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Dirac positive-energy wave equation with para-Bose internal variables

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Dirac's new infinite-component positive-energy relativistic wave equation is generalized by introducing para-Bose internal dynamical variables. Electromagnetic interaction of Dirac's particle is studied and the origin of the difficulty in the original formulation identified. It is seen to be evaded by our generalization, a special case of which for para-Bose variables of order 2 is discussed elsewhere. Our equation also describes a spinless positive-energy unique-mass particle in the absence of external fields. We outline the new Dirac theory in $1+1$ and $2+1$ dimensions.

I. INTRODUCTION

In the conventional treatment of relativistic wave equations we are used to the appearance of positive- and negative-energy solutions in a symmetric way. For finite-component manifestly covariant relativistic wave equations however constructed, this is a consequence of invariance under the complex Lorentz group, one of whose elements is the strong spacetime reflection. Dirac[1,2] has proposed a new relativistic wave equation, hereafter called the Dirac equation, which is not symmetrical between positive and negative energies. This equation describes a spinless particle of unique nonzero mass with positive-definite energy; and there exists a conserved four-vector current with a positive-definite density. An important but unwelcome feature of this new equation is that the conserved particle current cannot interact minimally with an external electromagnetic field because the replacement $p_\mu \to \pi_\mu = p_\mu - eA_\mu$ leads to algebraic inconsistencies.

Attempts have been made to generalize the Dirac equation. Kapuscik[3] has formulated a general class of wave equations from which he derives the Dirac equation as a special case. Biedenharn *et al.*[4] have generalized the Dirac equation to describe particles of mass m and spin s, where s can take on any of the values of $0, \frac{1}{2}, 1, \ldots$. These generalizations also allow only positive-energy solutions, and have a conserved current with positive density. As for the Dirac equation, equally so for the generalized equations of Biedenharn *et al.*, a minimal electromagnetic interaction cannot be consistently introduced.

The novel feature of the Dirac equation is that, in addition to a spacetime coordinate subject to the usual action of the Poincaré group, there are also internal degrees of freedom involving two harmonic oscillators. Biedenharn *et al.*[5] attempted an interpretation of such additional dynamical variables as arising from a system of two subparticles interacting with each other through harmonic forces. One may also seek an interpretation of the Dirac equation or its generalization as an extended particle with a Gaussian type of distribution.

An attempt to construct a multilocal field theory to describe extended particles was made by Yukawa quite some time ago.[6] In the last few years, interest in the Yukawa type of multilocal field theory has been revived in connection with quark confinement. In particular, a multilocal field theory describing subparticles such as quarks interacting with each other harmonically has been discussed extensively.[7,8] An important motivation for constructing a multilocal field theory is to have a divergence-free quantum field theory.

In the formulation by Dirac[1,2] (or by Biedenharn *et al.*[4,5]) the internal degrees of freedom involve two independent sets of bosonic variables. In this paper, we present and study a generalization of the Dirac equation obtained by replacing the boson variables by paraboson variables. We shall refer to the resulting equation as the generalized Dirac equation. Since both Bose and para-Bose oscillators for 2 degrees of freedom lead to an SO(3,2) structure, many features of the Dirac equation might carry over to the generalized case. In fact, we will see that our generalized Dirac equation with para-Bose internal degrees of freedom is also a relativistically invariant wave equation describing a particle with fixed mass, zero spin, and positive energy. Furthermore, in this case minimal coupling to an external electromagnetic field becomes possible. This possibility was realized for a special case of the para-Bose variables of order 2 in a recent paper.[9]

In Sec. II we study the algebraic structure of the para-Bose system for 2 degrees of freedom. A class of representations for this system is also presented. In Sec. III we prove the relativistic invariance of the generalized Dirac equation. For the para-Bose representations used in this paper, we show that the generalized Dirac equation describes spin-zero particles alone. The problem of introducing minimal external electromagnetic interaction is examined in Sec. IV. It is found that owing to the algebraic structure of the para-Bose system no algebraic inconsistencies arise. Some concluding remarks are given in Sec. V. In Appendix A, we demonstrate the emergence of an $SO(4,2)\otimes U(1)$ algebra for para-Bose variables on 2 degrees of freedom, using Green's ansatz.[10] In Appendix B we describe briefly the structure of the Dirac equation and its generalization in $1+1$ and $2+1$ spacetime dimensions.

II. THE PARA-BOSE SYSTEM WITH 2 DEGREES OF FREEDOM

In terms of "position" and "momentum" variables q_r, p_r, $r=1,2$, the commutation relations defining two independent boson oscillators are

$$[q_r, p_s] = i\delta_{rs}\,, \quad [q_r, q_s] = [p_r, p_s] = 0\,. \tag{1}$$

Such variables constitute the internal degrees of freedom in the Dirac equation. For convenience, let us denote by ξ the column vector

$$\xi = \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix}\,. \tag{2}$$

Then Eq. (1) appears as

$$[\xi_a, \xi_b] = i\beta_{ab}\,, \quad a,b=1,\ldots,4\,,$$

$$\beta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\,. \tag{3}$$

For some purposes it is more convenient to use the non-Hermitian annihilation and creation operators in place of the Hermitian q's and p's:

$$a_s = \frac{1}{\sqrt{2}}(q_s + ip_s)\,,$$
$$a_s^\dagger = \frac{1}{\sqrt{2}}(q_s - ip_s)\,, \quad s=1,2\,. \tag{4}$$

Then the Bose relations (1) take the form

$$[a_r, a_s^\dagger] = \delta_{rs}\,, \quad [a_r, a_s] = [a_r^\dagger, a_s^\dagger] = 0\,. \tag{5}$$

It is convenient to make a special choice of the γ matrices at this point. We take

$$\gamma_0 = \beta = i\rho_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma_1 = \rho_3\sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$
$$\gamma_2 = -\rho_1 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma_3 = -\rho_3\sigma_1 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{6}$$

All the γ_μ are real and obey

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\,, \quad g_{00} = -1\,. \tag{7}$$

The fifth matrix is given by

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -i\rho_3\sigma_2 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \gamma_5{}^2 = -1\,. \tag{8}$$

The three matrices γ_j, $j=1,2,3$, are Hermitian, while γ_0 and γ_5 are anti-Hermitian. Because of the transposition property

$$\gamma_\mu^T = -\beta\gamma_\mu\beta^{-1}, \quad \gamma_5^T = \beta\gamma_5\beta^{-1}\,, \tag{9}$$

we find that the "vector" and "tensor" matrices $\beta\gamma_\mu$, $\beta[\gamma_\mu, \gamma_\nu]$ are ten independent symmetric real matrices, while the "scalar," "axial vector," and "pseudoscalar" matrices β, $\beta\gamma_5$, $\beta\gamma_5\gamma_\mu$ are six independent antisymmetric real ones.

It is an immediate consequence of the Bose relations (3) that, if we construct the ten independent symmetric bilinears in ξ in this way,

$$S_{\mu\nu} = \tfrac{1}{8}\xi^T\beta[\gamma_\mu, \gamma_\nu]\xi\,, \quad V_\mu = -\tfrac{1}{4}\xi^T\beta\gamma_\mu\xi\,, \tag{10}$$

then (i) $S_{\mu\nu}$ and V_μ are Hermitian, (ii) on commutation with ξ we have

$$[S_{\mu\nu}, \xi] = \frac{i}{4}[\gamma_\mu, \gamma_\nu]\xi\,, \quad [V_\mu, \xi] = \frac{-i}{2}\gamma_\mu\xi\,, \tag{11}$$

and (iii) among themselves the $S_{\mu\nu}$ and V_μ reproduce the commutation relations of the Lie algebra of $SO(3,2)$:

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\mu\rho}S_{\nu\sigma} - g_{\nu\rho}S_{\mu\sigma} + g_{\mu\sigma}S_{\rho\nu} - g_{\nu\sigma}S_{\rho\mu})\,,$$
$$[S_{\mu\nu}, V_\rho] = i(g_{\mu\rho}V_\nu - q_{\nu\rho}V_\mu)\,, \tag{12}$$
$$[V_\mu, V_\nu] = -iS_{\mu\nu}\,.$$

We recall here the following well-known further features of this construction based on the Bose structure (3): (i) if the commutation relations (3) are realized irreducibly on a Hilbert space $\mathcal{H}_0$, then on exponentiation the $S_{\mu\nu}$ and V_μ generate a unitary $SO(3,2)$ representation [more correctly an $Sp(2,2)$ representation] on $\mathcal{H}_0$, in which an element g in $SO(3,2)$ is represented by a unitary operator $U(g)$; (ii) this $SO(3,2)$ representation, which is characteristic of the fact that we started with Bose operators, is the direct sum of two irreducible unitary representations, each of which is a "remarkable" representation[11]; (iii) on restriction to the Lorentz subgroup $SO(3,1)$ generated by the

$S_{\mu\nu}$, we get a unitary representation $\Lambda \in SO(3,1) \rightarrow U(\Lambda)$ acting on $\mathcal{H}_0$; (iv) this SO(3,1) representation, again characteristic of the underlying Bose[12] structure, is the direct sum of the two Majorana representations thus each of the two irreducible SO(3,2) representations on $\mathcal{H}_0$ remains irreducible on restriction to SO(3,1); (v) because of Eqs. (11), under the unitary transformations $U(g)$, $U(\Lambda)$, the ξ are transformed linearly

$$U(g)\xi U(g)^{-1} = S(g)^{-1}\xi\,, \qquad U(\Lambda)\xi U(\Lambda)^{-1} = S(\Lambda)^{-1}\xi\,, \tag{13}$$

where $S(g)$ is a real 4×4 irreducible nonunitary representation of SO(3,2) while $S(\Lambda)$ is the 4×4 reducible nonunitary spinor representation of SO(3,1) that occurs in the old Dirac equation; (vi) the following additional relations obtain

$$U(\Lambda)V_\mu U(\Lambda)^{-1} = \Lambda^\nu{}_\mu V_\nu\,, \qquad U(\Lambda)S_{\mu\nu}U(\Lambda)^{-1} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu S_{\rho\sigma}\,, \tag{14a}$$

$$S(\Lambda)^{-1}\gamma_\mu S(\Lambda) = \Lambda_\mu{}^\nu \gamma_\nu\,. \tag{14b}$$

In the Dirac equation, the "wave function" $\psi(x)$ is a vector in $\mathcal{H}_0$ for each x.

The para-Bose system for 2 degrees of freedom can now be defined: it is a set of four Hermitian operators ξ which obey the trilinear commutation relations (11), where the bilinear $S_{\mu\nu}$ and V_μ are formed from ξ by Eqs. (10) again. The trivial solution corresponds to setting all ξ_a equal to zero. The Bose solution corresponds to obeying Eq. (3), of which Eqs. (11) will then be consequences. A nontrivial para-Bose system obeys Eqs. (11) but not Eq. (3). Such a system may be reducible or irreducible. Green's ansatz provides us with reducible nontrivial para-Bose systems.[10] Unlike Eq. (3), Eqs. (11) possess infinitely many inequivalent solutions, some of which are described in the sequel.

For a nontrivial para-Bose system, defined on a Hilbert space $\mathcal{H}_0$ say, we may note several important properties. There will be some SO(3,2) unitary representation generated by $S_{\mu\nu}$ and V_μ, and a related SO(3,1) unitary representation generated by $S_{\mu\nu}$, acting on $\mathcal{H}_0$. We may denote the corresponding operators by $U(g)$ and $U(\Lambda)$, with the understanding that the nature of these group representations certainly depends on the specific para-Bose system chosen. All this happens because from Eqs. (10) and (11), we can derive Eqs. (12) as consequences. The system of Eqs. (13) and (14) remains valid in the para-Bose case, with no changes in the 4×4 matrices $S(g)$, $S(\Lambda)$. Since in a nontrivial para-Bose system the commutator $[\xi_a, \xi_b]$ is an operator and not a c number, we can use the six antisymmetric matrices β, $\beta\gamma_5$, and $\beta\gamma_5\gamma_\mu$ to set up corresponding bilinears:

$$S = \frac{-i}{4}\xi^T\beta\xi\,, \qquad P = \frac{-i}{4}\xi^T\beta\gamma_5\xi\,, \qquad A_\mu = \frac{-i}{4}\xi^T\beta\gamma_5\gamma_\mu\xi\,. \tag{15}$$

In terms of q and p we have

$$S = \frac{-i}{4}([q_1,p_1]+[q_2,p_2])\,, \quad P = \frac{-i}{4}([q_1,p_2]-[q_2,p_1])\,,$$
$$A_0 = \frac{-i}{4}([q_2,q_1]+[p_1,p_2])\,, \quad A_1 = \frac{-i}{4}([q_1,p_2]+[q_2,p_1])\,,$$
$$A_2 = \frac{-i}{4}([q_2,q_1]-[p_1,p_2])\,, \quad A_3 = \frac{-i}{4}([q_1,p_1]-[q_2,p_2])\,. \tag{16}$$

Like $S_{\mu\nu}$ and V_μ, the operators S, P, and A_μ are all Hermitian. For the Bose solution we see immediately that P and A_μ vanish and $S=\frac{1}{2}$. (As we will see later, it is exactly the vanishing of P and A_μ that leads to the inconsistency in the Dirac equation when interaction with an external electromagnetic field is introduced through minimal coupling.) It can easily be shown that A_μ and P transform as a five-component vector and S as a scalar under the SO(3,2) transformations $U(g)$ generated by $S_{\mu\nu}$, V_μ. Furthermore if we assume that the para-Bose operators are given by Green's ansatz,[10] we find that $S_{\mu\nu}$, V_μ, A_μ, and P form an SO(4,2) algebra with S generating an invariant Abelian transformation. This is shown in Appendix A.

The above description of the para-Bose ring was in terms of the Hermitian q's and p's, and moreover gave prominence to the SO(3,1) subgroup of the naturally occurring SO(3,2) structure. We now transcribe the description to deal with the more familiar oscillator operators $a, a^\dagger$, and expose the $SO(3)\otimes SO(2)$ subgroup of SO(3,2) to facilitate construction of a class of irreducible nontrivial para-Bose representations. Let us arrange the a's and $a^\dagger$'s into the column vector

$$\xi' = \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \\ a_2 \\ -a_1 \end{bmatrix}\,. \tag{17}$$

The Bose relations (3), (5) would take the form

$$[\xi'_a, \xi'_b] = i\beta'_{ab}\,, \quad \beta' = \rho_1\sigma_2\,. \tag{18}$$

If we use a new set of γ' matrices defined as

$$\gamma'_0 = i\rho_3\,, \quad \gamma'_1 = -\rho_1\sigma_1\,, \qquad \gamma'_2 = \rho_1\sigma_2\,, \quad \gamma'_3 = -\rho_1\rho_3\,, \tag{19}$$

then the symmetric bilinears $S_{\mu\nu}$ and V_μ appear as follows:

$$S_{\mu\nu}=\tfrac{1}{8}\,\xi^T\beta[\gamma_\mu,\gamma_\nu]\xi=\tfrac{1}{8}\,\xi'^T\beta'[\gamma'_\mu,\gamma'_\nu]\xi',$$
$$V_\mu=\frac{-1}{4}\,\xi^T\beta\gamma_\mu\xi=\frac{-1}{4}\,\xi'^T\beta'\gamma'_\mu\xi'. \tag{20}$$

The defining trilinear para-Bose commutation relations (11) then read in familiar form:

$$[a_r,\tfrac{1}{2}\{a_s^\dagger,a_u\}]=\delta_{rs}a_u,$$
$$[a_r,\tfrac{1}{2}\{a_s,a_u\}]=0, \tag{21a}$$

$$[a_r^\dagger,\tfrac{1}{2}\{a_s^\dagger,a_u\}]=-\delta_{ru}a_s^\dagger,$$
$$[a_r^\dagger,\tfrac{1}{2}\{a_s^\dagger,a_u^\dagger\}]=0, \tag{21b}$$
$$[a_r,\tfrac{1}{2}\{a_s^\dagger,a_u^\dagger\}]=\delta_{rs}a_u^\dagger+\delta_{ru}a_s^\dagger,$$
$$[a_r^\dagger,\tfrac{1}{2}\{a_s,a_u\}]=-\delta_{rs}a_u-\delta_{ru}a_s.$$

In fact, the four Eqs. (21b) are consequences of the two Eqs. (21a). We rearrange $S_{\mu\nu}$ and V_μ into the following combinations:

$$J_j=\tfrac{1}{2}\epsilon_{jkl}S_{kl}=\tfrac{1}{4}\,(a^\dagger\sigma_j a+a^T\sigma_j^T a^{\dagger T}),$$
$$L_j=S_{0j}+iV_j=\tfrac{1}{2}\,a^\dagger\sigma_j c a^{\dagger T}, \tag{22}$$
$$M_j=S_{0j}-iV_j=-\tfrac{1}{2}\,a^T c\sigma_j a,$$
$$K=V_0=\tfrac{1}{4}\,(a^\dagger a+a^T a^{\dagger T}).$$

Here we have used the notation that

$$a=\begin{pmatrix}a_1\\a_2\end{pmatrix},\quad a^\dagger=(a_1^\dagger a_2^\dagger),\quad c=i\sigma_2. \tag{23}$$

Writing the SO(3, 2) generators in the pattern of Eq. (22) exhibits the structure with respect to the maximal compact subgroup which is generated by $\vec{J}$ and K: both $\vec{L}$ and $\vec{M}$ are SO(3) vectors while they, respectively, raise and lower the eigenvalues of K by unity. The SO(3, 2) commutation relations (12) can be given equally well in this form:

$$[J_j,J_k]=i\epsilon_{jkl}J_l,\quad [J_j,K]=0, \tag{24a}$$

$$[J_j,L_k]=i\epsilon_{jkl}L_l,\quad [J_j,M_k]=i\epsilon_{jkl}M_l,$$
$$[K,L_j]=L_j,\quad [K,M_j]=-M_j, \tag{24b}$$

$$[L_j,L_k]=[M_j,M_k]=0,$$
$$[L_j,M_k]=-2\delta_{jk}k-2i\epsilon_{jkl}J_l, \tag{24c}$$

while $\vec{J}$ and k are Hermitian, $\vec{L}^\dagger=\vec{M}$.

Let us now study a class of representations of the para-Bose ring for 2 degrees of freedom. We restrict ourselves for the purpose of this paper to those representations which have a unique "vacuum" or ground state annihilated by a_1 and a_2; although the generalized Dirac equation will be covariant with the use of *any* para-Bose representation. As we will see in the next section, the generalized Dirac equation will describe a single spin-zero particle only if the vacuum is unique. The general irreducible para-Bose representations will be presented elsewhere. Let us define the vacuum $|0\rangle$ as follows:

$$a_s|0\rangle=0,\quad s=1,2. \tag{25}$$

From its assumed uniqueness and the commutation relations (21a) we have

$$a_r a_s^\dagger|0\rangle=c_{rs}|0\rangle, \tag{26}$$

where c_{rs} are numbers. Furthermore, since

$$[\tfrac{1}{2}\{a_r^\dagger,a_s\},a_u a_u^\dagger]=\delta_{su}a_u a_r^\dagger-\delta_{ru}a_s a_u^\dagger,\quad\text{no sum on }u, \tag{27}$$

we have

$$(\delta_{su}c_{ur}-\delta_{ru}c_{su})|0\rangle=0,\quad\text{no sum on }u, \tag{28}$$

which implies

$$c_{ur}=\eta\delta_{ur}. \tag{29}$$

Therefore, uniqueness of the vacuum implies existence of a (real nonnegative) η such that

$$a_r a_s^\dagger|0\rangle_\eta=\eta\delta_{rs}|0\rangle_\eta. \tag{30}$$

Here we have labeled the vacuum state by η, which may be called the para-Bose order. The case $\eta=0$ is the trivial solution in which all a_j and $a_j^\dagger$ vanish. (Incidentally this possibility is excluded for the Bose ring.) Adapting a method given by Greenberg and Messiah,[13] we may show that in all other representations we must have

$$\eta\geq 1. \tag{31}$$

One computes the norm of the vector

$$|\Phi\rangle=\tfrac{1}{2}\,(a_1^\dagger a_2^\dagger-a_2^\dagger a_1^\dagger)|0\rangle_\eta \tag{32}$$

to obtain

$$\langle\Phi|\Phi\rangle=2_\eta\langle 0|a_2a_1|\Phi\rangle=\eta(\eta-1). \tag{33}$$

Since η cannot be negative it follows that if $\eta\neq 0$, it must not be less than unity. The case $\eta=1$ corresponds to the ordinary Bose solution since in this case $|\Phi\rangle$ itself vanishes. We have described earlier in this section the SO(3, 2) algebra generated by the Bose ring with 2 degrees of freedom; it has been studied by us elsewhere in connection with Majorana's infinite component relativistic equation.[14] The integer values of η are associated with the realization of para-Bose systems in terms of Green's ansatz[10] discussed in Appendix A. Odd integer values of η lead to "remarkable" representations.[11]

In the general case it can be shown that

$$J_j|0\rangle_\eta=0,\quad K|0\rangle_\eta=\tfrac{1}{2}\,\eta|0\rangle_\eta. \tag{34}$$

From the vacuum we can construct states which are simultaneous eigenstates of J_3, $\vec{J}^2$, and K. Indeed let us define

$$|jmn\rangle_\eta = \frac{1}{(N_{jmn})^{1/2}} \tfrac{1}{2}\{a_1^\dagger, a_2\}^{j+m} \quad (a_2^\dagger)^{2j} \tfrac{1}{2}[a_1^\dagger, a_2]^n |0\rangle_\eta \,, \quad N_{jmn} = \frac{(j+m)!\,2j!}{(j-m)!} N_{j-jn} \,,$$

$$N_{j-jm} = \begin{array}{c|cc} 2j \diagdown n & + & - \\ + & \begin{array}{c}(2j)!!\,(n+\eta+2j-2)!! \\ \times \dfrac{(n+2j+1)!!\,(\eta+n-3)!!\,n!!}{(2j+1)!!\,(\eta-2)!}\end{array} & \begin{array}{c}(2j)!!\,(n+\eta+2j-1)!! \\ \times \dfrac{(n+2j)!!\,(\eta+n-2)!!\,(n-1)!!}{(2j+1)!!\,(\eta-2)!}\end{array} \\ - & \begin{array}{c}(2j-1)!!\,(n+\eta+2j-1)!! \\ \times \dfrac{(n+1)!!\,(\eta+n-3)!!\,n!!}{(2j)!!\,(\eta-2)!}\end{array} & \begin{array}{c}(2j-1)!!\,(n+\eta+2j-1)!! \\ \times \dfrac{(n+2j+1)!!\,(\eta+n-2)!!\,(n-1)!!}{(2j)!!\,(n-2)!}\end{array} \end{array} \tag{35}$$

where the + means even and − means odd. Then we can show that these are normalized eigenstates of $J_3, \vec{J}^2, K$ with eigenvalues m, $j(j+1)$, and $(j+n+\eta/2)$, respectively. Here j takes the values $0, \frac{1}{2}, 1, \ldots$ and independently n takes the values $0, 1, 2, \ldots$. Of course m runs over $j, j-1, \ldots, -j+1, -j$. These states $|jmn\rangle_\eta$ span a Hilbert space $\mathcal{H}_0$ carrying an irreducible para-Bose representation with a unique vacuum and characterized by $\eta > 1$. The representation is completely specified by giving the action of a_j and $a_j^\dagger$ on these basis states. These equations are

$$a_1|jmn\rangle = \alpha_{jmn}|j-\tfrac{1}{2}\,m-\tfrac{1}{2}\,n\rangle + \beta_{jmn}|j+\tfrac{1}{2}\,m-\tfrac{1}{2}\,n-1\rangle \,,$$

$$a_2|jmn\rangle = \gamma_{jmn}|j-\tfrac{1}{2}\,m+\tfrac{1}{2}\,n\rangle + \rho_{jmn}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\,n-1\rangle \,,$$

$$a_1^\dagger|jmn\rangle = \alpha_{j+1/2\,m+1/2\,n}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\,n\rangle + \beta_{j-1/2\,m+1/2\,n+1}|j-\tfrac{1}{2}\,m+\tfrac{1}{2}\,n+1\rangle \,,$$

$$a_2^\dagger|jmn\rangle = \gamma_{j+1/2\,m-1/2\,n}|j+\tfrac{1}{2}\,m-\tfrac{1}{2}\,n\rangle + \rho_{j-1/2\,m-1/2\,n+1}|j-\tfrac{1}{2}\,m-\tfrac{1}{2}\,n+1\rangle \,,$$

$$\alpha_{jmn} = (m+j)\left(\frac{N_{j-1/2\,m-1/2\,n}}{N_{j-1/2\,m+1/2\,n}}\right)^{1/2}\gamma_{jmn} \,, \quad \beta_{jmn} = (m+j)(-2j-1)\left(\frac{N_{j+1/2\,m-1/2\,n-1}}{N_{j+1/2\,m+1/2\,n-1}}\right)^{1/2} ,$$

$$\gamma_{jmn} = \begin{array}{c|cc} 2j \diagdown n & + & - \\ + & \left[\dfrac{(n+2j+1)(j-m)}{2j+1}\right]^{1/2} & \left[\dfrac{(\eta+n+2j-1)(j-m)}{2j+1}\right]^{1/2} \\ - & \left[\dfrac{(n+\eta+2j-1)(j-m)}{2j}\right]^{1/2} & \left[\dfrac{(n+2j+1)(j-m)}{2j}\right]^{1/2} \end{array} \tag{36}$$

$$\rho_{jmn} = \begin{array}{c|cc} 2j \diagdown n & + & - \\ + & -\left[\dfrac{n(m+j+1)}{2j+1}\right]^{1/2} & -\left[\dfrac{(m+j+1)(\eta+n-2)}{2j+1}\right]^{1/2} \\ - & \left[\dfrac{n(m+j+1)}{2j+2}\right]^{1/2} & \left[\dfrac{(m+j+1)(\eta+n-2)}{2j+2}\right]^{1/2} \end{array} .$$

III. THE GENERALIZED DIRAC EQUATION

The equation is given by

$$(\gamma_\mu \partial^\mu + m)\xi\psi(x) = 0 \,, \tag{37}$$

with ξ any solution of the para-Bose algebra (other than the Bose solution) in some Hilbert space $\mathcal{H}_0$. Here ψ has only one "component," that is, $\psi(x)$ is a scalar function of x with values in $\mathcal{H}_0$. We have a *one*-component wave function obeying *four* partial differential equations.

Let us first study the relativistic invariance of this equation. For this we specify that under the inhomogeneous Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,, \tag{38}$$

the wave function $\psi(x)$ changes according to

$$\psi'(x') = U(\Lambda)\psi(x) \,. \tag{39}$$

Here $U(\Lambda)$ is that unitary Lorentz group representation in $\mathcal{H}_0$ that, according to the previous section, is generated by the bilinear $S_{\mu\nu}$ in ξ. With the help of Eqs. (13), (14b), (38), and (39), we see that if $\psi(x)$ obeys the generalized Dirac equation, then so will $\psi'(x')$:

$$\begin{aligned}(\gamma_\mu \partial'^\mu + m)\xi\psi'(x') &= (\gamma_\mu \Lambda^\mu_\nu \partial^\nu + m)\xi U(\Lambda)\psi(x)\\ &= [S(\Lambda)\gamma_\nu S(\Lambda)^{-1}\partial^\nu + m]U(\Lambda)S(\Lambda)\xi\psi(x)\\ &= S(\Lambda)U(\Lambda)(\gamma_\nu\partial^\nu + m)\xi\psi(x) = 0\,. \end{aligned} \tag{40}$$

Therefore, like the Dirac equation, our generalization of it is also relativistically invariant: the replacement of the internal Bose variables by para-Bose variables in no way spoils this property. The only difference is that the specific Lorentz group representation $U(\Lambda)$ by which ψ transforms depends on the para-Bose solution used.

Next let us consider the plane-wave solutions

$$\psi(x) = e^{iP\cdot x}u(P)\,, \quad u(P) \in \mathcal{H}_0\,. \tag{41}$$

Assuming P is timelike, in the rest frame we have

$$\begin{aligned}&(iP^0 p_1 + mq_1)u = 0\,, \quad (iP^0 p_2 + mq_2)u = 0\,,\\ &(-iP^0 q_1 + mp_1)u = 0\,, \quad (-P^0 q_2 + mp_2)u = 0\,.\end{aligned} \tag{42}$$

These equations imply $(P^0)^2 = m^2$, so we must have $P^0 = \pm m$. At this point we restrict the analysis to the class of para-Bose representations given in Sec. II, having a unique vacuum state. For the negative-energy case, $P^0 = -m$, Eqs. (42) require

$$a_1^\dagger u = a_2^\dagger u = 0\,, \tag{43}$$

and clearly no such vector u exists in $\mathcal{H}_0$. For $P^0 = m$, Eqs. (42) become

$$a_1 u = a_2 u = 0\,, \tag{44}$$

with the unique (up to a factor) solution $u = |0\rangle_\eta$.

The eigenvalue of $\vec{\mathfrak{J}}^2$ gives us the (nonorbital) angular momentum. From Eq. (34) we have for the vacuum state $j=0$. Since we must interpret angular momentum in the rest frame as the spin, we see that our generalized Dirac equation describes a spin-zero particle with mass m and positive energy, just as does the Dirac equation, *provided* we consider para-Bose representations with unique vacuum. It may be easily checked that our generalized Dirac equation has no spacelike or lightlike solutions.[14]

IV. MINIMAL ELECTROMAGNETIC INTERACTION

We now study the problem of minimal coupling to an external electromagnetic field. The interaction is introduced by simply replacing ∂_μ by $\pi_\mu = \partial_\mu - ieA_\mu$, when Eq. (37) becomes

$$(\gamma_\mu \pi^\mu + m)\xi\psi(x) = 0 \tag{45}$$

or

$$\begin{aligned}&\tau_a\psi(x) = 0\,,\\ &\tau_a = (\gamma_\mu\pi^\mu + m)_{ab}\xi_b\,, \quad a = 1, \ldots, 4\,.\end{aligned} \tag{46}$$

It was stated by Dirac that his equation becomes inconsistent in the presence of such coupling. This fact has been explicitly proved by Biedenharn *et al.*[4] by a series of elegant calculations which differ somewhat from the following analysis.

The consistency of Eqs. (46) requires that

$$[\tau_a, \tau_b]\psi(x) = 0\,. \tag{47}$$

In what follows we will show that this consistency condition leads to constraint equations which imply for the Dirac equation either $F_{\mu\nu} = 0$ or $\psi = 0$; but for the generalized Dirac equation no such consequences are implied. The commutator in Eq. (47) may be decomposed into two parts:

$$\begin{aligned}[\tau_a, \tau_b] &= [(\gamma_\mu\pi^\mu + m)_{ac}\xi_c, (\gamma_\nu\pi^\nu + m)_{bd}\xi_d]\\ &= X_{ab} + Y_{ab}\,,\\ X_{ab} &= \frac{ie}{2}\gamma^\mu_{ac}\gamma^\nu_{bd}F_{\mu\nu}\{\xi_c, \xi_d\}\,,\\ Y_{ab} &= \tfrac{1}{2}\{(\gamma_\mu\pi^\mu + m)_{ac}, (\gamma_\nu\pi^\nu + m)_{bd}\}[\xi_c, \xi_d]\,.\end{aligned} \tag{48}$$

Both terms X_{ab} and Y_{ab} are skew symmetric in a and b, and so each may be expanded in terms of the six independent skew-symmetric matrices β, $\beta\gamma_5$, and $\beta\gamma_5\gamma_\lambda$ with unique (operator) coefficients:

$$\begin{aligned}X_{ab} &= \mathfrak{A}\beta_{ab} + \mathfrak{B}(\beta\gamma_5)_{ab} + \mathfrak{C}^\lambda(\beta\gamma_5\gamma_\lambda)_{ab}\,,\\ Y_{ab} &= [\mathfrak{A}'_{cd}\beta_{ab} + \mathfrak{B}'_{cd}(\beta\gamma_5)_{ab} + \mathfrak{C}'^\lambda_{cd}(\beta\gamma_5\gamma_\lambda)_{ab}][\xi_c, \xi_d]\,.\end{aligned} \tag{49}$$

It is somewhat tedious but straightforward to compute these coefficients; they are

$$\begin{aligned}\mathfrak{A} &= -ieF^{\mu\nu}S_{\mu\nu}\,,\\ \mathfrak{B} &= \frac{ie}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}\,,\\ \mathfrak{C}^\lambda &= ie\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}V_\rho\,,\end{aligned} \tag{50a}$$

$$\begin{aligned}\mathfrak{A}'_{cd} &= -(\pi^2 - m^2)\beta_{cd}\,,\\ \mathfrak{B}'_{cd} &= (\pi^2 + m^2)(\beta\gamma_5)_{cd} + 2m(\beta\gamma_5\gamma_\mu\pi^\mu)_{cd}\,,\\ \mathfrak{C}'^\lambda_{cd} &= 2m\pi^\lambda(\beta\gamma_5)_{cd} + [\pi^\rho\pi^\lambda + \pi^\lambda\pi^\rho\\ &\quad + (m^2 - \pi^2)g^{\lambda\rho}](\beta\gamma_5\gamma_\rho)_{cd}\,.\end{aligned} \tag{50b}$$

Since β, $\beta\gamma_5$, and $\beta\gamma_5\gamma_\lambda$ are linearly independent, the consistency conditions (47) reduce to the following system of equations on ψ:

$$\{ieF^{\mu\nu}S_{\mu\nu}+(\pi^2-m^2)\beta_{cd}[\xi_c,\xi_d]\}\psi=0\,, \tag{51a}$$

$$\left\{\frac{ie}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}+[(\pi^2+m^2)\beta\gamma_5+2m\beta\gamma_5\gamma_\mu\pi^\mu]_{cd}[\xi_c,\xi_d]\right\}\psi=0\,, \tag{51b}$$

$$(ie\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}V_\rho+\{2m\pi^\lambda\beta\gamma_5+[\pi^\lambda\pi^\rho+\pi^\rho\pi^\lambda+(m^2-\pi^2)g^{\lambda\rho}]\beta\gamma_5\}[\xi_c,\xi_d])\psi=0\,. \tag{51c}$$

It is easy to show that (45) implies

$$(m\beta\gamma_5+\beta\gamma_5\gamma_\mu\pi^\mu)_{cd}[\xi_c,\xi_d]\psi=0\,. \tag{52}$$

On using Eq. (52) in Eqs. (51), and also introducing the "antisymmetric bilinear" S, P, A_λ as defined in Eq. (15), the consistency requirements on ψ take the compact form

$$[ieF^{\mu\nu}S_{\mu\nu}+8(\pi^2-m^2)S]\psi=0\,, \tag{53a}$$

$$[ie\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}+8(\pi^2-m^2)P]\psi=0, \tag{53b}$$

$$[ie\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}V_\rho+8ieF^{\lambda\mu}A_\mu-8(\pi^2-m^2)A^\lambda]\psi=0\,. \tag{53c}$$

In the field-free case, i.e., $F_{\mu\nu}=0$, these conditions reduce to

$$(\partial_\mu\partial^\mu-m^2)S\psi=(\partial_\mu\partial^\mu-m^2)P\psi=(\partial_\mu\partial^\mu-m^2)A^\lambda\psi=0\,. \tag{54}$$

These are indeed satisfied since the allowed solutions of Eq. (37) lie on the mass shell. In particular, for the Dirac equation with Bose variables when $S=\frac{1}{2}$ while P and A_λ vanish, we just obtain the Klein-Gordon equation for ψ; this was of course obtained by Dirac.

With an external field but with Bose internal variables, i.e., for Dirac's equation, Eqs. (53) become

$$[ieF^{\mu\nu}S_{\mu\nu}+4(\pi^2-m^2)]\psi=0\,, \tag{55a}$$

$$ie\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}S_{\rho\sigma}\psi=0\,, \tag{55b}$$

$$ie\epsilon^{\lambda\mu\nu\rho}F_{\mu\nu}V_\rho\psi=0\,. \tag{55c}$$

Equations (55b) and (55c) are the constraint equations found by Biedenharn *et al.*[4] One can see that these constraints are inadmissible by considering the simple case of a constant external magnetic field along, say, the third axis. These constraints imply that ψ is annihilated by S_{03}, V_3, and V_0. However, V_0 is positive definite. Therefore, we deduce that $F_{\mu\nu}$ itself should vanish: for the Dirac equation we cannot introduce minimal coupling to an external electromagnetic field.

For our generalized Dirac equation with nontrivial para-Bose internal degrees of freedom, Eqs. (53b) and (53c) do not disappear as $F_{\mu\nu}\to 0$; correspondingly when $F_{\mu\nu}$ is nonzero, these are equations of motion for ψ, involving the spacetime derivatives of ψ, and not merely algebraic constraints on ψ. Hence minimal electromagnetic coupling is no longer forbidden.

V. CONCLUDING REMARKS

The Dirac equation of 1971, while built on a fascinating algebraic structure, was seriously flawed in its inability to consistently interact in minimal fashion with electromagnetism. This problem has been solved in this paper via a generalization in which the Bose internal variables give way to para-Bose variables generalizing our earlier work.[9] Our equation does not belong to the general pattern considered by Kapuscik.[3]

The particular para-Bose representations used here are distinguished by the existence of a unique vacuum, and so the generalized Dirac equation describes spin-zero particles. However, other para-Bose representations exist, with several independent vectors being all annihilated by a_1 and a_2. With their use one expects to describe via the generalized Dirac equation particles with nonzero spin, of course again able to consistently interact with the Maxwell field. In this manner we would have been able to obtain the higher spin analog of the Dirac equation without introducing an ever-increasing number of spacetime derivatives in the equation (cf. Ref. 4).

The essential complication arising with the use of para-Bose variables is that simple descriptions of the internal space $\mathcal{H}_0$ no longer exist; for instance, we cannot assume q_1 and q_2 diagonal. For the same reason, a limiting semiclassical description, along the lines indicated by Dirac,[2] seems considerably harder for our equation.

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APPENDIX A: PARA-BOSE ALGEBRA

Let us combine the ten SO(3, 2) generators $S_{\mu\nu}$, V_μ of Eq. (10) into a set $S_{AB}=-S_{BA}$, A, B

$=0, 1, 2, 3, 5$ by identifying V_μ with $S_{\mu 5}$. Then the commutation relations (12) are

$$[S_{AB}, S_{CD}] = i(g_{AC}S_{BD} - g_{BC}S_{AD} + g_{AD}S_{CB} - g_{BD}S_{CA}), \tag{A1}$$

$$g_{00} = g_{55} = -1, \quad g_{11} = g_{22} = g_{33} = 1 .$$

By making use of the defining para-Bose relations (11), which in fact give (A1), we can now show that S_{AB} and A_B (where $A_5 = P$) obey

$$[S_{AB}, A_C] = i(g_{AC}A_B - g_{BC}A_A) . \tag{A2}$$

Thus A_B transforms as a five-vector under SO(3, 2), whatever representation of the para-Bose system is chosen.

Let us consider the cases with integer η which arise on reduction of Green's construction.[10] We assume the form

$$\xi_a = \sum_{r=1}^{\eta} \xi_a^r , \tag{A3}$$

where the ξ_a^r satisfy the mixed relations

$$\{\xi_a^r, \xi_b^s\} = 0, \quad r \neq s, \quad [\xi_a^r, \xi_b^r] = i\beta_{ab} . \tag{A4}$$

We immediately see that

$$\{\xi_a, \xi_b\} = \sum_r \{\xi_a^r, \xi_b^r\},$$
$$[\xi_a, \xi_b] = i\eta\beta_{ab} + 2\sum_{r\neq s} \xi_a^r\xi_b^r . \tag{A5}$$

Therefore we may write S_{AB}, A_B, and S as

$$S_{\mu\nu} = \tfrac{1}{8}\sum_r \xi^{rT}\beta[\gamma_\mu, \gamma_\nu]\xi^r ,$$
$$V_\mu = S_{\mu 5} = \tfrac{1}{4}\sum_r \xi^{rT}\beta\gamma_\mu\xi^r ,$$
$$A_\mu = \frac{-i}{4}\sum_{r\neq s} \xi^{rT}\beta\gamma_5\gamma_\mu\xi^s , \tag{A6}$$
$$P = A_5 = \frac{-i}{4}\sum_{r\neq s} \xi^{rT}\beta\gamma_5\xi^s ,$$
$$S = \tfrac{1}{2}\eta - \frac{i}{4}\sum_{r\neq s} \xi^{rT}\gamma\xi^s .$$

We have already the attractive set of commutation relations (A1) and (A2). Let us now find the commutators among the A_B. To simplify the notation we write

$$A_B = M_{ab}(B)\sum_{r\neq s} \xi_a^r\xi_a^s . \tag{A7}$$

For each $B = 0, 1, 2, 3, 5$, $M_{ab}(B)$ is an antisymmetric matrix. We then have

$$[A_A, A_B] = M_{ab}(A)M_{cd}(B)\sum_{r\neq s}\sum_{u\neq v}[\xi_a^r\xi_b^s, \xi_c^u\xi_d^v] . \tag{A8}$$

With the help of Eq. (A4) we can easily see that the last commutator here vanishes if all four superscripts r, s, u, v are distinct. For terms with one equal pair of superscripts, for example $r = u$, $s \neq v$, we have

$$[\xi_a^r\xi_b^s, \xi_c^r\xi_d^v] = (\xi_a^r\xi_c^r + \xi_c^r\xi_a^r)\xi_d^v\xi_b^s . \tag{A9}$$

Similar expressions result for the other three ways in which just one of the pair (r, s) coincides with one of the pair (u, v). Taking this set of four kinds of terms together and invoking the antisymmetry of $M(A)$ and $M(B)$, we find that these terms cancel against one another and drop out. For $r = u$, $s = v$, we have

$$[\xi_a^r\xi_b^s, \xi_c^r\xi_d^s] = [\xi_c^r, \xi_a^r]\xi_b^s\xi_d^s + \xi_c^r\xi_a^r[\xi_d^s, \xi_b^s]$$
$$= i\beta_{ca}\xi_b^s\xi_d^s + i\beta_{db}\xi_c^r\xi_a^r . \tag{A10}$$

The case $r = v$, $s = u$ behaves similarly and gives a factor of 2 since $M(B)$ is antisymmetric. Therefore the only surviving terms in (A8) are

$$2M_{ab}(A)M_{cd}(B)\sum_{r\neq s}[\xi_a^r\xi_b^s, \xi_c^r\xi_d^s]$$
$$= 2i\sum_{r\neq s}[\xi^{sT}M(A)^T\beta^T M(B)\xi^s + \xi^{rT}M(B)\beta M(A)^T\xi^r]$$
$$= 2i(\eta - 1)\sum_r \xi^{rT}[M(A)\beta M(B) - M(B)\beta M(A)]\xi^r . \tag{A11}$$

Now the matrix standing between ξ^{rT} and ξ^r is *symmetric*, so we can exploit the first Eqs. (A4) to rewrite (A11) as

$$[A_A, A_B] = 2i(\eta - 1)\xi^T[M(A)\beta M(B) - M(B)\beta M(A)]\xi , \tag{A12}$$

where the *total* ξ of Eq. (A3) appears. By taking $M(\mu) = (-i/4)\beta\gamma_5\gamma_\mu$ and $M(5) = (-i/4)\beta\gamma_5$, we get

$$[A_\mu, A_\nu] = \frac{i}{8}(\eta - 1)\xi^T\beta[\beta_\mu, \gamma_\nu]\xi = i(\eta - 1)S_{\mu\nu} ,$$
$$[A_\mu, A_5] = \frac{i}{4}(\eta - 1)\xi^T\beta\gamma_\mu\xi = i(\eta - 1)S_{\mu 5} . \tag{A13}$$

Thus, the set of 15 Hermitian operators S_{AB} and $(\eta - 1)^{-1/2}A_B$ realize the Lie algebra of SO(4, 2). Since we can also show that S commutes with all of these 15 operators, the entire collection of 16 bilinears in ξ generate an SO(4, 2) $\otimes$ U(1) representation.

APPENDIX B: DIRAC'S NEW EQUATION IN (1 + 1) AND (2 + 1) DIMENSIONS

We have seen that the essential reason why Dirac's new equation became inconsistent under minimal interaction by new unsatisfactory constraints arising. We traced this to the vanishing of the several antisymmetric bilinears in the internal Bose variable ξ in Dirac's theory. The use of para-Bose internal variables ξ resolved this difficulty.

In this appendix we investigate Dirac's equation in a spacetime with only one or two spatial dimensions. The equation may be written

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right)\xi\psi = 0 . \tag{B1}$$

The demonstration of the relativistic invariance of the equation follows the arguments given by Dirac and in this paper.

The interesting fact is that in $1+1$ and $2+1$ dimensions, the Dirac matrices may be chosen as 2×2 matrices. This simplifies the calculations enormously since there are three symmetric matrices and only one antisymmetric matrix. In the $(2+1)$-dimensional case the symmetric matrices are $\beta\gamma^\mu$ which are the same apart from normalization as the matrices $\beta[\gamma_\nu, \gamma_\lambda]\epsilon^{\mu\nu\lambda}$, while the antisymmetric matrix is β. We may define

$$V^\mu = \xi^T \beta\gamma^\mu \xi , \quad S = \xi^T \beta \xi . \tag{B2}$$

These operators have irreducible representations (labeled 2 to 1) by a parameter η.[15,16] In all representations (apart from the trivial one) η is a positive (nonzero) number.

The consistency conditions are rather simple in this case:

$$[(\pi^2 - m^2)S + eV_\alpha B^\alpha]\psi = 0 , \tag{B3}$$

where

$$B^\alpha = \tfrac{1}{2}\,\epsilon^{\alpha\mu\nu}(\partial_\nu A_\mu - \partial_\mu A_\nu) . \tag{B4}$$

This is the same equation as obtained by acting on the differential equation (B1) by $\xi^T(\gamma^\nu\pi_\nu - m)$, and is, thus, an equation of motion rather than a constraint. Whether ξ is a para-Bose variable or a Bose variable this leads to a consistent equation of motion.

For the $(1+1)$-dimensional case we need to restrict μ to the values 0, 1 and replace $\beta\gamma^3$ by $\beta\sigma^{01}$ and use only $\partial_1 A_0 - \partial_0 A_1$ in (B3) and (B4). In this case also we have a consistent system.

In both cases it follows that in the absence of electromagnetic fields the system describes a particle of mass m, and positive energy [compare Eqs. (42)–(44) in the text].

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[1]P. A. M. Dirac, Proc. R. Soc. London A322, 435 (1971).
[2]P. A. M. Dirac, Proc. R. Soc. London A328, 1 (1972).
[3]E. Kapuscik, Acta Phys. Pol. B3, 451 (1972).
[4]L. C. Biedenharn, M.-Y. Han, and H. van Dam, Phys. Rev. D 8, 1735 (1973).
[5]L. C. Biedenharn and H. van Dam, Phys. Rev. D 9, 471 (1974).
[6]H. Yukawa, Phys. Rev. 77, 219 (1950).
[7]A. Z. Capri and C. C. Chiang, Prog. Theor. Phys. 59, 996 (1978).
[8]A. Z. Capri and C. C. Chiang, Prog. Theor. Phys. 59, 1376 (1978).
[9]N. Mukunda, C. C. Chiang, and E. C. G. Sudarshan, Proc. R. Soc. London (to be published).
[10]H. S. Green, Phys. Rev. 90, 270 (1953).
[11]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).
[12]E. Majorana, Nuovo Cimento 9, 335 (1932).
[13]O. W. Greenberg and A. M. L. Messiah, Phys. Rev. 138, B1155 (1965).
[14]E. C. G. Sudarshan and N. Mukunda, Phys. Rev. D 1, 571 (1970).
[15]N. Mukunda, E. C. G. Sudarshan, J. K. Sharma, and C. L. Mehta, J. Math. Phys. 21, 2386 (1980).
[16]J. K. Sharma, C. L. Mehta, N. Mukunda, and E. C. G. Sudarshan, J. Math. Phys. 22, 78 (1981).