evidently preserve the brackets (1) and so are canonical transformations. This canonical realization $(\Lambda, a) \to R(\Lambda, a)$ obviously has the following set of infinitesimal generators:

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad P_\mu = p_\mu . \tag{3}$$

The brackets among these generators reproduce the relations of the Lie algebra of the Poincaré group.

We now impose the constraint

$$K = p^2 - m^2 \approx 0 . \tag{4}$$

The function K serves two purposes: on the one hand it defines a constraint hypersurface Σ lying within Γ and having dimension 7; on the other hand it can be used as the generator of canonical transformations on Γ which have the property of mapping Σ into itself. As for the former role, it is clear, since K is invariant under the transformations $R(\Lambda, a)$, i.e., since

$$\{K, J_{\mu\nu}\} \approx 0, \quad \{K, P_\mu\} \approx 0 , \tag{5}$$

the region Σ defined by the vanishing of K is invariant under the canonical mappings $R(\Lambda, a)$:

$$R(\Lambda, a)\Sigma = \Sigma . \tag{6}$$

Turning to the second role, suppose we start with some point (x, p) in Σ and then apply to it the one-parameter family of canonical transformations generated by K; we then build up a line L, the orbit of (x, p) under this group of transformations. All of L will clearly lie in Σ. One can set up a system of differential equations with respect to an unspecified independent variable σ, say, by solving which we can find the line L:

$$\frac{dx^\mu(\sigma)}{d\sigma} \approx v\{x^\mu(\sigma), K\}, \quad \frac{dp^\mu(\sigma)}{d\sigma} \approx v\{p^\mu(\sigma), K\}, \qquad x^\mu(0)=x^\mu, \quad p^\mu(0)=p^\mu . \tag{7}$$

Here v is an arbitrary multiplier. As a result the line L is determined once (x, p) is given, but the precise value of σ to be assigned to each point on L is left free, since it depends on the choice of v.

The seven-dimensional Σ is thus the union of one-dimensional lines L, so the latter constitute a six-parameter family. The relations (5) which earlier led to the consequence (6) can now be "read" in another way: in its action on Σ, each transformation $R(\Lambda, a)$ will carry a line L onto another line L'.

Up to this point the development has been purely mathematical, and we have yet to make contact with the physical system we really wish to describe, namely, the single free particle. This is achieved by adjoining to the constraint (4) another one,

$$\chi(x, p, \tau) \approx 0 . \tag{8}$$

The purpose of this constraint is to assign to each point (x, p) on a line L a definite value of an evolution parameter τ; the choice we make for χ will determine whether τ is kinematically or dynamically determined. To serve this purpose χ must obey two conditions: (i) it must be explicitly dependent on τ, (ii) it must vary along L for fixed τ, i.e.,

$$\{\chi, K\} \neq 0 . \tag{9}$$

This will ensure that, for each τ, there is just one point on L at which χ vanishes.

The physical interpretation of the system as a free particle emerges only after both constraints K, χ are imposed. The true physical variables and brackets among them are obtained [for the choice of evolution parameter determined by (8)] when we pass from the Poisson brackets (1) to the Dirac brackets[6] determined by K and χ:

$$\{f, g\}^* = \{f, g\} - (\{f, K\}\{\chi, g\} - \{f, \chi\}\{K, g\})/\{\chi, K\} . \tag{10}$$

This is a nondegenerate system of brackets for a six-dimensional phase space, just the correct number of variables for a single particle. As a result of Eq. (5) we can see that the Dirac brackets between the ten quantities $J_{\mu\nu}, P_\mu$ reproduce the Lie relation of the Poincaré group, just as their Poisson brackets did. We can therefore use these same generators to set up a new realization of the Poincaré group, by transformations $R^*(\Lambda, a)$ that are canonical with respect to the Dirac bracket. It is these transformations $R^*(\Lambda, a)$, and not the original $R(\Lambda, a)$, that must be physically identified as representing changes of inertial frame $\mathcal{O} \to \mathcal{O}' = (\Lambda, a)\mathcal{O}$. The two transformations $R(\Lambda, a)$, $R^*(\Lambda, a)$ for the same element (Λ, a) in the Poincaré group are related in this way: they both map Σ onto Σ; if $R(\Lambda, a)$ maps a line L onto another L', then $R^*(\Lambda, a)$ also maps L onto L'; beyond this, $R^*(\Lambda, a)$ preserves the value of τ when it carries each point on L to its image on L', while this is not generally so for $R(\Lambda, a)$.

Now that Eq. (8) has supplied us with a definite evolution parameter along an L, the arbitrariness in the multiplier v in the differential equations (7) is lifted: v is fixed by the condition

$$\frac{d\chi}{d\tau} \approx \frac{\partial \chi}{\partial \tau} + v\{\chi, K\} \approx 0 \Rightarrow v \approx -\frac{\partial \chi}{\partial \tau}\Big/\{\chi, K\} . \tag{11}$$

We now have definite equations of evolution for any phase-space function $f(x, p, \tau)$ restricted to Σ:

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} - \frac{\{f, K\}}{\{\chi, K\}} \frac{\partial \chi}{\partial \tau} . \tag{12}$$

The following results now hold. It is always possible to find six independent functions of x, p, τ such that the Dirac brackets (1) among them are a nondegenerate system, and furthermore when these Dirac brackets are expressed in terms of these six functions there is no residual τ dependence; when both constraints (4) and (8) are valid, then any $f(x, p, \tau)$ appears as some function of these six independent variables and τ, thereby giving a new meaning to the term "explicit dependence on τ". Denoting by $\partial'/\partial\tau$ the partial derivative with respect to such explicit τ dependence in the new sense,[12] a "Hamiltonian" $\mathcal{H}$ will exist such that it acts as the generator of dynamical evolution via the Dirac brackets,

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} - \frac{\{f, K\}}{\{\chi, K\}} \frac{\partial \chi}{\partial \tau} \approx \frac{\partial' f}{\partial \tau} + \{f, \mathcal{H}\}^* . \tag{13}$$

The eleven generators for the single free particle are then the ten quantities (3) giving rise to the realization $R^*(\Lambda, a)$ of the Poincaré group, and the Hamiltonian $\mathcal{H}$; all eleven quantities are ultimately functions of the six independent degrees of freedom remaining after the two constraints K, χ have been imposed and one has switched from the original Poisson to the Dirac brackets.

The relation to a space-time description is straightforward. Let $\mathcal{O}$ and $\mathcal{O}'$ be two inertial frames connected by an infinitesimal element (Λ, a) of the Poincaré group, so that the space-time coordinates x^μ, x'^μ that are assigned to one and the same event P in $\mathcal{O}$ and in $\mathcal{O}'$ are related geometrically by

$$\mathcal{O}' = (\Lambda, a)\mathcal{O}: \quad x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu + a^\mu, \quad |\omega|, |a| \ll 1 . \tag{14}$$

We set up the generator

$$G = \tfrac{1}{2}\omega^{\alpha\beta}J_{\alpha\beta} - a^\alpha P_\alpha . \tag{15}$$

If the particle is in a state of motion corresponding to the line L in the constraint surface Σ, the points on L supply us with a one-parameter set of space-time position vectors $x^\mu(\tau)$ with which, in a frame $\mathcal{O}$, we can set up a world line in a space-time diagram—we plot spatial positions $\vec{x}(\tau)$ at laboratory time $x^0(\tau)$ in $\mathcal{O}$. The line L is carried, by the infinitesimal Dirac canonical transformation generated by G, into a line L' in a manner that preserves the τ values. The points on L' now supply us with a one-parameter set of space-time position vectors $x'^\mu(\tau)$ which *by definition* are what must be used to reconstruct a world line in $\mathcal{O}'$:

$$x'^\mu(\tau) \approx x^\mu(\tau) + \{G, x^\mu(\tau)\}^* . \tag{16}$$

The space-time constructions carried out in $\mathcal{O}$ and in $\mathcal{O}'$ describe one and the same *objectively* *real world line* if, for each τ, $x'^{\mu}(\tau)$ obtained by *canonical* means in (16) is related in the *geometrical* manner of Eq. (14) to $x^{\mu}(\tau+\delta\tau)$ for some *infinitesimal* $\delta\tau$:

$$x'^{\mu}(\tau)\approx x^{\mu}(\tau+\delta\tau)+\omega^{\mu\nu}x_{\nu}(\tau+\delta\tau)+a^{\mu}\,. \tag{17}$$

Here $\delta\tau$ is permitted to be a linear expression in $\omega^{\mu\nu}$ and a^{μ} with coefficients that could depend on dynamical variables. Combining Eqs. (13), (16), and (17) and retaining those linear in $\omega^{\mu\nu}$ and a^{μ} alone, the WLC is the condtion that there exist an expression for $\delta\tau$ such that

$$\{G,x^{\mu}\}^{*}=\omega^{\mu\nu}x_{\nu}+a^{\mu}+\left(\frac{\partial' x^{\mu}}{\partial\tau}+\{x^{\mu},\mathcal{H}\}^{*}\right)\delta\tau\,. \tag{18}$$

In this form, the WLC is written exclusively in terms of the final physical brackets and it shows explicitly that in general it is a condition to be obeyed jointly by G and $\mathcal{H}$.

We discuss two choices for χ, one corresponding to a kinematic choice for τ and the other to a dynamical choice. Suppose we take

$$\chi=x^{0}-\tau\,. \tag{19}$$

Then one finds that the Dirac bracket (10) is a nondegenerate one in the six variables $x^{j},p_{j},j=1,2,3$ and the brackets between them are τ independent:

$$\{x^{j},x^{k}\}^{*}=0,\quad \{x^{j},p_{k}\}^{*}=\delta^{j}_{k},\quad \{p_{j},p_{k}\}^{*}=0\,. \tag{20}$$

Now the coefficient v in the equation of motion (11) has the value

$$v=\tfrac{1}{2}p^{0},\quad p^{0}\approx(p^{2}+m^{2})^{1/2}\,. \tag{21}$$

The Hamiltonian $\mathcal{H}$ is to be such that via the Dirac bracket it reproduces the right equations of motion for x^{j} and p_{j} (its existence being guaranteed):

$$\dot{x}^{j}=\{x^{j},\mathcal{H}\}^{*}=\frac{p^{j}}{p^{0}},\quad \dot{p}^{j}=\{p^{j},\mathcal{H}\}^{*}=0\,. \tag{22}$$

Evidently we have

$$\mathcal{H}=(p^{2}+m^{2})^{1/2}=p^{0}\,. \tag{23}$$

We have automatically reviewed the description of the single free particle in Dirac's instant form: the basic variables are x^{j},p_{j}; the former are physically interpreted as position at some laboratory time τ; the basic brackets are given by (20); and the eleven generators are

$$J_{jk}=x_{j}p_{k}-x_{k}p_{j},\quad J_{0j}=\tau p_{j}-x_{j}(p^{2}+m^{2})^{1/2},$$
$$P_{j}=p_{j},\qquad P_{0}=(p^{2}+m^{2})^{1/2},\qquad \mathcal{H}=P^{0}\,. \tag{24}$$

The WLC (18), which is certainly expected to be satisfied, behaves as follows: there is no need for $\delta\tau$ to contain terms involving ω_{jk} and a_{j} which describe a Euclidean transformation. The remaining terms in (18) are

$$\omega^{0j}\{J_{0j},x^{\mu}\}^{*}-a^{0}\{P_{0},x^{\mu}\}^{*}\approx\omega^{0j}(\delta^{\mu}_{0}x_{j}-\delta^{\mu}_{j}\tau)+a^{0}\delta^{\mu}_{0}$$
$$+\left(\frac{\partial' x}{\partial\tau}+\{x^{\mu},\mathcal{H}\}^{*}\right)\delta\tau\,. \tag{25}$$

The choice $\mu=0$ serves to determine $\delta\tau$,

$$\delta\tau=-a^{0}-\omega^{0j}x_{j}\,. \tag{26}$$

The use of this in the remaining components of (25) then shows that the only nontrivial part of the WLC refers to pure Lorentz transformations and is

$$\{J_{0j},x^{k}\}^{*}=-\delta^{k}_{j}\tau-x_{j}\{x^{k},\mathcal{H}\}^{*}\,. \tag{27}$$

This is the form of the WLC originally used in proving the no-interaction theorem; and for the free particle case (24) it is clearly obeyed.

As another choice for χ we next consider

$$\chi=p\cdot x-m\tau\,. \tag{28}$$

The Dirac brackets can be computed and again one finds that x^{j},p_{j} are a complete set of variables with τ-independent brackets,

$$\{x^{j},x_{k}\}^{*}=(x^{k}p^{j}-x^{j}p^{k})/m^{2}\,,$$
$$\{x^{j},p_{k}\}^{*}=\delta^{j}_{k}-\frac{p^{j}p_{k}}{m^{2}},\quad \{p_{j},p_{k}\}^{*}=0\,. \tag{29}$$

The multiplier v is now

$$v=\tfrac{1}{2}m\,, \tag{30}$$

so $\mathcal{H}$ must be so chosen that through the brackets (29) it generates the equations of motion

$$\dot{x}^{j}=\{x^{j},\mathcal{H}\}^{*}=\frac{p^{j}}{m},\quad \dot{p}^{j}=\{p^{j},\mathcal{H}\}^{*}=0\,. \tag{31}$$

The solution is

$$\mathcal{H}=-m\ln\frac{p_{0}}{m},\quad p_{0}=(\vec{p}^{\,2}+m^{2})^{1/2}\,. \tag{32}$$

We have now *a new description of the free particle, not belonging to any of Dirac's forms*: the six basic variables x_{j},p_{j} have brackets (29); the x_{j} are not spatial position variables at some laboratory time but at that point on the world line which is τ units of proper time away from the point where p and x are orthogonal; the eleven generators are

$$J_{jk}=x_{j}p_{k}-x_{k}p_{j},\quad J_{0j}=(m\tau+\vec{p}\cdot\vec{x})\frac{p_{j}}{p_{0}}-x_{j}p_{0}\,,$$
$$P_{j}=p_{j},\quad P_{0}=p_{0},\quad \mathcal{H}=-m\ln\frac{p_{0}}{m}\,. \tag{33}$$

The brackets among $J_{\mu\nu}, P_\mu$ computed on the basis of (29) will have the proper values, and $\mathcal{H}$ *is not in the Poincaré algebra*. This time the WLC (18) has a different behavior than previously; there is no need for $\delta\tau$ to depend on $\omega_{\mu\nu}$ at all, it must be chosen only to satisfy

$$-\{a\cdot P, x^\mu\}^* = a^\mu + \frac{\partial' x}{\partial\tau} + \{x^\mu, \mathcal{H}\}^*\ \delta\tau\ . \quad (34)$$

The $\mu = 0$ case determines $\delta\tau$ to be $-a\cdot p/m$, and the remaining conditions are

$$\{x^j, P_k\}^* = \delta^j_k - \frac{p_k}{m}\{x^j, \mathcal{H}\}^*\ . \quad (35)$$

These nontrivial parts of the WLC refer to the behavior of x_j under spatial translations, and are of course obeyed.

Because of the simplicity of the present system (essentially the fact that each L is one-dimensional) one can easily show that the WLC (18) can be satisfied for any choice of χ. (This means that we can view different choices as various ways of describing the same physical system, namely, the free particle.) We can rewrite (18) in terms of the original Poisson brackets and then it reads

$$\{G,\chi\}\{K, x^\mu\} = -\frac{\partial\chi}{\partial\tau}\{x^\mu, K\}\delta\tau\ . \quad (36)$$

The choice

$$\delta\tau = \{G,\chi\}\frac{\partial\chi}{\partial\tau} \quad (37)$$

will always work. In any event we have demonstrated, with the simplest imaginable system, how one can arrive at new Hamiltonian relativistic descriptions not encompassed by Dirac's program. We have also seen that the methods of constrained Hamiltonian mechanics serve only as convenient tools to ultimately arrive at various descriptions of one particle, always characterized by six independent variables, suitable brackets, and generators; it is just that the system defined by (29) and (33) would have been awkward to define directly.

III. TWO INTERACTING PARTICLES

We begin with a sixteen-dimensional phase space Γ with variables $x_{\mu\alpha}$, $p_{\mu\alpha}$, $\alpha = 1, 2$, the only nonzero Poisson brackets being

$$\{x_{\mu\alpha}, p_{\nu\beta}\} = \delta_{\alpha\beta}\, g_{\mu\nu}\ . \quad (38)$$

The transformations

$$R(\Lambda, a):\ x_{\mu\alpha} \to x'_{\mu\alpha} = \Lambda_\mu{}^\nu x_{\nu\alpha} + a_\mu,\quad p_{\mu\alpha} \to p'_{\mu\alpha} = \Lambda_\mu{}^\nu p_{\nu\alpha} \quad (39)$$

are canonical, provide a realization of the Poincaré group, and have as generators

$$J_{\mu\nu} = \sum_\alpha (x_{\mu\alpha}p_{\nu\alpha} - x_{\nu\alpha}p_{\mu\alpha}),\quad P_\mu = \sum_\alpha p_{\mu\alpha}\ . \quad (40)$$

We must now choose two independent constraints K_1, K_2 such that each is invariant under $R(\Lambda, a)$ and they are a first-class pair. We make the *ansatz*[13]

$$K_1 = p_1{}^2 - m_1{}^2 + V,\quad K_2 = p_2{}^2 - m_2{}^2 + V\ , \quad (41)$$

incorporating a common "interaction term" in addition to the free-particle forms. The first-class condition is

$$\{K_1, K_2\} = \{p_1{}^2 - p_2{}^2, V\} = 2\left(p_1\cdot\frac{\partial}{\partial x_1} - p_2\cdot\frac{\partial}{\partial x_2}\right)V = 0\ . \quad (42)$$

The most general V is easily discovered. Invariance of V under $R(1, a)$ tells us that V is some function of x_1-x_2. The condition (42) then says that V is unchanged if x_1 and x_2 are changed to $x_1 + \epsilon p_1$, x_2-ϵp_2, respectively, with ϵ small. Thus V must be some function of the part of $x_1 - x_2$ that is transverse to $P = p_1 + p_2$. Writing

$$r^\mu = \tfrac{1}{2}(x_1^\mu - x_2^\mu),\quad r_\perp^\mu = r^\mu - p^\mu\frac{P\cdot r}{p^2}\ , \quad (43)$$

and finally invoking the invariance of V under $R(\Lambda, 0)$, we see that we can take for V any function of Lorentz scalars formed out of $r_\perp^\mu$, p_1, and p_2. For simplicity we shall consider V to be some function of $r_\perp^2$ alone, so we finally have the two first-class constraints

$$K_1 = p_1{}^2 - m_1{}^2 + V(\xi),\quad K_2 = p_2{}^2 - m_2{}^2 + V(\xi)\ ,$$
$$\xi = r_\perp{}^2 = r^2 - \frac{(P\cdot r)^2}{p^2}\ . \quad (44)$$

One could now choose to work with the two combinations $K_1 \pm K_2$, in one of which V is absent; but we will use the above set as given.

The region in Γ wherein both K_1 and K_2 vanish will be a fourteen-dimensional constraint hypersurface Σ. Clearly Σ is mapped onto itself by $R(\Lambda, a)$. K_1 and K_2 themselves generate commuting canonical transformations that also map Σ onto itself; these follow from the first-class property. Given any point (x, p) in Σ we can develop its "orbit" under the Abelian group of canonical transformations generated by K_1 and K_2. This will be a two-dimensional "sheet" S lying wholly in Σ. One can imagine getting S in this way; we solve the differential equations

$$\frac{dx^\mu_\alpha(\sigma)}{d\sigma}=\{x^\mu_\alpha(\sigma),K_\beta\}v_\beta,\quad \frac{dp^\mu_\alpha(\sigma)}{d\sigma}=\{p^\mu_\alpha(\sigma),K_\beta\}v_\beta, \tag{45}$$

$$x^\mu_\alpha(\sigma)=x^\mu_\alpha,\quad p^\mu_\alpha(0)=p^\mu_\alpha,$$

for all possible choices of the v_β and then collect together all the points of Σ that can be reached from (x,p) in this way. The fourteen-dimensional Σ thus breaks up into a union of two-dimensional sheets S, which must then form a twelve-parameter family. And this breakup of Σ with disjoint sheets is preserved by $R(\Lambda,a)$; each S will be mapped by $R(\Lambda,a)$ onto another entire sheet S'.

So far again the development has been mathematical; more needs to be done before we arrive at a physical system of two interacting particles. It is necessary to identify precisely what constitutes a "state of motion" for the two particles; and this must be such that, *in each state of motion and in each frame* $\mathcal{O}$ *a pair of world lines in space-time is unambiguously determined*. Reflection shows that it would be physically inappropriate to identify a sheet S as a state of motion of the two-particle system. It should be clear that $x_1{}^0$ and $x_2{}^0$, the "time components" of x^μ_α, vary independently over a sheet S, so on a given sheet we can view $\vec{x}_1$ and $\vec{x}_2$ (and p^μ_α as well) as functions of $x_1{}^0, x_2{}^0$. Except in the noninteracting case, we expect *each* of $\vec{x}_1, \vec{x}_2$ to depend on *both* of $x_1{}^0, x_2{}^0$. If therefore *all* of S were to be used to reconstruct space-time world lines for the two particles, we would end up with a *sheet* and not a *line* for each particle. In order that each state of motion lead uniquely to a world line for each particle it is necessary to choose some one-dimensional curve C in S and call that alone a state of motion; the rest of S is to be then discarded as being of no physical significance.

A curve C on each S can be specified by choosing one constraint $\chi_1(x,p)$ with no explicit dependence on any parameters,

$$\chi_1(x,p)\approx 0. \tag{46}$$

It is only necessary to ensure that the function χ is not constant over an S. To then assign a value of an evolution parameter τ to each point of C we must set up an explicitly τ-dependent second constraint

$$\chi_2(x,p,\tau)\approx 0. \tag{47}$$

The pair of constraints χ_α added to the earlier pair K_α then defines the physical interacting two-particle system: *the system is not really defined until the* χ_α *are chosen;* and one cannot view different choices of the χ_α as giving different descriptions of "the same physical system.[14]"

For this scheme to work it is clearly adequate that the set of four constraints K_α, χ_α be second class, i.e.,

$$\det|\{\chi_\alpha, K_\beta\}|\neq 0. \tag{48}$$

Let us now write

$$a_{\alpha\beta}\{\chi_\beta, K_\gamma\}=\delta_{\alpha\gamma}. \tag{49}$$

Then the true physical variables are obtained by imposing all four constraints

$$K_\alpha\approx 0,\quad \chi_\alpha\approx 0, \tag{50}$$

and the physical brackets are the Dirac brackets relative to K_α, χ_α:

$$\{f,g\}^*=\{f,g\}-a_{\alpha\beta}(\{f,K_\alpha\}\{\chi_\beta,g\}-\{f,\chi_\beta\}\{K_\alpha,g\})$$
$$-\{f,K_\alpha\}\,a_{\alpha\beta}\{\chi_\beta,\chi_{\beta'}\}\,a_{\alpha'\beta'}\{K_{\alpha'},g\}. \tag{51}$$

This system of brackets support a twelve-dimensional phase space. States of motion are the curves C on sheets S, with parameter of evolution τ. Since the K_α were constructed to have vanishing Poisson brackets with $J_{\mu\nu}$ and P_μ, the latter quantities continue to provide, via their Dirac brackets, a realization of the Lie algebra of the Poincaré group. They thus integrate to a realization of the Poincaré group by transformations $R^*(\Lambda,a)$ canonical with respect to the Dirac bracket. Each $R^*(\Lambda,a)$, like $R(\Lambda,a)$, maps Σ onto itself; moreover if $R(\Lambda,a)$ carries S to S', so does $R^*(\Lambda,a)$; beyond this, $R^*(\Lambda,a)$ takes the curve C in S determined by (46) to the curve C' determined similarly in S', and preserves the value of τ in the process. From now on the change of inertial frame $\mathcal{O}\to\mathcal{O}'=(\Lambda,a)\mathcal{O}$ is represented by the transformation $R^*(\Lambda,a)$ acting on the physical system.

The hitherto arbitrary quantities v_α in (45) are now fixed since the constraints $\chi_\alpha=0$ must be maintained:

$$\frac{d\chi_\alpha}{d\tau}\approx 0\Rightarrow v_\alpha=-a_{\alpha\beta}\frac{\partial\chi_\beta}{\partial\tau}=-a_{\alpha 2}\frac{\partial\chi_2}{\partial\tau}. \tag{52}$$

The resulting general equation of dynamical evolution along any C,

$$\frac{df}{d\tau}(x,p,\tau)=\frac{\partial f}{\partial\tau}-\{f,K_\alpha\}\,a_{\alpha 2}\frac{\partial\chi_2}{\partial\tau} \tag{53}$$

can be rewritten in Hamiltonian form with the Dirac bracket provided the meaning of explicit dependence on τ is suitably altered:

$$\frac{df}{d\tau}=\frac{\partial' f}{\partial\tau}+\{f,\mathcal{H}\}^*. \tag{54}$$

[We must pick twelve independent variables surviving after all constraints (50) are imposed, such that their Dirac brackets are τ independent when expressed in terms of themselves; if f in (54) is expressed in terms of such a set of independent variables, any residual τ dependence is what contributes in (54)]. Subject to discussion of the WLC, the relativistic system of two interacting particles is defined by the ten generators (40) and the Hamiltonian $\mathcal{H}$, all viewed as functions of twelve independent degrees of freedom left when on the original phase space Γ all four constraints (50) are operative.

The WLC are easy to set up. The point $x_\alpha(\tau)$, $p_\alpha(\tau)$ on C in some S leads, in a frame $\mathcal{O}$, to the pair of space-time points $x_1^\mu(\tau)$, $x_2^\mu(\tau)$. These are the points on the two world lines to which, in the considered state of motion, and in the frame $\mathcal{O}$, a common value τ of the parameter is assigned. In the frame $\mathcal{O}'=(\Lambda,a)\mathcal{O}$ with (Λ,a) infinitesimal, one assigns the common value τ to the pair of points

$$\mathcal{O}':\; x'^\mu_\alpha(\tau)\approx x^\mu_\alpha(\tau)+\{G,x^\mu_\alpha(\tau)\}^*, \tag{55}$$

G being formed again as in (15). The reconstructions in $\mathcal{O}'$ and $\mathcal{O}$ refer to the same world line if there exist two expressions $\delta_1\tau$, $\delta_2\tau$ such that

$$x'^\mu_\alpha(\tau)=x^\mu_\alpha(\tau+\delta_\alpha\tau)+\omega^{\mu\nu}x_{\nu\alpha}(\tau+\delta_\alpha\tau)+a^\mu,\quad \alpha=1,2. \tag{56}$$

(Thus the pair of points with a common τ value in $\mathcal{O}'$ *could* have different τ values in $\mathcal{O}$.) Thus the WLC is the requirement that there exist expressions $\delta_\alpha\tau$ so that

$$\{G,x^\mu_\alpha\}^*=\omega^{\mu\nu}x_{\nu\alpha}+a^\mu+\left(\frac{\partial' x^\mu_\alpha}{\partial\tau}+\{x^\mu_\alpha,\mathcal{H}\}^*\right)\delta_\alpha\tau. \tag{57}$$

All eleven generators of the system *are* again *involved.*

For the two (or more) particle case it is *not true* that expressions $\delta_\alpha\tau$ can be found to satisfy (57) for any choice of χ_α. This reinforces our view point that *the choice of χ_α is part of the definition of the physical system.* To examine if (57) can be obeyed for various choices of χ_α, it is simpler to rewrite it in terms of the original Poisson brackets

$$\{x^\mu_\alpha,K_\beta\}a_{\beta\gamma}\{G,\chi_\gamma\}\simeq\{x^\mu_\alpha,K_\beta\}a_{\beta2}\frac{\partial\chi_2}{\partial\tau}\,\delta_\alpha\tau,\quad \alpha=1,2. \tag{58}$$

Let us first see what happens with the choice

$$\chi_1=r^0=\tfrac12(x_1^0-x_2^0),\quad \chi_2=\tfrac12(x_1^0+x_2^0)-\tau. \tag{59}$$

This corresponds to adopting Dirac's instant form. The matrix in (48) is

$$(\{\chi_\alpha,K_\beta\})\simeq\begin{bmatrix} p_1^0 & -p_2^0\\ p_1^0+2P^0\dfrac{(P\cdot r)^2}{p^4}V'(\xi) & p_2^0+2P^0\dfrac{(P\cdot r)^2}{p^4}V'(\xi)\end{bmatrix}, \tag{60}$$

$$V'(\xi)=\frac{dV}{d\xi}(\xi).$$

The inverse matrix is then

$$(a_{\alpha\beta})\simeq\frac1D\begin{bmatrix} p_2^0+2P^0\dfrac{(P\cdot r)^2}{p^4}V'(\xi) & p_2^0\\ -p_1^0-2P^0\dfrac{(P\cdot r)^2}{p^4}V'(\xi) & p_1^0\end{bmatrix},$$
$$D=2p_1^0p_2^0+2(P^0)^2\frac{(P\cdot r)^2}{p^4}V'(\xi). \tag{61}$$

Owing to Euclidean invariance of the χ's, the Poisson brackets $\{G,\chi_\alpha\}$ simplify to

$$\{G,\chi_1\}\approx\omega^{0j}r_j,\quad \{G,\chi_2\}\approx a^0+\tfrac12\omega^{0j}(x_{j1}+x_{j2}). \tag{62}$$

If the $\delta_\alpha\tau$ exist, they must be linear expressions in a^0, ω^{0j} and must obey

$$\{x^\mu_\alpha,K_\beta\}a_{\beta1}\omega^{0j}r_j+\{x^\mu_\alpha,K_\beta\}a_{\beta2}[a^0+\tfrac12\omega^{0j}(x_{j1}+x_{j2})]\approx-\{x^\mu_\alpha,K_\beta\}a_{\beta2}\delta_\alpha\tau$$

or equivalently,

$$\{x^\mu_\alpha,K_\beta\}a_{\beta1}\omega^{0j}r_j\simeq-\{x^\mu_\alpha,K_\beta\}a_{\beta2}[\delta_\alpha\tau+a^0+\tfrac12\omega^{0j}(x_{j1}+x_{j2})],$$
$$\text{no sum on }\alpha. \tag{63}$$

If $\delta_\alpha\tau$ do exist, it must happen that the expressions

$$\delta_\alpha\tau+a^0+\tfrac12\omega^{0j}(x_{j1}+x_{j2})$$

are proportional to $\omega^{0j}r_j$ for both α values. Thus the $\delta_\alpha\tau$ will exist, and the WLC will be obeyed, if out of the four vectors

$$A^\mu_\alpha = D\{x^\mu_\alpha, K_\beta\}a_{\beta 1}, \quad B^\mu_\alpha = D\{x^\mu_\alpha, K_\beta\}a_{\beta 2}, \tag{64}$$

A^μ_1 is parallel to B^μ_1 and A^μ_2 to B^μ_2. These vectors are

$$\begin{aligned} A_1 &= 2\left[p_2^0 + 2P^0 \frac{(P\cdot r)^2}{p^4} V'(\xi)\right]p_1 \\ &\quad + 2(p_1^0 - p_2^0)\frac{P\cdot r}{p^2} V'(\xi) r_\perp, \\ B_1 &= 2P_2^0 p_1 - 2P^0 \frac{P\cdot r}{p^2} V'(\xi) r_\perp, \\ A_2 &= -2\left[p_1^0 + 2P^0 \frac{(P\cdot r)^2}{p^4} V'(\xi)\right]p_2 \\ &\quad + 2(p_1^0 - p_2^0)\frac{P\cdot r}{p^2} V'(\xi) r_\perp, \\ B_2 &= 2p_1^0 p_2 - 2P^0 \frac{P\cdot r}{p^2} V'(\xi) r_\perp. \end{aligned} \tag{65}$$

In the absence of any interaction, $V'=0$, A_1 and B_1 are parallel, and so are A_2 and B_2. Thus for free particles the WLC (57) or (58) can be obeyed. But if V' is nonzero, then the WLC demand that $r_\perp$ be parallel to p_1 as well as to p_2, and these conditions cannot be met, since it is not a consequence of the given constraints that p_1 is parallel to p_2. In this way *the no-interaction theorem reappears* in the present framework.

On the other hand a dynamical choice of evolution parameter can easily be made for which the WLC are obeyed. In fact, Eq. (58) shows the way. Let us choose χ_1 explicitly invariant under $R(\Lambda, a)$:

$$\chi_1 = P\cdot r, \quad \chi_2 = \tfrac{1}{2}P\cdot(x_1+x_2) - \tau. \tag{66}$$

Then for both values of α, if we set

$$\delta_\alpha \tau = -\{G, \chi_2\} = -a\cdot P, \tag{67}$$

the WLC are obeyed, whatever be the interaction potential V. As expected, it is only the behavior under space-time translations that leads to a nontrivial WLC with the χ_α of (66). The system of constraints K_α, χ_α of Eqs. (44) and (66), the Dirac brackets (51), the ten generators (40), and the $\mathcal{H}$ of Eq. (54) taken together describe an interacting two-particle system with objectively real world lines, in the framework of a dynamically determined evolution parameter. Since the choice of twelve independent physical variables, their brackets, and the generator $\mathcal{H}$ all can involve the interaction V, the constraint formalism is a convenient technical means to set up the system in an implicit fashion.

It is trivial to remark that *any* choice of χ_1 and χ_2 such that χ_1 is invariant under $R(\Lambda, a)$ and χ_2 alone carries a τ dependence explicitly gives a system with acceptable world lines. One need only choose

$$\delta_1\tau = \delta_2\tau = \{G, \chi_2\}/(\partial\chi_2/\partial\tau) \tag{68}$$

to obey the WLC (58). However, we stress that the choices of V, χ_1 and χ_2 all together define the particular two-particle system in interaction.

IV. THE N-PARTICLE PROBLEM

The model of Sec. III can be generalized from 2 to N particles for any N. We start with variables $x_{\mu\alpha}$, $p_{\mu\alpha}$, $\alpha = 1, 2, \ldots, N$ giving us an $8N$-dimensional phase space Γ, the only nonzero Poisson brackets being

$$\{x_{\mu\alpha}, p_{\nu\beta}\} = g_{\mu\nu}\delta_{\alpha\beta}, \quad \alpha,\beta = 1, 2, \ldots, N. \tag{69}$$

We must next choose N-independent first-class constraints K_α, each invariant under the canonical transformations $R(\Lambda, a)$ (which act in the obvious way and have the obvious expressions $J_{\mu\nu}$, P_μ for generators). We assume there is a common "potential" V and take

$$K_\alpha = p_\alpha{}^2 - m_\alpha{}^2 + V(x,p), \quad \alpha = 1, 2, \ldots, N. \tag{70}$$

The first-class conditions are

$$\{K_\alpha, K_\beta\} = 0 \Rightarrow \left(p_\alpha\cdot\frac{\partial}{\partial x_\alpha} - p_\beta\cdot\frac{\partial}{\partial x_\beta}\right)V = 0. \tag{71}$$

Moreover V must be invariant under $R(\Lambda, a)$. The $(N-1)$ independent conditions (71) on V can be expressed in this way: For each of the values $\alpha = 1, 2, \ldots, N-1$, V must be unchanged by the infinitesimal changes

$$x_\alpha \to x_\alpha + \epsilon p_\alpha, \quad x_{\alpha+1} \to x_{\alpha+1} - \epsilon p_{\alpha+1}, \tag{72}$$

the remaining x's being held fixed. Translation invariance restricts V to be a function only of the $(N-1)$ differences $y_1 = x_1 - x_2$, $y_2 = x_2 - x_3, \ldots$ $y_{N-1} = x_{N-1} - x_N$, and of all the p_α. We must extract from the set of $(N-1)$ four-vectors $(y_1, y_2, \ldots, y_{N-1})$ just those combinations (dependent on the p_α also) that are unchanged under each of the $(N-1)$ transformations (72). Let us write, in view of (72), the set $y_1, \ldots, y_{N-1}$ as a column vector with $4(N-1)$ components, and also set up $(N-1)$ similar column vectors made up from the p_α in this way:

$$\underline{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1}\end{bmatrix},\quad \underline{z}_1=\begin{bmatrix} p_1+p_2 \\ -p_2 \\ 0 \\ \vdots \\ 0\end{bmatrix},\quad \underline{z}_2=\begin{bmatrix} -p_2 \\ p_2+p_3 \\ -p_3 \\ 0 \\ \vdots \\ 0\end{bmatrix},$$

$$\underline{z}_3=\begin{bmatrix} 0 \\ -p_3 \\ p_3+p_4 \\ -p_4 \\ 0 \\ \vdots \\ 0\end{bmatrix},\ \ldots,\ \underline{z}_{N-1}=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -p_{N-1} \\ p_{N-1}+p_N\end{bmatrix}. \tag{73}$$

Then we need to form expressions linear in y that are unchanged if we were to add y to any linear combination of $z_1, \ldots, z_{N-1}$: V can then be any Lorentz-invariant function of such expressions and the p_α. Let us equip these $4(N-1)$-component quantities with a metric in which we sum up the Lorentz scalar products of corresponding component four-vectors: thus we have

$$y \cdot z_1 = y_1 \cdot (p_1+p_2) - y_2 \cdot p_2\,,$$
$$z_1 \cdot z_2 = -(p_1+p_2)\cdot p_2 - p_2 \cdot (p_2+p_3), \ldots . \tag{74}$$

If the symmetric $(N-1)\times(N-1)$ matrix $\underline{z}_\alpha \cdot \underline{z}_\beta$ has an inverse $B_{\alpha\beta}$ (these are all functions of the p's), then the $4(N-1)$-component object

$$\underline{y} - \underline{z}_\alpha B_{\alpha\beta} \underline{z}_\beta \cdot \underline{y} \tag{75}$$

has the desired property. It gives us the $(N-1)$ Lorentz four-vectors each being linear in the y_α and unchanged under every transformation (72). V can then be any Lorentz-invariant function of these $(N-1)$ four-vectors and the N vectors p_α. (It must be noted that in place of (74) one could have used other definitions of a metric for the quantities (73); but one finds that one is not led to anything essentially new.)

The constraint hypersurface Σ in Γ defined by

$$K_\alpha \approx 0 \tag{76}$$

is $7N$ dimensional; in it the K_α generate an N-parameter Abelian group of canonical transformations, giving rise to orbits in the form of N-dimensional "sheets" S. So the sheets form a $6N$-parameter family. Again *it is inappropriate to identify a sheet with a state of motion* of an N-particle system. One must give a rule for selecting a *one-dimensional curve* C on each S, parametrize it with a variable τ, and view each such curve above *as a state of motion*. Then as τ varies along a given C, in a given frame $\mathcal{O}$ the quantities $x_\alpha^\mu(\tau)$ give a set of N world lines. The specification of C and then of τ is achieved by adding a set of N constraints

$$\chi_\alpha(x,p)\approx 0, \quad \alpha=1,2,\ldots,N-1, \quad \chi_N(x,p,\tau)\approx 0\,, \tag{77}$$

to (76). Only χ_N carries explicit τ dependence, and the K_α, χ_α must together form a second-class system. In Eqs. (49) and (51) if we now let $\alpha, \beta, \ldots$ run from 1 to N, we can write the replacements for (52) and (54). There will exist s dynamical generator $\mathcal{H}$ such that along any C one has a general equation of motion

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} - \{f, K_\alpha\} a_{\alpha N} \frac{\partial \chi_N}{\partial \tau} \approx \frac{\partial' f}{\partial \tau} + \{f, \mathcal{H}\}^*. \tag{78}$$

The eleven generators $J_{\mu\nu}$, P_μ, and $\mathcal{H}$ used within the Dirac bracket system, and all constraints K_α, χ_α operative, *define* the relativistic N-particle system in interaction. Of course changes of inertial frame are implemented by transformations $R^*(\Lambda, a)$.

The WLC is, as before, given by the requirement: For any infinitesimal transformation (Λ, a) there must exist expressions $\delta_\alpha \tau$, $\alpha = 1, \ldots, N$, such that

$$\{G, x_\alpha^\mu\}^* \approx \omega^{\mu\nu} x_{\nu\alpha} + a^\mu + \left(\frac{\partial' x_\alpha^\mu}{\partial \tau} + \{x_\alpha^\mu, \mathcal{H}\}^*\right)\delta_\alpha \tau\,,$$
$$\alpha = 1, \ldots, N. \tag{79}$$

For computational ease we rewrite this as

$$\{x_\alpha^\mu, K_\beta\} a_{\beta\gamma} \{G, \chi_\gamma\}$$
$$\approx \{x_\alpha^\mu, K_\beta\} a_{\beta N} (\partial \chi_N / \partial \tau)\, \delta_\alpha \tau, \quad \alpha = 1, 2, \ldots, N. \tag{80}$$

A simple choice of χ_α yielding a physical system obeying these WLC is

$$\chi_1 = P\cdot(x_1 - x_2), \quad \chi_2 = P\cdot(x_2 - x_3),$$
$$\chi_{N-1} = P\cdot(x_{N-1} - x_N), \quad \chi_N = P\cdot x_1 - \tau\,, \tag{81}$$

for then the value $\delta_\alpha \tau = -a \circ P$ for all α fits the bill. Note that τ in (81) has the dimensions of action.

V. SPECIAL CASES OF N-PARTICLE SYSTEMS

It should be clear from the development so far presented that the real reason we have succeeded in not being bound by the no-interaction theorem is this: We have enlarged the framework of relativistic Hamiltonian dynamics in an essential way by allowing for a dynamically determined evolution parameter. But this new framework so greatly increases the possibilities of constructing models with interaction that we devote this section to looking at some novel possibilities.

(a) Both in the two-particle and N-particle models so far discussed the evolution parameter τ had the following meaning: In any state of motion observed in a frame $\mathcal{O}$, τ was the time as measured in the center-of-momentum frame. [In (81) this involved measuring τ in units of $\sqrt{p^2}$.] Since P_μ is a constant of motion, keeping $\mathcal{O}$ fixed we see that for each state of motion the center-of-momentum frame is also an *inertial frame*, but one that depends on P_μ and *so on the particular state of motion.*[15] This is the case then with χ_α chosen as in Eqs. (66) and (81). But we can consider another kind of system with the choices

$$K_\alpha = p_\alpha^{\ 2} - m_\alpha^{\ 2} + V(x,p), \quad \alpha = 1,2,\ldots,N,$$
$$\chi_\alpha = p_1\cdot(x_\alpha - x_{\alpha+1}), \quad \alpha = 1,2,\ldots,N-1, \tag{82}$$
$$\chi_N = p_1\cdot x_1 - \tau .$$

The K_α can be taken to be the same set of first-class functions as in Sec. IV; so the transition from Γ to Σ and the subsequent breakup of Σ into sheets S is exactly as before. But thereafter *the model changes*. The curve C we pick out on each S to identify with a state of motion is no longer the same as before; so *even with the same "potential" V we have now a completely different physical system* of N-interacting particles: the Dirac brackets and $\mathcal{H}$ will all be different from the case with Eq. (81). In particular, since in the present model the first particle is not free, i.e., p, is not a constant of motion, τ *no longer has the meaning of being the time in some* (dynamically determined) *inertial frame*. Nevertheless all the requirements of special relativity and also the WLC remain fulfilled. In particular the WLC (80) is now satisfied with the common choice

$$\delta_\alpha\tau = -a\cdot p_1, \quad \alpha = 1,2,\ldots,N. \tag{83}$$

(b) The case we just considered was this: One of the particles in the N-particle system, subject to interaction similar to the rest, "carries a clock with it" and that gives us a parameter τ. We can easily make a model in which this "time keeper" is free but is not the "center of momentum" of the total system. To have a system of N particles in interaction, we adopt the formalism of the previous section appropriate to a system of $(N+1)$ particles, and make the choice

$$K_\alpha = p_\alpha^{\ 2} - m_\alpha^{\ 2} + V(x_1,\ldots,x_N,p_1,\ldots,p_N) \qquad \alpha = 1,2,\ldots,N,$$
$$K_{N+1} = p_{N+1}^{\ \ 2} - m_{N+1}^{\ \ 2}, \tag{84}$$
$$\chi_\alpha = p_{N+1}\cdot(x_\alpha - x_{\alpha+1}), \quad \alpha = 1,2,\ldots,N,$$
$$\chi_{N+1} = p_{N+1}\cdot x_{N+1} - m_{N+1}\tau .$$

The function V is of the kind we investigated in Sec. IV. We have then an initial Γ of dimension $8(N+1)$; the surface Σ defined by the vanishing of $k_1, k_2, \ldots, k_{N+1}$ is of dimension $7(N+1)$; these first class K's generate in Σ sheets S of dimension $(N+1)$; and the final Dirac bracket system refers to a system with $6(N+1)$ phase-space variables. The following bracket relations are checked:

$$\{K_\alpha, K_{N+1}\} = 0, \quad \alpha = 1,2,\ldots,N,$$
$$\{K_{N+1}, \chi_\alpha\} = 0, \quad \alpha = 1,2,\ldots,N-1,$$
$$\{K_{N+1}, \chi_N\} \approx 2m_{N+1}^{\ \ 2}, \tag{85}$$
$$\{K_\alpha, \chi_{N+1}\} = 0, \quad \alpha = 1,2,\ldots,N,$$
$$\{K_{N+1}, \chi_{N+1}\} \approx -2m_{N+1}^{\ \ 2}.$$

The equation of dynamical evolution

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial\tau} + \sum_{\alpha=1}^{N}\{f, K_\alpha\}v_\alpha + \{f, K_{N+1}\}v_{N+1} \tag{86}$$

has coefficients v which are fixed by the requirement that each χ be preserved in time. In this way we find

$$\frac{d\chi_{N+1}}{d\tau} \approx 0 \Rightarrow v_{N+1} = \frac{1}{2m_{N+1}}. \tag{87}$$

For the equations of motion for x_{N+1}, p_{N+1} we have

$$\frac{dx_{N+1}}{d\tau} \approx \frac{p_{N+1}}{m_{N+1}}, \quad \frac{dp_{N+1}}{d\tau} \approx 0. \tag{88}$$

Thus the $(N+1)$th particle indeed moves freely, but it supplies its own "rest-frame time" as the evolution parameter for the entire system. In particular, let us note that the physical changes of inertial frame $\mathcal{O}\to\mathcal{O}' = (\Lambda, a)\mathcal{O}$ are represented by Dirac canonical transformation $R^*(\Lambda, a)$ in the final phase space of all $(N+1)$ particles with the generators being

$$J_{\mu\nu} = \sum_{\alpha=1}^{N+1}(x_{\mu\alpha}p_{\nu\alpha} - x_{\nu\alpha}p_{\mu\alpha}), \quad P_\mu = \sum_{\alpha=1}^{N+1}p_{\mu\alpha}, \tag{89}$$

even though it is true that the contributions to $J_{\mu\nu}$ and P_μ from particles $1,2,\ldots,N$ and from particle $(N+1)$ are separately constants of motion.

This particular model suggests that we may have succeeded in avoiding the no-interaction theorem merely by including the dynamical variables of a noninteracting but observing particle, in short *a passive observer*, in the overall framework. However, it is more appropriate to say that such a possibility *is one of many that arise in our extended form of relativistic dynamics*.

VI. CONCLUDING REMARKS

The work described in this paper is devoted to an analysis of relativistic theories of N interacting particles making use of the constant formalism of Dirac. The starting point is a collection of quadruplets of canonical pairs one for each particle. The four-momentum variables of each particle is constrained by one constraint each; and these N constraints are required to be first class. Even with these constraints the system is not yet a collection of particles since the allowed region Σ of phase space is $7N$-dimensional and mapped into itself by the Poincaré group $R(\Lambda, a)$ acting on the primitive $8N$-dimensional space. In view of the first-class nature of the N constraints K_α these constraints generate commuting canonical transformations. Any point in the $7N$-dimensional constrained surface Σ is taken into an N-dimensional sheet S by these transformations. On the other hand a collection of N particles would have an initial state labeled by $6N$ variables; and *dynamical evolution should be described by a one-dimensional curve* indexed by an evolution parameter τ. We shall have to introduce then N constraints χ_α which form a second-class system together with the K_α, and the parameter of evolution τ should enter at least one of the constraints. When *equipped with these constraints the system may be viewed as a system of N particles.*

We recognize that much of the spirit of our study follows the various discoveries and ideas of Dirac, we find that *our formalism for relativistic dynamics goes beyond the four forms of relativistic dynamics that were outlined by Dirac.* We need to allow dynamically (rather than kinematically) defined Lorentz frames. In the process we have been led to consider eleven distinguished generators corresponding to the Poincaré transformations and the dynamical evolution of the system. The dynamical evolution may be identified with the time-translation operator as a special but not useful choice when describing interacting particles. It is our belief that we have definitely made an advance in relation to the formulation of relativistic dynamics.

It would be incorrect and unsatisfactory to consider the choice of the χ constraints as "gauge" conditions.[14] The physics of the evolution of the system does critically depend upon the choice as well as the relativistic invariance.

The world-line condition is an added restriction to the dynamics going beyond that of relativistic invariance. The no-interaction theorem having been proved cannot be set aside or invalidated without abandoning the theory. The N-dimensional sheets are relativistically invariant, but they do not yet describe particles and their existence and characterization neither affirms nor denies the no-interaction theorem.

What we need therefore is to try to impose a requirement different from the ones that led to the no-interaction theorem. The world-line condition[4] discussed in the text is such a choice. With this choice and in the framework of the post-Dirac formulation of relativistic dynamics we are able to obtain dynamical results.

The requirement of cluster decomposition brings in a host of new problems.[16] Our discussion of these should serve to focus attention on the fact that these go beyond all the other requirements imposed on the system.

The particularly interesting results on the timekeeper particle may evoke different responses from different readers. On the one hand every clock is a more or less isolated subunit; and as such treating it as a "particle" (free or otherwise) or a subcollection of particles is no essential limitation. Therefore, one may argue, we have essentially overcome the limitations imposed by the no-interaction theorem. On the other hand, one could argue that as long as the frame of the clock is a dynamical system we have gone outside the dynamical framework for considering objective world lines which are purely geometrical. We leave it to the reader to decide how much of an advance we have made.

The authors themselves are not in complete agreement about the physical significance of the χ constraints. In this paper the attitude is taken that the dynamical system is not defined until the χ constraints have been specified. However, in the following paper, the point of view is taken that the dynamics of the interacting particles is given by the K constraints while the χ constraints define a *class of observers* by fixing the relationship among the particle coordinates at a definite value of the evolution parameter τ. The "kinematic" or "dynamic" choice of τ then appears as a relatively unimportant choice of clock and clock-rate. To some extent this difference in point of view is only semantic. It does, however, give additional insight into the meaning and significance of the WLC.

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