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Non-Abelian monopoles break color. II. Field theory and quantum mechanics

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Many grand unified theories (GUT's) predict non-Abelian monopoles which are sources of non-Abelian (and Abelian) magnetic flux. In the preceding paper, we discussed in detail the topological obstructions to the global implementation of the action of the "unbroken symmetry group" H on a classical test particle in the field of such a monopole. In this paper, the existence of similar topological obstructions to the definition of H action on the fields in such a monopole sector, as well as on the states of a quantum-mechanical test particle in the presence of such fields, are shown in detail. Some subgroups of H which *can* be globally realized as groups of automorphisms are identified. We also discuss the application of our analysis to the SU(5) GUT and show in particular that the non-Abelian monopoles of that theory break color and electroweak symmetries.

I. INTRODUCTION

In a typical grand unified theory (GUT),[1] a gauged grand unifying group G is spontaneously broken by a suitable Higgs field to an "unbroken" subgroup H. The group G is simply connected while H is not, which leads to the existence of magnetic monopoles[2] in such models. A remarkable feature of many of these monopoles, which distinguishes them from the Dirac monopoles, is that they are sources not only of Abelian magnetic fluxes but also of non-Abelian magnetic fluxes. In this paper, we study the classical field theory of these non-Abelian monopoles as well as the quantum mechanics of a test particle in the field of such monopoles. Our main conclusions can be summarized as follows. (1) Although the subgroup H, defined as the little group of the Higgs field at spatial infinity, is perfectly well defined as an abstract group, still it is impossible to realize all the transformations of H either on the fields which describe the non-Abelian monopole or on the states of the test particle. Any attempt to do so is likely to map a finite-energy configuration into an infinite-energy configuration. (2) The transformations which *can* be implemented[3] consist of several different subgroups K_T, $K_{T'}$,... of H with very different actions on the fields or states. (3) In the GUT scenario $SU(5)\to SU(3)_C\times U(1)_{em}$, one of these subgroups is $K_T=SU(2)_C\times U(1)_{Y_C}\times U(1)_{em}$ while in the scenario $SU(5)\to SU(3)_C\times SU(2)_{WS}\times U(1)$, one of these subgroups is $K_T=SU(2)_C\times U(1)\times U(1)\times U(1)$. [Here $SU(2)_C$ acts on the first two quarks (say), $U(1)_{Y_C}$ is generated by the color hypercharge Y_C and the remaining U(1)'s are generated by elements in the Cartan subalgebra of SU(5)]. In either case color SU(3) cannot be implemented, while in the second case the electroweak group also suffers the same fate.

Preliminary accounts of our investigation have been reported elsewhere.[4] We have also already treated the *classical* mechanics of a test particle in the field of GUT monopoles (and the associated differential geometry) in detail[5] and shown that similar difficulties are encountered in that system as well. Analogous conclusions have been reached by other authors.[6] *All these results show an important structural result in any field theory which predicts non-Abelian monopoles by means of a suitable Higgs mechanism: In the presence of these monopoles, the symmetry group of the theory is not the little group of the Higgs field at spatial infinity, rather it is a different set of transformations.*

From a physical point of view, it is important to know if the effects we describe are associated with any energy scale. Our discussion suggests that being consequences of topology they are not correlated with any such scale. They cannot, however, be perceived in any experiment which explores only a small portion of the two-sphere surrounding the monopole, that is to say when the solid angle subtended by the experimental set-up at the monopole is negligible in comparison with 4π.

The plan of the paper is as follows. In Sec. II, we consider the monopoles produced in the symmetry breakdown $G \to H$. An elementary and suggestive discussion is presented which shows in a clear and simple manner the origins of the topological obstructions to the realization of the action of H when the monopole is non-Abelian. A preliminary identification of the realizable subgroup of H is also carried out. These calculations are done in the U gauge which is particularly suited to display such obstructions. Section III studies the problem with greater generality with special attention to the characterization of the subgroups of H which survive as symmetry transformations. Here we discover a surprising result: there are in general several subgroups K_T, $K_{T'}$, ... of the abstract group H which enjoy this property. The action of these subgroups is in general space dependent, meaning that each element acts via a particular gauge transformation, and differs from naive expectations. Thus, for example, the *same* element of the abstract group H may belong to K_T and $K_{T'}$, and because K_T and $K_{T'}$ as transformation groups act differently, this s may have different actions depending on whether we focus attention on K_T or $K_{T'}$. (We postpone to another work further discussion of these distinct actions and the infinite-parameter group to which they lead.) In Sec. IV, we consider a generic non-Abelian monopole and show that a general (illegal) transformation of H maps a monopole configuration of finite energy into one of infinite energy. Section V examines the quantum mechanics of a test particle in a background non-Abelian monopole field. It shows that a generic H transformation can map a state with finite mean energy into one with infinite mean energy, indicating that the full group of H transformations is physically pathological. Section VI concludes the paper with some miscellaneous remarks. It is emphasized in particular that irreducible color (or in general H) multiplets of quantum test particles consist of both bosons and fermions in the presence of spherically symmetric non-Abelian monopoles, therefore color transformations on such a test particle do not commute with angular momentum and appear inconsistent with superselection rules. (This problem does not arise for multiplets with respect to any of K_T, $K_{T'}$,. . . .) We interpret this fact as additional evidence that the concept of color partially breaks down in the presence of non-Abelian monopoles.

II. LOSS OF H ACTION AND THE REALIZABLE SUBGROUP: ELEMENTARY DISCUSSION

In a conventional grand unified theory, a simply connected unifying group G is spontaneously broken by a Higgs field Φ to a subgroup H. This subgroup is not as a rule simply connected, and as a consequence the theory predicts magnetic monopoles. In this section, we show that there are topological obstructions to the implementation of the action of H (defined as the little group of Φ at spatial infinity) on the fields when the monopole is non-Abelian. Now it is well known that the topology of the monopole is coded in the asymptotic behavior of the Higgs and gauge fields. It is thus adequate for us to examine the fields at large spatial distances where they can be approximated by their asymptotic values. It is understood hereafter that the radial variable r is confined to such large values $r \geq r_1$. The Higgs and gauge fields for $r \geq r_1$ are denoted by $\Phi(\hat{x})$ and $W_j(\vec{x})$ where $\hat{x} = \vec{x}/r$.

The discussion will be phrased in the U gauge. (For other gauges, see Ref. 4.) The passage to the U gauge has been recapitulated in detail in our previous work,[5] here we shall only summarize the results. Let $\mathscr{O}$ denote the region of space where the above-mentioned asymptotic approximation is valid:

$$\mathscr{O} = \{ \vec{x} \in \mathrm{R}^3 \mid r = (\vec{x} \cdot \vec{x})^{1/2} \geq r_1 \} . \tag{2.1}$$

We divide $\mathscr{O}$ into two coordinate patches $\mathscr{O}_{N,S}$ where $\mathscr{O}_N$ ($\mathscr{O}_S$) does not contain the negative (positive) z axis:

$$\begin{aligned} &\mathscr{O} = \mathscr{O}_N \cup \mathscr{O}_S , \\ &\mathscr{O}_{N,S} = \mathrm{R}_+ \times \Sigma_{N,S} , \\ &\mathrm{R}_+ = \{ r \mid r \geq r_1 \} , \\ &\Sigma_{N,S} = \{ \hat{x} \mid \hat{x} \neq (0,0,\mp 1) \} . \end{aligned} \tag{2.2}$$

Then in the U gauge, we have the following. (i) All over $\mathscr{O}$, the field Φ is a constant:

$$\Phi(\hat{x}) = \Phi = \text{independent of } \hat{x} . \tag{2.3}$$

The little group of Φ is a fixed subgroup H of G. (ii) The gauge field is described by a pair of potentials $\vec{W}_{N,S}$ which are defined and smooth on $\mathscr{O}_{N,S}$. On $\mathscr{O}_N \cap \mathscr{O}_S$, they are gauge transforms of each other by a transition function $h(\phi)$, ϕ being the azimuthal angle

$$W_{Nj}(\vec{x}) = h(\phi)^{-1} \left[W_{Sj}(\vec{x}) - \frac{i}{e} \nabla_j \right] h(\phi), \quad \vec{x} \in \mathscr{O}_N \cap \mathscr{O}_S . \tag{2.4}$$

Here e is the coupling constant. If there is another H-multiplet field present, it also has a pair of sections $\psi_{N,S}$ defined on $\mathscr{O}_{N,S}$ such that

$$\psi_N(\vec{x}) = D[h(\phi)^{-1}] \psi_S(\vec{x}), \quad \vec{x} \in \mathscr{O}_N \cap \mathscr{O}_S , \tag{2.5}$$

where $h \to D(h)$ defines the appropriate representation of H. (iii) $h(\phi)$ describes a closed curve in H as ϕ increases from 0 to 2π. The homotopy class of this curve is characteristic of the topology of the monopole sector. In the trivial (no monopole) sector, this curve can be deformed to a point. Further, in any topological sector, by suitable gauge transformations, $h(\phi)$ can be changed to $h_T(\phi)$ which lies on a one-parameter subgroup of H:

$$h(\phi) \to h_T(\phi) = e^{i\phi T}, \quad T \in \text{Lie algebra } \underline{H} \text{ of } H\ . \tag{2.6}$$

(Note however that two different generators T and T' with quite different spectra can lead to homotopically equivalent h_T and $h_{T'}$ which therefore describe the same monopole sector.) Hereafter, we assume that the transition function has the form $h_T(\phi)$.

If the monopole is Abelian, the curve $h_T(\phi)$ can be assumed to be entirely in a U(1) factor of H. For a non-Abelian monopole, such a choice of h_T is not possible.

We can illustrate these considerations in a simple way for the GUT breakdown

$$G = \mathrm{SU}(5) \to H = [\mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}]/Z_3 \equiv \mathrm{U}(3)\ .$$

(Elsewhere, we have omitted writing discrete factors like Z_3.) Here Z_3 is generated by the element

$$c = (e^{i2\pi/3}\mathbb{1}, e^{-i2\pi/3})\ , \tag{2.7}$$

of $\mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}$, where $\mathbb{1}$ is the 3×3 unit matrix. In the triplet representation of U(3) [which realizes U(3) faithfully], a possible choice for the transition function to describe the elementary monopole is

$$h_T(\phi) = e^{i\phi T}\ ,$$
$$T = -\frac{1}{\sqrt{3}}\lambda_8 + \frac{1}{3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ , \tag{2.8}$$

while for the elementary antimonopole, it is $e^{-i\phi T}$. (By definition of elementarity, all monopoles in this system are composites of these elementary systems.) The components of T in the Lie algebras $\underline{\mathrm{SU}}(3)_C$ and $\underline{\mathrm{U}}(1)_{\mathrm{em}}$ are $-\lambda_8/\sqrt{3}$ and $\frac{1}{3}$. Thus T has a non-Abelian component. Further, the projection of $h_T(\phi)$ in $\mathrm{U}(1)_{\mathrm{em}}$, being $\exp(i\phi/3)$ $(0 \le \phi \le 2\pi)$, is not closed so that $h_T(\phi)$ cannot be deformed to lie entirely in $\mathrm{U}(1)_{\mathrm{em}}$. It follows that this monopole is non-Abelian.

Let us return to general considerations. The source of the topological obstructions is in the transition rules (2.4) and (2.5). Any transformation of the fields must respect these rules. Now if s is a generic element of H and it acts rigidly (with no $\vec{x}$ dependence) on the fields, the latter become

$$\vec{\mathbf{W}}'_{N,S}(\vec{x}) = s\vec{\mathbf{W}}_{N,S}s^{-1}\ ,$$
$$\psi'_{N,S}(\vec{x}) = D[s]\psi_{N,S}(\vec{x})\ , \tag{2.9}$$

and fulfill

$$W'_{Nj}(\vec{x}) = [sh_T(\phi)s^{-1}]^{-1}\left[W'_{Sj}(\vec{x}) - \frac{i}{e}\nabla_j\right][sh_T(\phi)s^{-1}]\ ,$$
$$\psi'_N(\vec{x}) = D[sh_T(\phi)^{-1}s^{-1}]\psi'_S(\vec{x})\ . \tag{2.10}$$

This is compatible with (2.4) and (2.5) only if

$$sh_T(\phi)s^{-1} = h_T(\phi)\ . \tag{2.11}$$

If $h_T(\phi)$ lies entirely in a U(1) factor of H, that is, for Abelian or Dirac monopoles, (2.11) is fulfilled. But for non-Abelian monopoles, h_T does not have this property and (2.11) is fulfilled only by the subgroup K_T of H which commutes with T. In the GUT scenario $\mathrm{SU}(5) \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}$, if T is as in (2.8), K_T is seen to be $\mathrm{SU}(2)_C \times \mathrm{U}(1)_{Y_C} \times \mathrm{U}(1)_{\mathrm{em}}$ where $\mathrm{SU}(2)_C$ acts on the first two quarks and $\mathrm{U}(1)_{Y_C}$ is generated by the color hypercharge. In the scenario $\mathrm{SU}(5) \to \mathrm{SU}(3)_C \times \mathrm{SU}(2)_{\mathrm{WS}} \times \mathrm{U}(1)$, the T associated with the elementary monopole in the $\underline{5}$ representation is

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\ . \tag{2.12}$$

Consequently, K_T is $\mathrm{SU}(2)_C \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

III. LOSS OF H ACTION AND THE REALIZABLE SUBGROUP: GENERAL DISCUSSION

In the preceding section, we assumed that the action of H on the fields was rigid, with no $\vec{x}$ dependence. In a gauge theory, however, such rigidity is not necessary so that we can envisage a more general H action. Thus for $s \in H$, we can try to construct the $\vec{x}$-dependent automorphisms

$$s \to k_A(\vec{x},s) \in H, \quad \vec{x} \in \mathscr{O}_A, \quad A = N,S\ ,$$
$$k_A(\vec{x},s)k_A(\vec{x},s') = k_A(\vec{x},ss') \tag{3.1}$$

with the convention

$$k_N(N,s) = s, \quad N = (0,0,1)\ . \tag{3.2}$$

The action of s on the fields is the gauge transform of their sections in $\mathscr{O}_N$ and $\mathscr{O}_S$ by $k_N(\vec{x},s)$ and $k_S(\vec{x},s)$.

Consistency with the transition rules (2.4) and (2.5) puts a condition on k_A:

$$k_S(\vec{x},s) = h_T(\phi)k_N(\vec{x},s)h_T(\phi)^{-1}, \quad \vec{x} \in \mathscr{O}_N \cap \mathscr{O}_S\ . \tag{3.3}$$

We can assume without loss of generality that Σ_N [Eq. (2.2)] is all of the two-sphere except the south pole S. Then, given k_N, Eq. (3.3) defines k_S on all of $\mathscr{O}_S$ except the negative z axis. Since k_S should have a well-defined value as we approach the negative z axis, (3.3) requires that

$$\lim_{\vec{x} \to r(0,0,-1)} h_T(\phi)k_N(\vec{x},s)h_T(\phi)^{-1} = \text{independent of } \phi\ , \tag{3.4}$$

where the limit is taken along a fixed azimuth. Thus to realize the action of H, we have to (a) construct the automorphisms (3.1) and (b) verify (3.4).

In physical applications, the group H is (locally) a product of semisimple and U(1) factors. For such groups, all automorphisms continuously connected to the identity are inner. Since the automorphisms k_N must be inner because of (3.2), we can write

$$k_N(\vec{x},s)=h_N(\vec{x})sh_N(\vec{x})^{-1},h_N(\vec{x})\in H\ , \tag{3.5}$$

$$h_N[r(0,0,1)]=\text{identity } e \text{ of } H\ . \tag{3.6}$$

Thus in view of (3.4) and since $h_T(0)=e$,

$$[h_T(\phi)\hat{h}_N(\phi)]s[h_T(\phi)\hat{h}_N(\phi)]^{-1}=\hat{h}_N(0)s\hat{h}_N(0)^{-1}\ , \tag{3.7}$$

where $\hat{h}_N(\phi)$ is the limit of $h_N(\vec{x})$ as we approach the negative z axis along the fixed azimuth ϕ. (There is no loss of generality in assuming that this limit is independent of r.) If we rewrite (3.7) in the form

$$\begin{aligned} &s^{-1}c(\phi)s=c(\phi)\ ,\\ &c(\phi)=\hat{h}_N(0)^{-1}h_T(\phi)\hat{h}_N(\phi)\ , \end{aligned} \tag{3.8}$$

the following important result is immediately seen: *All the transformations of H are globally realizable if the monopole is Abelian so that the transition function is homotopic to a closed curve in the center of H.* For (3.8) shows that the closed curve $c(\phi)$ is in the center of H. Further the curve $\hat{h}_N(\phi)$ can be shrunk to a point through the configurations $h_N(\vec{x})$ by varying the polar angle [see (3.5) and (3.6)]. Therefore $c(\phi)$ is homotopic to $h_T(\phi)$ and can equally well be used to describe the monopole in question. The result follows.

We can also study (3.7) to determine the subgroups of H which *can* be realized as symmetry transformations. For the choice $h_N(\vec{x})\equiv$ identity, such a subgroup is just the commutant K_T of T. But there are other solutions as well obtained by choosing nontrivial h_N and they lead to symmetry transformations with $\vec{x}$ dependence *even* in the U gauge. We postpone the general study of (3.7) to later work. Here we shall only show that there are in fact these other solutions.

There is no unique association of the monopole to the transition function. Two transition functions $h_T=e^{i\phi T}$ and $h_{T'}=e^{i\phi T'}$ will describe the same monopole sector if $e^{i\phi T}\,e^{-i\phi T'}$ is a homotopically trivial closed curve. In such a case there is a function $h(\theta,\phi)\in H$ defined on $\mathscr{O}_N$ (and independent of r) such that

$$\begin{aligned} &h(0,\phi)=e\ ,\\ &h(\pi,\phi)=h_T(\phi)h_{T'}(\phi)^{-1}\ , \end{aligned} \tag{3.9}$$

θ being the polar angle. If $\psi'_{N,S}$ are the sections of a field in the gauge with transition function $h_{T'}$,

$$\psi'_N(\vec{x})=D[h_{T'}(\phi)^{-1}]\psi'_S(\vec{x}),\quad \vec{x}\in\mathscr{O}_N\cap\mathscr{O}_S\ , \tag{3.10}$$

then the following are its sections in the gauge with transition function $h_T(\phi)$:

$$\begin{aligned} &\psi_N(\vec{x})=D[h_T(\phi)^{-1}h(\theta,\phi)^{-1}h_T(\phi)]\psi'_N(\vec{x}),\quad \vec{x}\in\mathscr{O}_N\ ,\\ &\psi_S(\vec{x})=D[h(\theta,\phi)^{-1}h_T(\phi)h_{T'}(\phi)^{-1}]\psi'_S(\vec{x}),\quad x\in\mathscr{O}_S\ . \end{aligned} \tag{3.11}$$

In the gauge with transition function $h_{T'}$, the group $K_{T'}$ which commutes with $h_{T'}$, can be globally realized with an $\vec{x}$ independent action. In view of (3.11), this action becomes the following in the h_T gauge:

$$\begin{aligned} \psi_N(\vec{x})\to&D[h_T(\phi)^{-1}h(\theta,\phi)^{-1}h_T(\phi)sh_T(\phi)^{-1}\\ &\times h(\theta,\phi)h_T(\phi)]\psi_N(\vec{x})\ ,\\ \psi_S(\vec{x})\to&D[h(\theta,\phi)^{-1}h_T(\phi)h_{T'}(\phi)^{-1}sh_{T'}(\phi)\\ &\times h_T(\phi)^{-1}h(\theta,\phi)]\psi_S(\vec{x}),\quad s\in K_{T'}\ . \end{aligned} \tag{3.12}$$

The action of $K_{T'}$ in the h_T gauge is thus $\vec{x}$ dependent. [Note that $K_{T'}$ acts on the potentials $\vec{W}_{N,S}$ by gauge transforming them with the respective elements of H appearing in (3.12).]

In our previous paper,[5] we gave an example for K_T and $K_{T'}$ in the model $SU(5)\to[SU(3)_C\times U(1)_{em}]/Z_3$. Here we shall therefore give another illustration. Consider the breakdown $SU(3)\to[SU(2)\times U(1)]/Z_2\equiv U(2)$. In the defining $\underline{3}$ representation of SU(3), this U(2) has generators

$$\frac{\sqrt{3}}{2}\lambda_8,\quad \frac{\lambda_i}{2},\quad i=1,2,3\ . \tag{3.13}$$

Consider the following T and T':

$$\begin{aligned} &T=\tfrac{1}{2}(\lambda_3+\sqrt{3}\lambda_8)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\ ,\\ &T'=T+\lambda_3=\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\ . \end{aligned} \tag{3.14}$$

(In this particular example, T and T' happen to commute. It need not generally be so.) The closed curve $h_T(\phi)=e^{i\phi T}$ in U(2) has projections $\exp[i(\lambda_3/2)\phi]$ and $\exp[i(\sqrt{3}/2)\lambda_8\phi]$ in SU(2) and U(1), neither of which is closed showing that h_T describes a non-Abelian monopole. The closed curve $e^{i\phi T}\,e^{-i\phi T'}=e^{i\phi(T-T')}$ has the following projections in SU(2) and U(1):

$$\begin{aligned} &e^{-i\lambda_3\phi}\in SU(2)\ ,\\ &\text{identity}\in U(1)\ . \end{aligned} \tag{3.15}$$

The U(1) projection is thus a point while the SU(2) projection, being closed, is also deformable to a point [since SU(2) is simply connected]. Hence $e^{i\phi(T-T')}$ is homotopically trivial and h_T and $h_{T'}$ describe the same monopole.

The map $h(\theta,\phi)$ of (3.9) is easily constructed:

$$\begin{aligned} &h(\theta,\phi)=\begin{bmatrix}\cos^2\frac{\theta}{2}+\cos\phi\sin^2\frac{\theta}{2}-i\sin\frac{\theta}{2}\sin\phi & -(1-\cos\phi)i\sin\frac{\theta}{2}\cos\frac{\theta}{2} & 0\\ -(1-\cos\phi)i\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}+\cos\phi\sin^2\frac{\theta}{2}+i\sin\frac{\theta}{2}\sin\phi & 0\\ 0 & 0 & 1\end{bmatrix}\in SU(2)\ ,\\ &h(0,\phi)=e,\quad h(\pi,\phi)=e^{i\phi(T-T')}\ . \end{aligned} \tag{3.16}$$

The groups K_T and $K_{T'}$ are the same, and are $U(1)\times U(1)$ locally. A basis for their Lie algebras is $\lambda_3/2$, $\lambda_8/2$. We have now two actions of this subgroup of H in the h_T gauge: viewed as K_T, it acts rigidly while viewed as $K_{T'}$, its action is given by (3.12). This means that the same element of the group U(2) is being given different actions on the fields when it lies in $U(1)\times U(1)$.

IV. COLOR TRANSFORMATIONS CREATE INFINITE ENERGY

Consider a generic element $s\in H$. Let us realize its action on the fields as a gauge transformation using the $\vec{x}$-dependent automorphisms (3.1). It was claimed that such an action is illegal if k_S acquires a ϕ dependence along the negative z axis. One would like to know the sense in which the transformed fields are pathological. We show in this section that the transformed gauge field strength has a δ-function singularity along the negative z axis so that the transformed energy is infinite.

The proof of this result is very simple. The potential $\vec{W}_S$ after gauge transformation by k_S becomes $\vec{W}'_S$ where

$$W'_{Si}(\vec{x})dx_i=k_S(\vec{x},s)W_{Si}(\vec{x})dx_i k_S(\vec{x},s)^{-1} -\frac{i}{e}k_S(\vec{x},s)dk_S(\vec{x},s)^{-1} \ . \tag{4.1}$$

If $k_S(\vec{x},s)$ has a ϕ dependence as the negative z axis is approached, then in this limit the second term in (4.1) induces the singular term

$$[W'_{Si}(\vec{x})dx_i]_{\rm sing}=t\,d\phi \ , \tag{4.2}$$

in $W'_{Si}(\vec{x})dx_i$, t being a Lie-algebra-valued constant. The integral of $W'_{Si}\,dx_i$ along an infinitesimal closed loop $\mathscr{C}$ around the negative z axis is thus

$$\oint_{\mathscr{C}} W'_{Si}(\vec{x})dx_i=2\pi t \ . \tag{4.3}$$

By Stokes's theorem, the transformed field strength $F_{ij}(W'_S)$ must have a δ-function term $[F_{ij}(W'_S)]_{\rm sing}$ with support on the negative z axis:

$$[F_{ij}(W'_S)]_{\rm sing}=t\epsilon_{ij3}\pi\theta(-z)\delta(x)\delta(y) \ . \tag{4.4}$$

Thus the transformed energy density $\mathrm{Tr}F_{ij}(W'_S)^2/4$ is infinite along the negative z axis and the transformed energy is infinite.

In connection with the above calculation, one may raise the following objection. Since the new field strengths $F(W'_N)$ and $F(W'_S)$ are obtained from the old field strengths $F(W_N)$ and $F(W_S)$ by gauge transformations, and since energy density is gauge invariant, the new energy density as well as the new total energy must be exactly equal to the corresponding old quantities, hence the result must be incorrect. However, this argument is invalid for the following reason. While at every point off the negative z axis, $F(W'_S)$ is related to $F(W_S)$ by a gauge transformation, along the negative z axis where the gauge transformation is ill defined, they are not so related. It is indeed true that off the negative z axis the old and new energy densities are exactly equal. But the only valid way of computing the new total energy is to find expressions for $F(W'_N)$ and $F(W'_S)$ valid *over all of* $\mathscr{O}_N$ *and* $\mathscr{O}_S$, *respectively,* such that all the properties of the fields are faithfully reflected, and then to use them in the standard expression for energy density and total energy. This necessarily leads to δ-function terms in $F(W'_S)$ and hence to divergences in final energy. Thus our result expresses a real physical effect and not just a consequence of a particular mode of calculation.

Abouelsaood[6] has pointed out that a physical consequence of the preceding result is that there are no dyonic excitations associated with such transformations. In particular, there are no color multiplets of dyonic excitations of the monopole in the symmetry breakdown $SU(5)\rightarrow SU(3)_C\times SU(2)_{WS}\times U(1)$.

V. QUANTUM MECHANICS OF A TEST PARTICLE IN A NON-ABELIAN MONOPOLE FIELD

We shall now discuss the quantum mechanics of a test particle in a non-Abelian monopole field. For specificity, we consider the monopole produced in the GUT scenario $G=SU(5)\rightarrow H=SU(3)_C\times SU(2)_{WS}\times U(1)$. For simplicity, we shall also assume the following. (1) The monopole is elementary with the standard spherically symmetrical form. (2) The particle is spinless and nonrelativistic. Neither of these assumptions is essential to the conclusions, nor is it difficult to consider other G's and H's.

The Schrödinger equation for the test particle is

$$i\frac{\partial\psi}{\partial t}=\mathscr{H}\psi \ ,$$

where

$$\begin{aligned}\mathscr{H}&=\frac{\vec{\pi}^{\,2}}{2m} \ ,\\ \pi_j&=-iD_j \ ,\\ D_j&=\partial_j+ieD[W_j(\vec{x})] \ ,\\ i\,eD[W_j(\vec{x})]&=-\frac{i}{2r}f(r)\epsilon_{jkl}\hat{x}_k D[\tau_l] \ ,\end{aligned} \tag{5.1}$$

and where $\{D(l)\}$ denotes the representation of the Lie algebra $\underline{SU}(5)=\{l\}$ of SU(5) according to which ψ transforms, and τ_i is defined in the $\underline{\bar{5}}$ representation as

$$\tau_i=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&\multicolumn{2}{c}{\sigma_i}&0\\0&0&&&0\\0&0&0&0&0\end{bmatrix} \ ,$$

$$\sigma_i=\text{Pauli matrices} \ . \tag{5.2}$$

The infinities we shall later find reside in the angular features of $\mathscr{H}$ and ψ and are in no way dependent on the distance of the test particle from the monopole center. We shall therefore as usual simplify the discussion by replacing $f(r)$ by its asymptotic value:

$$f(r)=1 \ . \tag{5.3}$$

Thus we are confining our attention to the region $\mathscr{O}$.

Note that since we are using a single globally defined potential, we are not working in the U gauge. As is well known,[2,5] the transition to the U gauge is achieved by separate (r independent) gauge transformations $g_{N,S}(\hat{x})$ in $\mathscr{O}_{N,S}$ which rotate $\vec{\tau}\cdot\hat{x}$ to τ_3. Further, the transition function in the U gauge is just $e^{i\phi\tau_3}$. Thus we have

$$g_{N,S}(\hat{x})^{-1}\vec{\tau}\cdot\hat{x}g_{N,S}(\hat{x})=\tau_3\ ,$$
$$g_N(\hat{x})=g_S(\hat{x})e^{i\phi\tau_3},\quad r\hat{x}\in\mathscr{O}_N\cap\mathscr{O}_S\ , \tag{5.4}$$
$$T=\tau_3\ .$$

For purposes of simplicity, we shall only examine the consequences of rigid ($\vec{x}$ independent) transformations $s\in H$ in this U gauge, we shall also of course assume that they do not commute with τ_3. They correspond in the present gauge to the following transformations:

$$s_N(\hat{x})=g_N(\hat{x})sg_N(\hat{x})^{-1},\quad \vec{x}\in\mathscr{O}_N\ ,$$
$$s_S(\hat{x})=g_S(\hat{x})sg_S(\hat{x})^{-1},\quad \vec{x}\in\mathscr{O}_S\ . \tag{5.5}$$

They do not commute with $\vec{\tau}\cdot\hat{x}$. Further, there is a difficulty in implementing the transformations $s_{N,S}$ in the present gauge: they do not agree on $\mathscr{O}_N\cap\mathscr{O}_s$ since $[s,\tau_3]\neq 0$ and there is no room now to transform the fields (which are all globally defined) with a pair of functions $s_{N,S}$. In order to define these transformations, we therefore take Σ_N in $\mathscr{O}_N$ to be all of the two-sphere except $(0,0,-1)$, this defines $s_N(x)$ on all of $\mathscr{O}_N$ except the negative z axis. The limit of $s_N(x)$ on this axis is not well defined. Still we shall transform the fields by $s_N(\hat{x})$ ignoring this singularity and examine the consequences. (Of course, such conceptual difficulties arise because these transformations are not smoothly defined for all $\hat{x}$ in any gauge.)

The Hamiltonian $\mathscr{H}$ admits the following constants of motion.

(a) The angular momentum $\vec{J}$, where

$$J_i=-i(\vec{x}\times\vec{\nabla})_i+\tfrac{1}{2}D(\tau_i)\ . \tag{5.6}$$

(b) The helicity

$$\vec{J}\cdot\hat{x}\ . \tag{5.7}$$

Note that

$$\vec{J}\cdot\hat{x}=\tfrac{1}{2}D(\tau_i)\hat{x}_i\ . \tag{5.8}$$

Thus we can diagonalize $\mathscr{H}$ along with J^2, J_3, and $D(\tau_i)\hat{x}_i$. Let $|\mu,k\rangle$ denote the eigenstates of $D(\tau_i)\hat{x}_i$ in the SU(5) representation space:

$$\tfrac{1}{2}D(\tau_i)\hat{x}_i\,|\mu,k\rangle=\mu\,|\mu,k\rangle\ . \tag{5.9}$$

Here k is a degeneracy index.

Now $\vec{J}$ can be written as

$$\vec{J}=-i(\vec{x}\times\vec{D})+\tfrac{1}{2}D(\tau_i)\hat{x}_i\hat{x}\ , \tag{5.10}$$

where $\vec{D}$ is given in (5.1) and fulfills

$$[D_i,D_j]=i\epsilon_{ijk}\frac{x_k}{r^3}[\tfrac{1}{2}D(\tau_\alpha)\hat{x}_\alpha]\ . \tag{5.11}$$

Let $\vec{J}^{(D)}(\mu)$ and $m\vec{V}(\mu)$ be the restrictions of $\vec{J}$ and $-i\vec{D}$ on states of the form $F(\vec{x})\,|\mu,k\rangle$. (Note that $[\vec{D},\vec{J}\cdot\hat{x}]=0$.) That is, let

$$\vec{J}F(\vec{x})\,|\mu,k\rangle=\vec{J}^{(D)}(\mu)F(\vec{x})\,|\mu,k\rangle\ ,$$
$$-i\vec{D}F(\vec{x})\,|\mu,k\rangle=m\vec{V}(\mu)F(\vec{x})\,|\mu,k\rangle\ . \tag{5.12}$$

Then in view of (5.10) and (5.11),

$$\vec{J}^{(D)}(\mu)=m(\vec{x}\times\vec{V}(\mu))+\mu\hat{x}\ , \tag{5.13a}$$

$$[V_i(\mu),V_j(\mu)]=-i\frac{\mu}{m^2}\epsilon_{ijk}\frac{x_k}{r^3}\ . \tag{5.13b}$$

That is, on such states $\vec{J}$ becomes the angular momentum $\vec{J}^{(D)}(\mu)$ and $-i\vec{D}/m$ becomes the velocity operator $\vec{V}(\mu)$ of the Dirac charge-monopole system for the value of $eg=\mu$. (Here e is the electric and g the magnetic charge of the charge-monopole system.) Further

$$\vec{J}^{(D)}(\mu)\cdot\hat{x}=\mu \tag{5.14}$$

by virtue of (5.13a).

The eigenfunctions of $[J^{(D)}(\mu)]^2$ and $J_3^{(D)}(\mu)$ are known to be the monopole harmonics $D^j_{m,-\mu}$:[7]

$$[J^{(D)}(\mu)]^2D^j_{m,-\mu}(\theta,\phi)=j(j+1)D^j_{m,-\mu}(\theta,\phi)\ ,$$
$$J_3^{(D)}(\mu)D^j_{m,-\mu}(\theta,\phi)=mD^j_{m,-\mu}(\theta,\phi)\ . \tag{5.15}$$

The discussion shows that the eigenfunction ψ_E of $\mathscr{H}$ for energy E can be taken to have the form

$$\psi_E=f^j_{m,\mu}(r)D^j_{m,-\mu}(\theta,\phi)\,|\mu,k\rangle\ . \tag{5.16}$$

Since

$$\mathscr{H}=\frac{{p_r}^2}{2m}+\frac{1}{2mr^2}\{J^2-[\tfrac{1}{2}D(\tau_i)\hat{x}_i]^2\}\ , \tag{5.17}$$

$f^j_{m,\mu}D^j_{m,-\mu}$ fulfills

$$\left[\frac{{p_r}^2}{2m}+\frac{1}{2mr^2}\{[J^{(D)}(\mu)]^2-\mu^2\}\right]f^j_{m,\mu}(r)D^j_{m,-\mu}(\theta,\phi)$$
$$=Ef^j_{m,\mu}(r)D^j_{m,-\mu}(\theta,\phi)\ . \tag{5.18}$$

Here $[J^{(D)}(\mu)]^2$ is given by

$$[J^{(D)}(\mu)]^2=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right]-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$
$$-\frac{2i\mu}{1+\cos\theta}\frac{\partial}{\partial\phi}+\frac{2\mu^2}{1+\cos\theta}\ . \tag{5.19}$$

It is remarkable that in the wave function ψ_E, the space-time and internal symmetry properties are tied together as shown by the correlation between the second index of the D function and the index μ in the state $|\mu,k\rangle$. Any transformation $s_N(x)$ which does not commute with $\vec{\tau}\cdot\hat{x}$ will spoil this correlation and transform ψ_E into a state for which the mean value of energy is infinite. (This infinity is *not* due to the fact that the wave function is not normalizable, as we shall see below.) For let

$$\psi' = f^j_{m,\mu}(r) D^j_{m,-\mu}(\theta,\phi) \,|\, \rho,k\rangle, \quad \rho \neq \mu \tag{5.20}$$

be a state where the index correlation mentioned above is absent. (Such states are necessarily created from ψ_E by transformations of H which do not commute with $\vec{\tau}\cdot\hat{x}$.) Then

$$\mathscr{H}\psi' = \left[\frac{{p_r}^2}{2m} + \frac{1}{2mr^2}[J^{(D)}(\rho)^2 - \rho^2]\right] f^j_{m,\mu}(r) D^j_{m,-\mu}(\theta,\phi) \,|\, \rho,k\rangle$$

$$= E\psi' + \frac{1}{2mr^2}\{[J^{(D)}(\rho)^2 - \rho^2] - [J^{(D)}(\mu)^2 - \mu^2]\} f^j_{m,\mu}(r) D^j_{m,-\mu}(\theta,\phi) \,|\, \rho,k\rangle \;, \tag{5.21}$$

where we used (5.18). In the expectation value $(\psi',\mathscr{H}\psi')$, the first term contributes finitely to the angular integral while the contribution of the second term is, in view of (5.19),

$$\frac{1}{2mr^2}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi [f^j_{m,\mu}(r) D^j_{m,-\mu}(\theta,\phi)]^* \left[-2i\frac{(\rho-\mu)}{1+\cos\theta}\frac{\partial}{\partial\phi} + \frac{2(\rho^2-\mu^2)}{1+\cos\theta} - (\rho^2-\mu^2)\right][f^j_{m,\mu}(r) D^j_{m,-\mu}(\theta,\phi)] \;. \tag{5.22}$$

This is infinite unless $D^j_{m,-\mu}(\theta,\phi)=0$ for $\theta=\pi$. But $D^j_{m,-\mu}$ does *not* vanish at the south pole if $m\neq\mu$. (Further, all values of m in the range $-j\leq m\leq +j$ must certainly be allowed in order to maintain rotational invariance.) We can thus conclude that a transformation in H which does not commute with τ_3 in the U gauge transforms finite-energy states into states with infinite mean values for energy.

It should be evident that the conclusion is not affected if ψ_E is replaced by a normalizable wave packet

$$\int dE\,\alpha(E)\psi_E \;. \tag{5.23}$$

The background Yang-Mills potential W_i is not invariant under $s_N(\hat{x})$ in the present gauge, so that we would not of course expect $s_N(\hat{x})$ (or all rigid H transformations in the U gauge) to be a symmetry of the test particle Hamiltonian. But we would also not expect a rigid H transformation to produce states with infinite mean energies. Our result can be loosely interpreted in terms of an infinite potential barrier which inhibits such transformations.

In quantum mechanics with its emphasis on the Hilbert-space structure, continuity requirements are not so strict as in a classical field theory. It is therefore not obvious that a transformation $s_N(\hat{x})$ cannot be implemented on the states without bad consequences even if it is not well defined along the negative z axis. Since these $s_N(\hat{x})$ map states of finite energy into states with infinite mean energies, this possibility, and hence the possibility of implementing rigid H transformations in the U gauge which do not commute with T, are now ruled out.

VI. CONCLUDING REMARKS

In this paper, we have considered gauge theories based on a gauged symmetry group G which is spontaneously broken by a Higgs field to a subgroup H. The precise definition of H is that it is the little group of the Higgs field at spatial infinity. This fact naturally leads one to expect that H is also the unbroken symmetry group of *transformations* of these theories. However, in the presence of non-Abelian monopoles, we have seen that this is not the case: the group of automorphisms is instead a *local* (x-dependent) group which is not isomorphic to H. The group H is thus topologically broken, and in its stead we have a novel group of automorphisms. [We may remark here that even in electrodynamics the group of *automorphisms* on the algebra of observables is a local group: it is G/G_0 where G is the set of *all* gauge transformations and G_0 is the set of gauge transformations which reduce to identity at spatial infinity. However, G/G_0 is "spontaneously broken" to U(1) in the sense that only this U(1) is unitarily implementable. Similarly, in the monopole sectors of GUT's as well, only a subgroup of the group of automorphisms is expected to be unitarily implementable. We plan to study this question in a paper under preparation.[8]]

While the demonstration of these results has been carried out using topological reasoning, there is an alternative and intuitively compelling argument to see that H cannot be the symmetry group in non-Abelian monopole sectors. Locally, in any region of space far from the monopole which does not also enclose the monopole, it is possible to realize all the transformations in H. (For such a region can always be enclosed in one coordinate patch $\mathscr{O}_N$ or $\mathscr{O}_S$.) Thus locally test particles in a monopole field can be classified into irreducible H multiplets and we can inquire about the angular momentum properties of such multiplets. The remarkable fact then emerges that in the presence of spherically symmetric non-Abelian monopoles, irreducible H multiplets may contain both integer- and half-integer-spin particles. For instance, in the model $G=\mathrm{SU}(5)\rightarrow H=\mathrm{SU}(3)_C\times \mathrm{U}(1)_{\mathrm{em}}$, the $\bar{5}$ multiplet $(d_1^C,d_2^C,d_3^C,e^-,\nu_e)_L$ splits under H into a triplet $(d_1^C,d_2^C,d_3^C)_L$ and two singlets e_L^-,ν_{eL}. As we saw in Sec. V, the potential of the spherically symmetric monopole couples only to d_{3L}^C and e_L^-, their angular momenta are thereby changed by the addition of an extra $\frac{1}{2}$ unit, while the angular momenta of the remaining particles are not affected. Thus in the color $\bar{3}$ multiplet, $d_{1,2L}^C$ act like fermions while d_{3L}^C acts as a boson and H does not commute with angular momentum in the presence of a non-Abelian monopole, suggesting that the concept of H as a symmetry group may be invalid.

It is easy to see that an H singlet composed of H nonsinglet test particles will not emerge as an H singlet on scattering by a classical non-Abelian monopole field. Thus in the model of the preceding paragraph, if we scatter an H singlet formed of three of the $\overline{3}$ multiplets from the spherically symmetric monopole field, the constituents d_{1L}^C and d_{2L}^C are not scattered at all while d_{3L}^C is scattered. This scrambles the phase relations between the constituents of the singlet so that the outgoing wave function is not going to be a singlet. Since color-confining forces are color singlets, they cannot bind the emerging constituents into a singlet. Thus the free existence of a colored monopole implies the existence of other colored objects.

The effect we have just now discussed will not of course constitute a problem for color confinement if color were a well-defined symmetry for monopoles. For, the fields which describe the classical non-Abelian monopole (in particular, the gauge potential) are not color invariant even locally, so neither is the corresponding quantum state. If color were a well-defined symmetry, we could then project out the color-singlet component of this quantum state and call that the physically correct quantum monopole state. The effect described in the preceding paragraph would then disappear. However color is not a well-defined symmetry for monopoles so that there seems to be no way to construct a color-singlet monopole state globally by such a method.

If non-Abelian monopoles do not exist as free particles in the standard GUT's, if they are confined and only Abelian monopoles are observable, then the conclusion that color and electroweak symmetries are broken in these GUT's by non-Abelian monopoles can be avoided. The dynamical reasons for such confinement however remain to be explored.

In this paper, we have examined the topological problems arising in the identification of the group of automorphisms in the presence of monopoles. The deeper question of the unitary implementation of this group of automorphisms will be treated elsewhere.[8]

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