Anisotropic Gaussian Schell-model beams: Passage through optical systems and associated invariants

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Anisotropic Gaussian Schell-model (AGSM) fields and their transformation by first-order optical systems (FOS's) forming Sp(4,$\mathbb{R}$) are studied using the generalized pencils of rays. The fact that Sp(4,$\mathbb{R}$), rather than the larger group SL(4,$\mathbb{R}$), is the relevant group is emphasized. A convenient geometrical picture wherein AGSM fields and FOS's are represented, respectively, by antisymmetric second-rank tensors and de Sitter transformations in a $(3+2)$-dimensional space is developed. These fields are shown to separate into two qualitatively different families of orbits and the invariants over each orbit, two in number, are worked out. We also develop another geometrical picture in a $(2+1)$-dimensional Minkowski space suitable for the description of the action of axially symmetric FOS's on AGSM fields, and the invariants, now seven in number, are derived. Interesting limiting cases forming coherent and quasihomogeneous fields are analyzed.

I. INTRODUCTION

Ever since the classic paper of Walther[1] on the connection between coherence and radiometry, there has been an enormous interest in the properties of radiation fields generated by partially coherent planar (scalar) sources.[2] In these situations the source is adequately described by the source-plane cross-spectral density. Several model sources have been studied in detail, Gaussian quasihomogeneous sources[3] and Gaussian Schell-model sources[4] having received particular attention. Many useful results have been established, and it has been shown[5] in particular that sources which are highly incoherent in a global sense can produce beams which are as directional as (fully coherent) laser beams.

A recent development of considerable interest is the introduction of the notion of generalized light rays[6] in statistical wave optics in an attempt to clarify the relationship[7] between radiative transfer theory[8] and electrodynamics. This notion leads to a ray picture of wave optics which is exact at the level of the two-point correlation function and is applicable equally well to coherent, partially coherent, and incoherent fields. In paraxial situations these generalized rays behave in an extremely simple way both under free propagation and on action by optical systems.[9]

In a previous paper[10] we employed the method of generalized rays to analyze the behavior of axially symmetric or, as we shall say, isotropic Gaussian Schell-model (IGSM) fields under the action of axially symmetric first-order optical systems (FOS's).[11] We demonstrated that this method enables us to answer any question related to this class of problems and circumvents the elaborate calculations involving integrals that arise in using conventional wave-optical methods. In particular we showed that such FOS's induce one-to-one maps on the family of IGSM fields, the degree of global coherence being invariant under these maps; and that for each IGSM field there is a corresponding one-parameter subgroup of FOS's which leave it invariant. [Axially symmetric FOS's themselves are naturally identified with elements of the group SL(2,$\mathbb{R}$).] We also developed an elegant geometrical picture wherein each IGSM field is represented by a timelike vector and each axially symmetric FOS by a proper Lorentz transformation in a fictitious $(2+1)$-dimensional Minkowski space, which makes all these results easily visualizable and also leads to a generalization of the Kogelnik "*abcd* law"[12] to partially coherent IGSM fields.

There have been attempts to generalize the notion of Gaussian Schell-model sources to include anisotropies in the source-plane intensity and coherence distributions. It has been shown that these anisotropic Gaussian Schell-model (AGSM) sources produce isotropic far-zone intensity distributions, provided that the source parameters meet certain conditions,[13] and "bladelike" shape-invariant fields if they obey certain other conditions.[14] Coherent anisotropic Gaussian beams have also been analyzed[15] and it has been shown that for such beams there always exists a transverse plane over which the intensity distribution is isotropic. It should be noted, however, that the AGSM fields discussed in these works are restricted in the sense that the anisotropies in the intensity and coherence distributions share the same principal axes.

The purpose of the present paper is to extend our treatment of IGSM fields to the AGSM case, using again the method of generalized rays and developing a suitable geometrical interpretation. It turns out that the inclusion of anisotropy leads to a considerable increase in structure and complexity of the problem. In particular the three-dimensional Minkowski-space representation of the IGSM case needs to be extended to a five-dimensional de Sitter—space representation[16] for both fields and FOS's.

In Sec. II we define the family of AGSM fields we shall be concerned with, both through the cross-spectral densities and through the generalized ray density distributions. We stress that the ten-parameter family of fields we introduce is minimal in the sense that it is carried into itself under the most general physically permissible two-dimensional FOS. Section III defines precisely the class of FOS's to be admitted, in particular explaining why the group $Sp(4,\mathbb{R})$ rather than the larger $SL(4,\mathbb{R})$ is the relevant one, and calculates the action of such FOS's on an AGSM field. It is stressed that all FOS's considered can be constructed using a finite number of (asymmetric) lenses plus free propagations. Section IV develops in detail the group-theoretical aspects of the family of AGSM fields, classifies them into subfamilies invariant under the action of all FOS's, and calculates the invariants associated with each subfamily. The subfamilies consist of one single-parameter continuous collection of AGSM fields, and one two-parameter continuous collection. The geometrical representation in the de Sitter space is also developed. In Sec. V we take up the action of axially symmetric FOS's, corresponding to an $SL(2,\mathbb{R})$ subgroup in $Sp(4,\mathbb{R})$, on AGSM fields; we give a parametrization of the fields appropriate for this purpose, and calculate the associated invariants. Section VI considers limiting fields possessing complete coherence or quasihomogeneity. Section VII contains concluding remarks, and there are three appendixes.

II. AGSM FIELDS AND ASSOCIATED GENERALIZED RAY DENSITY DISTRIBUTIONS

We restrict attention to time-stationary fields[17] so that different frequency components can be treated independently of one another. Thus we present our analysis for a fixed frequency ω which is suppressed. Let us choose a Cartesian coordinate system with the positive z axis along the beam axis. The field in a transverse plane $z=z_0$ is specified through its cross-spectral density in that plane. Suppressing z_0 and denoting by $\underline{\rho}$ the transverse two-vector part $(x,y)^T$ (a column matrix) of the three-vector $(x,y,z_0)^T$, we have an AGSM field if the cross-spectral density assumes the form

$$\begin{aligned}
&\Gamma(\underline{\rho}_1;\underline{\rho}_2)=[I(\underline{\rho}_1)I(\underline{\rho}_2)]^{1/2}g(\underline{\rho}_1-\underline{\rho}_2)e^{-i\phi(\underline{\rho}_1;\underline{\rho}_2)}\,,\\
&I(\underline{\rho})=(A/2\pi)(\det\underline{\sigma}_I)^{-1}\exp[-\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_I{}^2)^{-1}\underline{\rho}]\,,\\
&g(\underline{\rho})=\exp[-\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_g{}^2)^{-1}\underline{\rho}]\,,\\
&\phi(\underline{\rho}_1;\underline{\rho}_2)=\tfrac{1}{2}(\underline{\rho}_1-\underline{\rho}_2)^T\underline{K}(\underline{\rho}_1+\underline{\rho}_2)\,.
\end{aligned}\tag{2.1}$$

The intensity distribution $I(\underline{\rho})$ is characterized by a real symmetric positive-definite matrix $\underline{\sigma}_I$, and $(\underline{\sigma}_I{}^2)^{-1}$ denotes the matrix inverse of $\underline{\sigma}_I{}^2$. The modulus $g(\underline{\rho})$ of the normalized degree of coherence is likewise characterized by another real symmetric positive-definite matrix $\underline{\sigma}_g$. It is on account of their physical significance that we require these two 2×2 matrices to be positive definite. The phase ϕ of the normalized degree of coherence is determined by the real 2×2 matrix $\underline{K}$. By integrating $I(\underline{\rho})$ over the transverse plane it is easily seen that A is the total irradiance.

Our interest is in the action of two-dimensional FOS's on such fields; since such systems are lossless, the parameter A is invariant under such action, and so it is suppressed in the following in counting the number of independent parameters in an AGSM field. It is then evident that the set of AGSM fields as we have defined them is a ten-parameter family, three parameters each in $\underline{\sigma}_I$ and $\underline{\sigma}_g$, and four in $\underline{K}$. Throughout this paper it is understood that, except where explicitly stated otherwise, $\underline{\sigma}_I$, $\underline{\sigma}_g$, $\underline{K}$, and $\underline{\gamma}$ to be defined presently are 2×2 real matrices.

The intensity and transverse coherence in an AGSM field are anisotropic Gaussian with different sets of principal axes in general. Furthermore we allow the phase ϕ to involve an arbitrary real 2×2 matrix $\underline{K}$. The AGSM fields of earlier authors[13,14] obtain when $\underline{\sigma}_I$, $\underline{\sigma}_g$, and $\underline{K}$ are simultaneously diagonal. Thus our definition of the AGSM family in (2.1) is a multiple generalization of earlier definitions in that we allow $\underline{\sigma}_I$ and $\underline{\sigma}_g$ to have different sets of principal axes, and we do not require $\underline{K}$ to be symmetric. As will be evident in the sequel, the definition (2.1) with the stated properties of $\underline{\sigma}_I$, $\underline{\sigma}_g$, and $\underline{K}$ is the minimal one necessary in order that the family of AGSM fields be carried into itself under the action of arbitrary two-dimensional FOS's.

To appreciate the nature of the phase term in (2.1) we recall that the amplitude transmittance of a general (astigmatic) thin lens is a quadratic phase function

$$e^{-i\phi_l(\underline{\rho})}=\exp(-\tfrac{1}{2}i\underline{\rho}^T\underline{M}\underline{\rho})\,,\tag{2.2}$$

where $\underline{M}$ is a real symmetric 2×2 matrix representing the optical thickness curvature of the lens. Its action on the cross-spectral density is as follows:

$$\Gamma(\underline{\rho}_1;\underline{\rho}_2)\rightarrow\Gamma'(\underline{\rho}_1;\underline{\rho}_2)=e^{i[\phi_l(\underline{\rho}_1)-\phi_l(\underline{\rho}_2)]}\Gamma(\underline{\rho}_1;\underline{\rho}_2)\,.\tag{2.3}$$

Now if $\underline{K}^S$ and $\underline{K}^A$, are, respectively, the symmetric and antisymmetric parts of $\underline{K}$, we can rewrite the phase function ϕ of the AGSM field as

$$\phi(\underline{\rho}_1;\underline{\rho}_2)=\tfrac{1}{2}\underline{\rho}_1^T\underline{K}^S\underline{\rho}_1-\tfrac{1}{2}\underline{\rho}_2^T\underline{K}^S\underline{\rho}_2+\underline{\rho}_1^T\underline{K}^A\underline{\rho}_2\,.\tag{2.4}$$

It is now clear that the $\underline{K}^S$ terms in ϕ are exactly what would be imparted to Γ by an appropriately astigmatic thin lens. It will be shown later (in Sec. V) that even under free propagation an equiphase AGSM field can pick up the $\underline{K}^A$-type term in ϕ. These remarks justify from a physical point of view the inclusion of the phase term in (2.1) with an arbitrary real matrix $\underline{K}$.

For any Γ, the generalized ray density distribution function W is defined by the Wigner-Moyal transform[18]

$$W(\underline{\rho};\underline{p})=(2\pi)^{-2}\int d^2\rho'\,\Gamma(\underline{\rho}+\tfrac{1}{2}\underline{\rho}';\underline{\rho}-\tfrac{1}{2}\underline{\rho}')e^{i\underline{p}^T\underline{\rho}'}\,.\tag{2.5}$$

Here $\underline{p}$ is a transverse two-dimensional ray vector, and $W(\underline{\rho};\underline{p})$ represents the intensity of the generalized pencils of rays going in the direction $(\underline{p},p_z=(k^2-\underline{p}^2)^{1/2})\approx(\underline{p},k)$ through the space point $(\underline{\rho},z_0)$. For the AGSM field (2.1) we have

$$\Gamma(\underline{\rho}+\tfrac{1}{2}\underline{\rho}';\underline{\rho}-\tfrac{1}{2}\underline{\rho}')$$
$$=(A/2\pi)(\det\sigma_I)^{-1}\exp[-\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_I{}^2)^{-1}\underline{\rho}]$$
$$\times\exp[-\tfrac{1}{2}(\underline{\rho}')^T\underline{\gamma}^{-2}\underline{\rho}'-i(\underline{\rho}')^T\underline{K}\underline{\rho}]\ , \qquad (2.6)$$

where the real symmetric positive-definite 2×2 matrix γ is defined by

$$(\underline{\gamma}^2)^{-1}=(\underline{\sigma}_g{}^2)^{-1}+\tfrac{1}{4}(\underline{\sigma}_I{}^2)^{-1}\ . \qquad (2.7)$$

Substitution of (2.6) in (2.5) gives

$$W(\underline{\rho};\underline{p})=[A/(2\pi)^2]\frac{\det\underline{\gamma}}{\det\underline{\sigma}_I}$$
$$\times\exp[-\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_I{}^2)^{-1}\underline{\rho}$$
$$-\tfrac{1}{2}(\underline{p}-\underline{K}\,\underline{\rho})^T\underline{\gamma}^2(\underline{p}-\underline{K}\,\underline{\rho})]\ . \qquad (2.8)$$

There are two different but equivalent ways of compactly expressing the quadratic in the exponential, each convenient for a particular purpose. Let us put $\underline{\rho}$ and $\underline{p}$ together to form a four-component column vector q as follows:

$$\underline{q}=\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}\ . \qquad (2.9)$$

Then we define a 4×4 real symmetric matrix $\underline{G}$ by setting

$$\underline{q}^T\underline{G}\,\underline{q}=\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_I{}^2)^{-1}\underline{\rho}+\tfrac{1}{2}(\underline{p}-\underline{K}\,\underline{\rho})^T\underline{\gamma}^2(\underline{p}-\underline{K}\,\underline{\rho})\ . \qquad (2.10)$$

Since it is easily shown that

$$\det\underline{G}=\frac{1}{16}\left[\frac{\det\underline{\gamma}}{\det\underline{\sigma}_I}\right]^2\ , \qquad (2.11)$$

we see that the W function takes the form

$$W(\underline{\rho};\underline{p})=(A/\pi^2)(\det\underline{G})^{1/2}\exp(-\underline{q}^T\underline{G}\,\underline{q})\ . \qquad (2.12)$$

The arrangement of the components of $\underline{\rho}$ and $\underline{p}$ as in (2.9) and the ensuing definition of $\underline{G}$ is particularly convenient for discussing the action of axially symmetric FOS's; this is taken up in Sec. V. However, for the purpose of reverting from W to Γ a more convenient arrangement is as follows:

$$\overline{\underline{q}}=\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}=\begin{bmatrix} \underline{\rho} \\ \underline{p} \end{bmatrix}\ . \qquad (2.13)$$

In that case we define a new symmetric matrix $\overline{\underline{G}}$ to represent the quadratic in the exponent in W:

$$\underline{q}^T\underline{G}\,\underline{q}=\overline{\underline{q}}^T\,\overline{\underline{G}}\,\overline{\underline{q}}\ . \qquad (2.14)$$

Since

$$\underline{q}=\underline{N}\overline{\underline{q}}\ ,$$
$$\underline{N}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2.15)$$
$$\underline{N}^T=\underline{N}^{-1}=\underline{N}\ ,\quad \det\underline{N}=-1\ ,$$

we have the relations

$$\overline{\underline{G}}=\underline{N}\,\underline{G}\,\underline{N}\ ,$$
$$\det\overline{\underline{G}}=\det\underline{G}\ , \qquad (2.16)$$

and W can be written as

$$W(\underline{\rho};\underline{p})=(A/\pi^2)(\det\overline{\underline{G}})^{1/2}\exp(-\overline{\underline{q}}^T\overline{\underline{G}}\,\overline{\underline{q}})\ . \qquad (2.17)$$

For a given AGSM field Γ characterized by definite $\underline{\sigma}_I$, $\underline{\sigma}_g$, and $\underline{K}$, unique 4×4 matrices $\underline{G}$, and $\overline{\underline{G}}$ are determined [and incidentally $\underline{G}$ and $\overline{\underline{G}}$ determine each other uniquely by (2.16)]. From (2.10) it is evident that $\underline{G}$ and $\overline{\underline{G}}$ are real, symmetric positive-definite matrices. Moreover from (2.7) and (2.11) it follows that $\det\underline{G}$ is bounded above by unity. Thus the following three properties of $\underline{G}$ (and equally well of $\overline{\underline{G}}$) are immediate and obvious:

$$\underline{G}^T=\underline{G}\ , \qquad (2.18a)$$

$\underline{G}$ is positive definite, (2.18b)

$$0<\det\underline{G}<1\ . \qquad (2.18c)$$

However, these do not exhaust all the properties of $\underline{G}$ or $\overline{\underline{G}}$. If one starts with a $\overline{\underline{G}}$ obeying all of (2.18), sets up W by (2.17), and then recovers Γ by inverting (2.5), one must ensure that the Γ so obtained is physically acceptable. Let the 4×4 $\overline{\underline{G}}$ be split into 2×2 blocks in this way:

$$\overline{\underline{G}}=\begin{bmatrix} \underline{A} & \underline{C} \\ \underline{C}^T & \underline{B} \end{bmatrix}\ . \qquad (2.19)$$

It is clear that both $\underline{A}$ and $\underline{B}$ are real, symmetric, 2×2 positive-definite matrices. The $\underline{\sigma}_I$, $\underline{\sigma}_g$, and $\underline{K}$ that result from a given choice of $\overline{\underline{G}}$ are easily read off by comparing $\overline{\underline{q}}^T\overline{\underline{G}}\,\overline{\underline{q}}$ with the expression on the right-hand side of (2.10):

$$(\underline{\sigma}_I{}^2)^{-1}=2(\underline{A}-\underline{C}\,\underline{B}^{-1}\underline{C}^T)\ , \qquad (2.20a)$$
$$(\underline{\sigma}_g{}^2)^{-1}=\tfrac{1}{2}(\underline{B}^{-1}-\underline{A}+\underline{C}\,\underline{B}^{-1}\underline{C}^T)\ , \qquad (2.20b)$$
$$\underline{K}=-\underline{B}^{-1}\underline{C}^T\ . \qquad (2.20c)$$

Now the positive definiteness of $\overline{\underline{G}}$ is adequate to ensure the same property for $\underline{\sigma}_I$ as determined by (2.20a) since from (2.10) and (2.14) we have for arbitrary $\underline{\rho}$

$$\tfrac{1}{2}\underline{\rho}^T(\underline{\sigma}_I{}^2)^{-1}\underline{\rho}=(\underline{\rho}^T,\underline{\rho}^T\underline{K}^T)\overline{\underline{G}}\begin{bmatrix} \underline{\rho} \\ \underline{K}\underline{\rho} \end{bmatrix}\ . \qquad (2.21)$$

However, the requirement that $\underline{\sigma}_g$ be positive definite is an added condition on $\overline{\underline{G}}$; it is a matrix condition and reads

$$\underline{\rho}^T(\underline{B}^{-1}-\underline{A}+\underline{C}\,\underline{B}^{-1}\underline{C}^T)\underline{\rho}>0\quad\text{for all }\underline{\rho}\ . \qquad (2.22)$$

It is easy in principle but awkward in practice to express this condition in terms of $\underline{G}$, so we shall be content to leave it in this form. Thus the complete set of properties characterizing $\overline{\underline{G}}$ is (2.18) (with $\underline{G}\to\overline{\underline{G}}$) and (2.22), and we have established the following result: *There is a one-to-one correspondence between the members of the ten-parameter family of AGSM fields, and 4×4 real symmetric positive-definite matrices $\overline{\underline{G}}$ obeying (2.18c) and (2.22).*

An easy and geometrically appealing way of visualizing the set of allowed matrices $\underline{G}$, and hence of handling the condition (2.22), will emerge from the work of Secs. III and IV and Appendixes A and C.

To recover and make contact with the description of IGSM fields in Ref. 10, let us first invert (2.20) to express $\underline{A}$, $\underline{B}$, and $\underline{C}$, in terms of $\underline{\sigma}_I$, $\underline{\sigma}_g$, and $\underline{K}$, or, rather, in terms of σ_I, $\underline{\gamma}$, and $\underline{K}$:

$$\begin{aligned} \underline{A}&=\tfrac{1}{2}[(\underline{\sigma}_I{}^2)^{-1}+\underline{K}^T\underline{\gamma}^2\underline{K}]\ ,\\ \underline{B}&=\tfrac{1}{2}\underline{\gamma}^2\ .\\ \underline{C}&=-\tfrac{1}{2}\underline{K}^T\underline{\gamma}^2\ . \end{aligned} \tag{2.23}$$

The IGSM limit is obtained by making each of the matrices $\underline{\sigma}_I$, $\underline{\sigma}_g$, $\underline{\gamma}$, and $\underline{K}$ multiples of the 2×2 unit matrix. Let this be indicated by $\underline{\sigma}_I\to\sigma_I\mathbb{1}_{2\times2}$, $\underline{\sigma}_g\to\sigma_g\mathbb{1}_{2\times2}$, $\underline{\gamma}\to\gamma\mathbb{1}_{2\times2}$, and $\underline{K}\to(k/R)\mathbb{1}_{2\times2}$ where only for the present discussion σ_I, σ_g, and γ are real positive numerical quantities. Then the cross-spectral density (2.1) becomes exactly the three-parameter IGSM field analyzed in Ref. 10, and $\underline{A}$, $\underline{B}$, and $\underline{C}$ also become multiples of the 2×2 unit matrix. Equation (2.7) is read as defining the number γ in terms of the numbers σ_I and σ_g exactly as in Ref. 10, so $\gamma<2\sigma_I$, and $\overline{\underline{G}}$ is effectively a 2×2 matrix. In this limit we have the correspondences

$$\begin{aligned} &\det\underline{\sigma}_I\to\sigma_I^2\ ,\\ &\det\underline{\gamma}\to\gamma^2\ ,\\ &\det\underline{G}\to(\gamma/2\sigma_I)^4\ , \end{aligned} \tag{2.24}$$

and so the bound on det $\underline{G}$ becomes obvious. Moreover, if we examine the condition (2.22) in the IGSM limit, we see that it is automatically obeyed once the bound on $\det\underline{G}$ is obeyed. This explains why a condition like (2.22) additional to the conditions (2.18) did not appear in the work of Ref. 10.

III. LENSLIKE SYSTEMS AND THE SP(4,$\mathbb{R}$) GROUP OF FOS'S

Given any cross-spectral density Γ over a transverse plane, the total irradiance can be obtained from the function $W(\underline{\rho};\underline{p})$ written as $W(\underline{q})$ by integration over all four components of $\underline{q}$. For the AGSM field with $W(\underline{q})$ given in (2.12), we recover

$$\int W(\underline{q})\,d^4q=A\ . \tag{3.1}$$

A priori, it may appear that the most general linear system which maps generalized rays onto themselves in a one-to-one way and which preserves the total irradiance would be described by a 4×4 real matrix $\underline{S}$ acting on $\underline{q}$ and $W(\underline{q})$ as follows:

$$\begin{aligned} &S:\ \underline{q}\to\underline{q}'=\underline{S}\,\underline{q}\ ,\\ &W(\underline{q})\to W'(\underline{q})=W(\underline{S}^{-1}\underline{q})\ ,\\ &\det\underline{S}=1\ . \end{aligned} \tag{3.2}$$

The determinant condition on $\underline{S}$ ensures that the total irradiance is preserved, and means that $\underline{S}$ is an element of the group SL(4,$\mathbb{R}$). As in geometrical optics, $\underline{S}$ could be called the ray-transfer matrix of the system.

However, it turns out that each $\underline{S}\in$SL(4,$\mathbb{R}$) transforms some physically acceptable field into an unphysical one: Examples are given in Appendix A. On the other hand, there is a subgroup Sp(4,$\mathbb{R}$)$\subset$SL(4,$\mathbb{R}$), the symplectic group in four real dimensions, such that every $\underline{S}\in$Sp(4,$\mathbb{R}$) maps each physically acceptable cross-spectral density Γ into another acceptable one. The action of such $\underline{S}$ is again given by (3.2), and this statement is proved in Appendix A. For these reasons we consider hereafter only those ray transfer matrices $\underline{S}$ and corresponding FOS's which belong to the group Sp(4,$\mathbb{R}$).

In accordance with the way the column vector $\underline{q}$ has been defined in (2.9), we introduce a real, 4×4 antisymmetric matrix $\underline{\beta}$ as

$$\underline{\beta}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\ . \tag{3.3}$$

The group Sp(4,$\mathbb{R}$) consists of all real 4×4 matrices $\underline{S}$ with unit determinant which obey[19]

$$\underline{S}^T\underline{\beta}\,\underline{S}=\underline{\beta}\ . \tag{3.4}$$

In other words, they preserve an antisymmetric real bilinear form in four dimensions. Free propagation through a positive distance D is represented by the ray-transfer matrix[20]

$$\underline{F}(D)=\begin{bmatrix}1&D/k&0&0\\0&1&0&0\\0&0&1&D/k\\0&0&0&1\end{bmatrix}\ ,\quad k=\omega/c \tag{3.5}$$

and it can be verified that

$$[\underline{F}(D)]^T\underline{\beta}\underline{F}(D)=\underline{\beta}\ , \tag{3.6}$$

i.e., $\underline{F}(D)\in$Sp(4,$\mathbb{R}$). A thin astigmatic lens with the phase function (2.2) is represented by the ray-transfer matrix[20]

$$\underline{L}(\underline{M})=\underline{N}\begin{bmatrix}\mathbb{1}_{2\times2}&\underline{0}_{2\times2}\\-\underline{M}&\mathbb{1}_{2\times2}\end{bmatrix}\underline{N}\ , \tag{3.7}$$

with $\underline{N}$ given in (2.15). It can be verified that $\underline{L}(\underline{M})$ also belongs to the group Sp(4,$\mathbb{R}$), for any symmetric 2×2 real matrix $\underline{M}$. With the help of a Lie-group-theoretic argument, it has been shown in Ref. 20 that *every element $\underline{S}\in$Sp(4,$\mathbb{R}$) can be represented as the product of a finite number of matrices of the forms $\underline{F}(D)$ and $\underline{L}(M)$,* for suitable choices of D's and $\underline{M}$'s.[21] [This too is a proof that every $\underline{S}\in \mathrm{Sp}(4,\mathbb{R})$ maps physically acceptable Γ's into other acceptable ones.] For this reason, it is natural to refer to the group $\mathrm{Sp}(4,\mathbb{R})$ of FOS's as the family of lens-like systems.

We have seen in Sec. II that each AGSM field Γ corresponds to some 4×4 real matrix $\underline{G}$ obeying (2.18) and (2.22) (the latter stated in terms of $\overline{\underline{G}}$), and conversely. Combining (2.12) with (3.2) we now see that if $\underline{S}\in \mathrm{Sp}(4,\mathbb{R})$ is some FOS, it produces the following change in G:

$$S\in \mathrm{Sp}(4,\mathbb{R}):\ \underline{G}\to\underline{G}'=(\underline{S}^{-1})^T\underline{G}\,\underline{S}^{-1}\ . \tag{3.8}$$

While it is obvious that $\underline{G}'$ also obeys (2.18), the fact that (2.22) is also maintained is somewhat harder to see; a direct attempt to check this fact is rather awkward. The proof that (2.22) is preserved is given in Appendix A. Thus the change $\underline{G}\to\underline{G}'$ given in (3.8) maps the set of allowed $\underline{G}$'s onto themselves in a one-to-one manner. Under these mappings induced by the group $\mathrm{Sp}(4,\mathbb{R})$, the set of allowed $\underline{G}$'s (and so the set of all AGSM fields) naturally splits into disjoint equivalence classes, $\underline{G}$ and $\underline{G}'$ being equivalent if and only if there is some $\underline{S}\in \mathrm{Sp}(4,\mathbb{R})$ relating them in the manner of (3.8). The determination of these equivalence classes, or "orbits" in the space of matrices $\underline{G}$, and of the associated invariants, is taken up in Sec. IV.

A general $\underline{S}\in \mathrm{Sp}(4,\mathbb{R})$ describes an anisotropic, i.e., a nonaxially-symmetric, FOS, since in the transformation rule $\underline{q}\to\underline{q}'=\underline{S}\,\underline{q}$ of (3.2) the pair $(x,p_x)^T$ is not transformed in the same way as the pair $(y,p_y)^T$. The subset of axially symmetric FOS's is an $\mathrm{SL}(2,\mathbb{R})$ subgroup within the larger $\mathrm{Sp}(4,\mathbb{R})$ group. In our description elements $\underline{S}\in \mathrm{SL}(2,\mathbb{R})\subset \mathrm{Sp}(4,\mathbb{R})$ describing such FOS's have the form

$$\begin{gathered}\underline{S}=\begin{bmatrix} \underline{s} & \underline{0}_{2\times 2}\\ \underline{0}_{2\times 2} & s\end{bmatrix},\\ \underline{s}=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in \mathrm{SL}(2,\mathbb{R})\ ,\\ ad-bc=1\ .\end{gathered} \tag{3.9}$$

The ray-transfer matrix $\underline{F}(D)$ in (3.5) describing free propagation is an example of (3.9), and so $\underline{F}(D)\in \mathrm{SL}(2,\mathbb{R})$. For an axially symmetric lens with optical power g, the matrix $\underline{M}$ in (2.2) and (3.7) becomes $kg\mathbb{1}_{2\times 2}$, so that the corresponding ray-transfer matrix $\underline{L}(\underline{M})$ in (3.7) becomes

$$\underline{L}(g)=\begin{bmatrix} 1 & 0 & 0 & 0\\ -kg & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -kg & 1\end{bmatrix}. \tag{3.10}$$

This again is an element of the $\mathrm{SL}(2,\mathbb{R})$ subgroup of $\mathrm{Sp}(4,\mathbb{R})$. Naturally there are more invariants associated with the action of axially symmetric FOS's on AGSM fields than with the action of general FOS's. This is analyzed in Sec. V.

IV. SO(3,2) REPRESENTATION, ORBITS, AND INVARIANTS

As we have recalled in the Introduction, IGSM fields and axially symmetric FOS's acting on them can be elegantly represented by (timelike) vectors and Lorentz transformations in a fictitious $(2+1)$-dimensional Minkowski space.[10] When we allow anisotropies in both, the geometrical representation gets enlarged to one in a fictitious $(3+2)$-dimensional de Sitter space. This is because the groups $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}(3,2)$ share the same Lie algebra—the former is the twofold spinor covering group of the latter.[16]

The infinitesimal generators of the group $\mathrm{Sp}(4,\mathbb{R})$ are real 4×4 matrices $\underline{J}$ which obey

$$\underline{J}^T\underline{\beta}+\underline{\beta}\,\underline{J}=\underline{0}_{4\times 4}\ ,$$

i.e.,

$$(\underline{\beta}\,\underline{J})^T=\underline{\beta}\,\underline{J}\ . \tag{4.1}$$

This arises by expressing $\underline{S}$ in (3.4) as the exponential of $\underline{J}$, and using the antisymmetry of $\underline{\beta}$. It follows that with the $\underline{G}$ matrix of an AGSM field we can associate in a unique manner some $\mathrm{Sp}(4,\mathbb{R})$ generator matrix by

$$\begin{gathered}\underline{G}=\underline{\beta}\,\underline{J}\ ,\\ \underline{J}=-\underline{\beta}\,\underline{G}\ .\end{gathered} \tag{4.2}$$

(Not all generator matrices J will lead to acceptable $\underline{G}$'s, however, since $\underline{G}$ has to obey other conditions besides symmetry.) The transformation $\underline{G}\to\underline{G}'$ of (3.8) for some FOS $\underline{S}\in \mathrm{Sp}(4,\mathbb{R})$ appears as the adjoint action of $\mathrm{Sp}(4,\mathbb{R})$ on its Lie algebra:

$$S\in \mathrm{Sp}(4,\mathbb{R}):\ \underline{J}\to\underline{J}'=\underline{S}\,\underline{J}\,\underline{S}^{-1}\ . \tag{4.3}$$

As mentioned earlier, any $\mathrm{Sp}(4,\mathbb{R})$ transform of a $\underline{G}$ matrix representing a physical AGSM field describes another physical AGSM field. It follows that the set of all allowed $\underline{G}$'s corresponds via (4.2) to a certain collection of entire orbits in the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$ under the adjoint action.

In order to set up a basis for the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$, and make explicit the connection to SO(3,2), we begin with a special real representation of the Dirac algebra in four dimensions:

$$\begin{gathered}\underline{\beta}=\underline{\gamma}_0=i\mathbb{1}_{2\times 2}\otimes\underline{\sigma}_2\ ,\quad \underline{\gamma}_1=\underline{\sigma}_3\otimes\underline{\sigma}_1\ ,\\ \underline{\gamma}_2=\underline{\sigma}_1\otimes\underline{\sigma}_1\ ,\quad \underline{\gamma}_3=-\mathbb{1}_{2\times 2}\otimes\underline{\sigma}_3\ .\end{gathered} \tag{4.4}$$

Here we use Kronecker products of two Pauli matrices. Thus we have explicitly

$$\begin{aligned}\beta=\underline{\gamma}_0=i\mathbb{1}_{2\times 2}\otimes\underline{\sigma}_2&=\begin{bmatrix} i & 0\\ 0 & i\end{bmatrix}\otimes\underline{\sigma}_2\\ &=\begin{bmatrix} i\underline{\sigma}_2 & 0\underline{\sigma}_2\\ 0\underline{\sigma}_2 & i\underline{\sigma}_2\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix},\end{aligned} \tag{4.5a}$$

$$\underline{\gamma}_1=\underline{\sigma}_3\otimes\underline{\sigma}_1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$
$$=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{bmatrix}, \qquad (4.5b)$$

$$\underline{\gamma}_2=\underline{\sigma}_1\otimes\underline{\sigma}_1\begin{bmatrix}\underline{0}_{2\times2} & \underline{\sigma}_1\\ \underline{\sigma}_1 & \underline{0}_{2\times2}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}, \qquad (4.5c)$$

$$\underline{\gamma}_3=-\underline{\mathbb{1}}_{2\times2}\otimes\underline{\sigma}_3=\begin{bmatrix}-\underline{\sigma}_3 & 0_{2\times2}\\ \underline{0}_{2\times2} & -\sigma_3\end{bmatrix}$$
$$=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \qquad (4.5d)$$

The four $\underline{\gamma}_\mu$ ($\mu, \nu, \ldots$ run over 0,1,2,3) obey

$$\{\underline{\gamma}_\mu,\underline{\gamma}_\nu\}=2\eta_{\mu\nu}\,, \qquad (4.6a)$$
$$(\underline{\gamma}_\mu)^T=-\underline{\beta}\,\underline{\gamma}_\mu\underline{\beta}^{-1}\,, \qquad (4.6b)$$

where the diagonal metric $\eta_{\mu\nu}$ is "spacelike:" $\eta_{00}=-1$, $\eta_{11}=\eta_{22}=\eta_{33}=+1$. We next introduce indices $A,B,\ldots$ going over the values 0,1,2,3,5 and labeling components of vectors and tensors in a five-dimensional space, and define ten independent real 4×4 matrices $\underline{\Sigma}_{AB}=-\underline{\Sigma}_{BA}$ by

$$\underline{\Sigma}_{\mu\nu}=-\tfrac{1}{4}[\underline{\gamma}_\mu,\underline{\gamma}_\mu]\,,$$
$$\underline{\Sigma}_{\mu5}=\tfrac{1}{2}\underline{\gamma}_\mu\,. \qquad (4.7)$$

They obey

$$(\underline{\Sigma}_{AB})^T=-\underline{\beta}\,\underline{\Sigma}_{AB}\underline{\beta}^{-1}\,, \qquad (4.8)$$

i.e., these matrices are a basis for the Lie algebra of the group Sp(4,$\mathbb{R}$). Another way of saying the same thing is this: $\underline{\beta}\,\underline{\Sigma}_{AB}$ are ten independent real symmetric 4×4 matrices. Thus *each $\underline{J}$ can be expanded as a linear combination of the $\underline{\Sigma}_{AB}$, and each $\underline{G}$ as a linear combination of $\underline{\beta}\,\underline{\Sigma}_{AB}$.*

The commutation relations among the $\underline{\Sigma}_{AB}$ are

$$[\underline{\Sigma}_{AB},\underline{\Sigma}_{CD}]=\eta_{AC}\underline{\Sigma}_{BD}-\eta_{BC}\underline{\Sigma}_{AD}+\eta_{AD}\underline{\Sigma}_{CB}-\eta_{BD}\underline{\Sigma}_{CA}\,, \qquad (4.9)$$

where the only additional nonvanishing component of η_{AB} is $\eta_{55}=-1$. (Therefore the dimensions 0 1 2 3 5 have the signatures $-\,+\,+\,+\,-$.) This system of commutation relations shows that the groups SP(4,$\mathbb{R}$) and SO(3,2) are locally isomorphic.

Any generator matrix $\underline{J}$ of Sp(4,$\mathbb{R}$) can be expressed as

$$\underline{J}=\tfrac{1}{2}\xi^{AB}\underline{\Sigma}_{AB}\,,$$
$$\xi^{AB}=-\xi^{BA}\,, \qquad (4.10)$$

involving ten real independent expansion coefficients ξ^{AB}. Correspondingly we write the matrix $\underline{G}$ of an AGSM field as

$$\underline{G}=\tfrac{1}{2}\xi^{AB}\underline{\beta}\,\underline{\Sigma}_{AB}\,. \qquad (4.11)$$

The action (3.8) of a FOS $\underline{S}\in$Sp(4,$\mathbb{R}$) on $\underline{G}$ results in the following change in $\underline{\xi}$:

$$\underline{G}'=(\underline{S}^{-1})^T\underline{G}\,\underline{S}^{-1}\Longrightarrow(\xi')^{AB}=[\underline{\Lambda}(\underline{S})]^A{}_C[\underline{\Lambda}(\underline{S})]^B{}_D\xi^{CD}\,, \qquad (4.12)$$

where $\underline{\Lambda}(\underline{S})$ is a 5×5 real SO(3,2) pseudorotation matrix. Thus we see the following: *each AGSM field corresponding to a definite $\underline{G}$ can be pictured as some real second-rank antisymmetric tensor in a five-dimensional de Sitter space; and each FOS $\underline{S}\in$Sp(4,$\mathbb{R}$) acts as a de Sitter transformation on this space.*

On account of the fact that SO(3,2) is a rank-two group, there are two independent invariants that can be formed from the "components" ξ^{AB} of $\underline{G}$. One of them is

$$\xi^2\equiv\tfrac{1}{2}\xi^{AB}\xi_{AB}\,. \qquad (4.13)$$

(The metric η_{AB} is used to raise and lower de Sitter indices $A,B,\ldots$.) To get the other one, it is useful to define the "Pauli-Lubanski" vector[22] $\underline{\zeta}^T$ such that

$$\zeta_A=\tfrac{1}{8}\epsilon_{ABCDE}\xi^{BC}\xi^{DE}\,, \qquad (4.14)$$

where ϵ_{ABCDE} is the five-dimensional Levi-Cività symbol with $\epsilon_{01235}=1$. The transformation (4.12) of ξ^{AB} induces

$$(\zeta')_A=[\underline{\Lambda}(\underline{S})]_A{}^B\zeta_B\,, \qquad (4.15)$$

i.e., $\underline{\zeta}^T$ is a five-component vector. The second invariant we can form from $\underline{G}$ is then

$$\zeta^2\equiv\zeta^A\zeta_A\,. \qquad (4.16)$$

The determinant of $\underline{G}$, which is obviously invariant under (3.8), is expressible in terms of the two independent invariants (4.13) and (4.16):

$$\det\underline{G}=\tfrac{1}{16}(\tfrac{1}{2}\xi^{AB}\xi_{AB})^2-\tfrac{1}{4}\zeta^A\zeta_A$$
$$=\tfrac{1}{16}(\xi^2)^2-\tfrac{1}{4}\zeta^2\,. \qquad (4.17)$$

One can obtain this relation by considering, for example, the special case[23] of diagonal $\underline{G}$, when the only nonvanishing elements of $\underline{\xi}$ are ξ^{03}, ξ^{05}, ξ^{13}, and ξ^{15}.

Under the adjoint action of Sp(4,$\mathbb{R}$) on the generator matrices $\underline{J}$, as given in (4.3), the space of generator matrices (in the Lie algebra) splits into disjoint (i.e., nonintersecting) orbits. If the antisymmetric tensor ξ^{AB} is used as a system of ten independent coordinates for the Lie algebra, ξ^{AB} varies over an orbit as $\underline{\Lambda}(\underline{S})$ in (4.12) varies over the group SO(3,2), while ξ^2 and ζ^2 stay constant over each orbit. Each "point" on an orbit arises from an arbitrarily chosen "representative point" on that orbit by means of a suitable de Sitter transformation $\Lambda(S)$. As stated previously, either all the matrices $\underline{G}$ associated with all the points on an orbit represent physical AGSM fields and obey conditions (2.18) and (2.22), or none do. That is, *each orbit is either allowed or disallowed in its entirety.*

To discover the allowed orbits, the following must be done: First, all the distinct orbits in the Lie algebra of Sp(4, $\mathbb{R}$) must be classified; second, a convenient representative element $\underline{J}^{(0)}$ must be chosen on each orbit; third, one must examine the matrices $\underline{G}^{(0)}$ associated with these representative elements, and see which ones obey conditions (2.18) and (2.22). The results of such an analysis are briefly described in Appendix C; we draw upon those results here, and pick out just the physically allowed orbits and discuss their properties.

There are in all 17 distinct families or types of orbits in the Sp(4,R) Lie algebra. Of these only two families lead to matrices $\underline{G}$ obeying (2.18) and (2.22). We shall call them family I and family II; the former is a one-parameter collection of distinct orbits, the latter a two-parameter collection. We first describe these two families of orbits, and then the AGSM fields corresponding to them.

Family I. For each value of a continuous parameter κ in the range $0<\kappa<1$, we have an allowed orbit. The representative matrices, $\underline{J}^{(0)}$ and $\underline{G}^{(0)}$, and the nonvanishing components of $(\underline{\xi}^{(0)})^{AB}$ are

$$\underline{J}^{(0)}=2\kappa\underline{\Sigma}_{50}\ ,\quad \underline{G}^{(0)}=2\kappa\underline{\beta}\,\underline{\Sigma}_{50}\ ,\qquad (\underline{\xi}^{(0)})^{50}=-(\underline{\xi}^{(0)})^{05}=2\kappa\ . \tag{4.18}$$

On each of these orbits the Pauli-Lubanski vector $\underline{\zeta}^T$ vanishes identically (*not* just at the representative points):

$$\underline{\zeta}^T=\underline{0}_{5\times 1}\ . \tag{4.19}$$

The invariants over these orbits take the values

$$\xi^2=4\kappa^2\ ,\qquad \det\underline{G}=\kappa^4\ . \tag{4.20}$$

The representative element $\underline{G}^{(0)}$ in (4.18) is a multiple of the 4×4 unit matrix:

$$\underline{G}^{(0)}=\underline{\overline{G}}^{(0)}=\kappa\underline{\mathbb{1}}_{4\times 4}\ . \tag{4.21}$$

Positivity of $\underline{G}^{(0)}$ leads to $\kappa>0$, while the condition (2.22) gives $\kappa<1$.

Family II. For each pair of values of two parameters κ_1, κ_2 obeying $0<\kappa_2<\kappa_1<1$, we have an allowed orbit. The representative matrices, $\underline{J}^{(0)}$ and $\underline{G}^{(0)}$, and the nonvanishing components of $(\underline{\xi}^{(0)})^{AB}$ are

$$\begin{aligned}&\underline{J}^{(0)}=(\kappa_1+\kappa_2)\underline{\Sigma}_{50}+(\kappa_1-\kappa_2)\underline{\Sigma}_{13}\ ,\\ &\underline{G}^{(0)}=(\kappa_1+\kappa_2)\underline{\beta}\underline{\Sigma}_{50}+(\kappa_1-\kappa_2)\underline{\beta}\underline{\Sigma}_{13}\ ,\\ &(\underline{\xi}^{(0)})^{50}=-(\underline{\xi}^{(0)})^{05}=(\kappa_1+\kappa_2)\ ,\\ &(\underline{\xi}^{(0)})^{13}=-(\underline{\xi}^{(0)})^{31}=\kappa_1-\kappa_2\ .\end{aligned} \tag{4.22}$$

On the orbit (κ_1,κ_2), at the representative point, the Pauli-Lubanski vector has only one nonvanishing component, namely,

$$\underline{\zeta}^{(0)}=(0,0,\kappa_2^2-\kappa_1^2,0,0)\ . \tag{4.23}$$

Therefore on each of these orbits, $\underline{\zeta}^T$ is a "spacelike vector." The invariants over the orbit (κ_1,κ_2) have values

$$\begin{aligned}&\xi^2=2(\kappa_1^2+\kappa_2^2)\ ,\\ &\zeta^2=(\kappa_1^2-\kappa_2^2)^2\ ,\\ &\det\underline{G}=\kappa_1^2\kappa_2^2\ .\end{aligned} \tag{4.24}$$

The representative element $\underline{G}^{(0)}$ in (4.22) is a diagonal matrix:

$$\underline{G}^{(0)}=\begin{bmatrix}\kappa_1&0&0&0\\0&\kappa_1&0&0\\0&0&\kappa_2&0\\0&0&0&\kappa_2\end{bmatrix},\qquad \underline{\overline{G}}^{(0)}=\begin{bmatrix}\kappa_1&0&0&0\\0&\kappa_2&0&0\\0&0&\kappa_1&0\\0&0&0&\kappa_2\end{bmatrix}. \tag{4.25}$$

Positivity of $\underline{G}^{(0)}$ and choice of orbit representatives lead to the restrictions $\kappa_1>\kappa_2>0$; the added condition (2.22) then gives $\kappa_1<1$, $\kappa_2<1$, so we have the final set of conditions on κ_1,κ_2 given above.

We may now elaborate on the physical significance and interpretations of the AGSM fields described by these two families of orbits. The $\underline{G}$ matrix describing a given AGSM field always has a $\underline{\zeta}$ vector that vanishes identically or else is spacelike in nature. In case $\underline{\zeta}=\underline{0}$, *and only then,* the given AGSM field arises from some IGSM field via a suitable FOS $\underline{S}\in\mathrm{Sp}(4,\mathbb{R})$. *Thus the* AGSM *fields of family I are all possible* IGSM *fields together with transforms of such fields by all possible FOS's.* In particular, if a given AGSM field has a $\underline{\zeta}$ vector which does not vanish identically, it is impossible to convert it into an IGSM field by any choice of $\underline{S}\in\mathrm{Sp}(4,\mathbb{R})$.

The invariants ξ^2 and ζ^2 obey

$$0<\xi^2<4\ ,\qquad \zeta^2=0\ , \tag{4.26}$$

over family I, and

$$\begin{aligned}&0<\xi^2<4\ ,\\ &0<\zeta^2<1\ ,\\ &\xi^2>2(\zeta^2)^{1/2}\ ,\end{aligned} \tag{4.27}$$

over family II. In both cases, the bounds [Eq. (2.18c)] on $\det\underline{G}$ hold.

Given *any* AGSM field with matrix $\underline{G}$, we can transform it by a suitable FOS in Sp(4,$\mathbb{R}$) to a form $\underline{G}^{(0)}$ which is diagonal, and which then has either the appearance (4.21) (if $\underline{\zeta}=\underline{0}_{1\times 5}$ to begin with) or (4.25) (if $\underline{\zeta}\neq\underline{0}_{1\times 5}$). Thus every AGSM field G can be transformed to an equiphase field $G^{(0)}$. In family I, this is an IGSM field with κ being the degree of global coherence common to x and y directions. In family II, the diagonal elements of $\underline{G}^{(0)}$ can be ordered as in (4.25); then κ_1 and κ_2 are the degrees of global coherence in the x and y directions, respectively, and they are definitely unequal. In that case, $\underline{G}^{(0)}$ represents an equiphase AGSM field for which both

$\underline{\sigma}_I$ and $\underline{\sigma}_g$ have common principal axes. Such a field can be called a separable equiphase AGSM field, and so *fields of family II are separable equiphase AGSM fields with unequal degrees of global coherence in the x and y directions and their transforms by all possible FOS's.* From (4.20) and (4.24) the degree(s) of global coherence can be expressed in terms of the geometrical invariants ξ^2 and ζ^2, and so they are themselves invariants.

Family I:

$$\kappa=(\tfrac{1}{4}\xi^2)^{1/2} . \tag{4.28}$$

Family II:

$$\kappa_1=[\tfrac{1}{4}\xi^2+\tfrac{1}{2}(\zeta^2)^{1/2}]^{1/2} ,$$
$$\kappa_2=[\tfrac{1}{4}\xi^2-\tfrac{1}{2}(\zeta^2)^{1/2}]^{1/2} .$$

Next we turn to an interesting application of the results of this section. We have seen that any AGSM matrix $\underline{G}$ can be brought to the diagonal form $\underline{G}^{(0}$ by a suitable FOS $\underline{S}_0\in\mathrm{Sp}(4,\mathbb{R})$:

$$S_0:\ \underline{G}\to\underline{G}^{(0)}=(\underline{S}_0^{-1})^T\underline{G}\underline{S}_0^{-1} . \tag{4.29}$$

Now if $\underline{G}$ belonged to family I to start with, then $\underline{G}^{(0)}$ is of the form (4.21) and there exists a four-parameter subgroup of FOS's which leaves $\underline{G}^{(0)}$ invariant. This subgroup, which we denote $\mathcal{G}_I$, is clearly the intersection of $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}(4,\mathbb{R})$, and is generated by $\underline{\Sigma}_{12}$, $\underline{\Sigma}_{13}$, $\underline{\Sigma}_{23}$, and $\underline{\Sigma}_{05}$. It follows that the four-parameter group $\underline{S}_0^{-1}\mathcal{G}_I\underline{S}_0\subset\mathrm{Sp}(4,\mathbb{R})$ leaves $\underline{G}$ invariant. On the other hand, if $\underline{G}$ belonged to family II, then $\underline{G}^{(0)}$ is of the type (4.25) and the FOS's which leave $\underline{G}^{(0)}$ invariant form a two-parameter subgroup $\mathcal{G}_{II}$ generated by $\underline{\Sigma}_{13}$ and $\underline{\Sigma}_{05}$, and $\underline{S}_0^{-1}\mathcal{G}_{II}\underline{S}_0$ leaves $\underline{G}$ invariant. Thus we have established the following result: *For every AGSM field belonging to family I (family II) there is a corresponding four-parameter (two-parameter) subgroup of FOS's which leaves it invariant.*

To conclude this section and to avoid any possible misunderstanding, let us state again the principal conclusions: each AGSM field belongs either to family I or to family II, and stays in that family under action by any FOS $\underline{S}\in\mathrm{Sp}(4,\mathbb{R})$. In the case of family I, we have the further invariant κ which cannot be altered by action of any FOS; in the case of family II we have two further invariants κ_1,κ_2 which again cannot be altered by action of any FOS. This is the situation for AGSM fields with finite matrix parameters $\underline{\sigma}_I$, $\underline{\sigma}_g$ and $\underline{K}$ leading to positive-definite matrices $\underline{G}$.

V. AXIALLY SYMMETRIC FOS'S AND THE SUBGROUP SL(2,$\mathbb{R}$)

The $\mathrm{SL}(2,\mathbb{R})$ subgroup of $\mathrm{Sp}(4,\mathbb{R})$ representing axially symmetric FOS's was identified in (3.9). Evidently its generators are those $\underline{\Sigma}_{AB}$ that do not involve the Pauli matrices in the first factor of the Kronecker products at all; these are $\underline{\Sigma}_{35}$, $\underline{\Sigma}_{50}$, and $\underline{\Sigma}_{03}$. We expect this $\mathrm{SL}(2,\mathbb{R})$ to determine some SO(2,1) subgroup of SO(3,2). Now SO(3,2) contains two qualitatively different types of SO(2,1) subgroups, typical examples being the ones acting on the subset of directions 0 1 2 with metric $-++$, and on the directions 3 5 0 with metric $+--$. It is seen that axially symmetric FOS's correspond to the SO(2,1) subgroup of SO(3,2) acting on components 3 5 0. It follows that under their action, an AGSM field possesses new invariants such as ξ^{12}, ζ^1, ζ^2, etc. We analyze the situation systematically.

Let the matrix $\underline{G}$ of an AGSM field be split into 2×2 blocks in the following way:

$$\underline{G}=\begin{bmatrix}\underline{U} & \underline{V}\\ \underline{V}^T & \underline{W}\end{bmatrix} ,$$
$$\underline{U}^T=\underline{U} ,\quad \underline{W}^T=\underline{W} . \tag{5.1}$$

$\underline{U}$ and $\underline{W}$ are positive-definite, 2×2 matrices and another condition representing (2.22) also holds. An axially symmetric FOS with the ray-transfer matrix (3.9) alters $\underline{U},\underline{V},\underline{W}$ in this way:

$$\underline{s}\in\mathrm{SL}(2,\mathbb{R}):\ \underline{\mathcal{U}}'=(\underline{s}^{-1})^T\underline{\mathcal{U}}\underline{s}^{-1} , \tag{5.2}$$

where $\underline{\mathcal{U}}'=\underline{U}',\underline{V}',\underline{W}'$ and $\underline{\mathcal{U}}=\underline{U},\underline{V},\underline{W}$, respectively. We can now carry over to this situation the geometrical analysis given in Ref. 10. $\underline{U}$ and $\underline{W}$ being real symmetric and positive definite, they can be represented by positive "timelike" vectors in the 3 5 0 subspace; and the transformation (5.2) amounts to a proper SO(2,1) pseudorotation on these vectors. As for $\underline{V}$, its antisymmetric part is invariant under $\mathrm{SL}(2,\mathbb{R})$, while its symmetric part yields an SO(2,1) vector. To do this in detail, the correspondence with the treatment of Ref. 10 is that the dimensions 0 1 2 there correspond, respectively, to the present dimensions 3 5 0, and there is an overall change of sign in the metric. We thus write the matrices $\underline{U}$, $\underline{V}$, $\underline{W}$ as

$$\underline{U}=\begin{bmatrix}u_3-u_5 & u_0\\ u_0 & u_3+u_5\end{bmatrix} ,$$
$$\underline{V}=\begin{bmatrix}v_3-v_5 & v_0-\tfrac{1}{2}\chi\\ v_0+\tfrac{1}{2}\chi & v_3+v_5\end{bmatrix} , \tag{5.3}$$
$$\underline{W}=\begin{bmatrix}w_3-w_5 & w_0\\ w_0 & w_3+w_5\end{bmatrix} .$$

If we use indices $a,b,\ldots$ to run over the values 3 5 0, we can say that matrix $\underline{G}$ consists of the three vectors $\underline{u}_a,\underline{v}_a,\underline{w}_a$ and one scalar χ with respect to the SO(2,1) action of axially symmetric FOS's. As stated earlier positivity of G tells us that $\underline{u}$ and $\underline{w}$ are positive timelike,

$$u^a u_a>0 ,\quad u^3>0 ,$$
$$w^a w_a>0 ,\quad w^3>0 , \tag{5.4}$$

while no such simple characterization of $\underline{v}$ seems possible. In Appendix B, we give expressions connecting ξ^{AB}, ζ_A to $\underline{u},\underline{v},\underline{w},\chi$. Isotropic Gaussian Schell-model fields correspond to $\underline{u}-\underline{w}=\underline{v}=0_{3\times1}$ and $\chi=0$.

We stress that when an AGSM field matrix $\underline{G}$ is conveniently represented by the three vectors $\underline{u},\underline{v},\underline{w}$ and one scalar χ, then under action by an axially symmetric FOS $\underline{s}\in\mathrm{SL}(2,\mathbb{R})\subset\mathrm{Sp}(4,\mathbb{R})$, all three vectors $\underline{u},\underline{v},\underline{w}$ experience the *same* three-dimensional Lorentz transformation, while χ is invariant. It follows that we can form seven invariants with respect to such action, namely,

$$\{u^2,v^2,w^2,\underline{u}^T\underline{v},\underline{v}^T\underline{w},\underline{w}^T\underline{u},\chi\}=\mathrm{SL}(2,\mathbb{R})\text{ invariants.} \tag{5.5}$$

This is consistent with $\underline{G}$ having ten parameters and SL(2,$\mathbb{R}$) being a three-parameter group.

To conclude this section, we give the proof promised in Sec. II that *even free propagation can generate an antisymmetric part* $\underline{K}^A$ *in the matrix* $\underline{K}$ *determining the phase function* $\phi(\underline{\rho}_1;\underline{\rho}_2)$ *of the AGSM field (2.1).* For this purpose let us start with an equiphase AGSM field, i.e., $\underline{K}=0_{2\times 2}$; assume also that $\underline{\sigma}_I$ is diagonal while $\underline{\sigma}_g$ and $\underline{\sigma}_I$ *do not commute.* This means that the principal axes of $\underline{\sigma}_I$ coincide with the transverse x and y coordinate axes, while the principal axes for $\underline{\sigma}_g$ and for $\underline{\gamma}$ are definitely different from the coordinate axes. With these assumptions we see from (2.10) that the quantities $\underline{u},\underline{v},\underline{w},\chi$ and the various matrices have the values

$$\underline{u}^T=(u_3,u_5,0)\ ,\quad \underline{v}^T=(v_3,v_3,0)\ ,\quad v_3\neq 0\ ,$$

$$\underline{w}^T=(w_3,w_5,0)\ ,\quad \chi=0\ ,$$

$$\tfrac{1}{2}(\underline{\sigma}_I{}^2)^{-1}=\begin{bmatrix} u_3-u_5 & 0 \\ 0 & w_3-w_5\end{bmatrix}, \tag{5.6}$$

$$\tfrac{1}{2}\underline{\gamma}^2=\begin{bmatrix} u_3+u_5 & 2v_3 \\ 2v_3 & w_3+w_5\end{bmatrix},$$

$$\underline{G}=\begin{bmatrix} u_3-u_5 & 0 & 0 & 0 \\ 0 & u_3+u_5 & 0 & 2v_3 \\ 0 & 0 & w_3-w_5 & 0 \\ 0 & 2v_3 & 0 & w_3+w_5\end{bmatrix}.$$

The free-propagation ray-transfer matrix $\underline{F}(D)$ of (3.5) corresponds to the SL(2,R) element

$$\underline{f}(D)=\begin{bmatrix} 1 & D \\ 0 & 1\end{bmatrix}, \tag{5.7}$$

where for the present discussion only the numerical unit of D is assumed to equal k. The corresponding three-dimensional Lorentz transformation can be read off from Eq. (4.5) of Ref. 10:

$$\underline{\Lambda}(D)=\begin{bmatrix} 1+\frac{1}{2}D^2 & -\frac{1}{2}D^2 & -D \\ \frac{1}{2}D^2 & 1-\frac{1}{2}D^2 & -D \\ -D & D & 1\end{bmatrix}. \tag{5.8}$$

This matrix is to be applied to the components of $\underline{u}$ arranged as a column $(\underline{u}_3,\underline{u}_5,\underline{u}_0)^T$, and similarly for $\underline{v}$ and $\underline{w}$. Thus after free propagation we find that $\underline{u},\underline{v},\underline{w},\chi$ of (5.6) change to

$$\underline{u}'=\underline{u}+(u_3-u_5)\begin{bmatrix}\frac{1}{2}D^2\\ \frac{1}{2}D^2\\ -D\end{bmatrix},$$

$$\underline{v}'=\underline{v}=\begin{bmatrix}v_3\\ v_3\\ 0\end{bmatrix}, \tag{5.9}$$

$$\underline{w}'=\underline{w}+(w_3-w_5)\begin{bmatrix}\frac{1}{2}D^2\\ \frac{1}{2}D^2\\ -D\end{bmatrix},$$

$$\chi'=\chi=0\ .$$

Now, *in general,* i.e., not merely for the specific AGSM field parameters chosen in (5.6), the matrices $\frac{1}{2}\underline{\gamma}^2$ and $-\frac{1}{2}\underline{\gamma}^2\underline{K}$ are given by (2.10) as

$$\tfrac{1}{2}\underline{\gamma}^2=\begin{bmatrix} u_3+u_5 & v_3+v_5 \\ v_3+v_5 & w_3+w_5\end{bmatrix},$$

$$-\tfrac{1}{2}\underline{\gamma}^2\underline{K}=\begin{bmatrix} u_0 & v_0+\frac{1}{2}\chi \\ v_0-\frac{1}{2}\chi & w_0\end{bmatrix}. \tag{5.10}$$

Substituting here the primed quantities of (5.9), we see that after free propagation through a distance D the particular equiphase AGSM field (5.6) has

$$\tfrac{1}{2}(\underline{\gamma}')^2=\begin{bmatrix} u_3+u_5+D^2(u_3-u_5) & 2v_3 \\ 2v_3 & w_3+w_5+D^2(w_3-w_5)\end{bmatrix},$$

$$-\tfrac{1}{2}(\underline{\gamma}')^2\underline{K}'=\begin{bmatrix} -D(u_3-u_5) & 0 \\ 0 & -D(w_3-w_5)\end{bmatrix}. \tag{5.11}$$

By assumption, $v_3\neq 0$, since $\underline{\sigma}_g$ and $\underline{\sigma}_I$ do not share the same principal axes. Therefore the two symmetric matrices appearing in (5.11) *do not commute;* by the same token the two symmetric matrices $(\underline{\gamma}')^{-2}$ and $(\underline{\gamma}')^2\underline{K}'$ do not commute. This means that $\underline{K}'$ has a nonzero antisymmetric part.

VI. COHERENT AND QUASIHOMOGENEOUS LIMITS

According to the basic definition (2.1), every AGSM field must necessarily involve *finite* matrices $(\underline{\sigma}_I{}^2)^{-1}$, $(\underline{\sigma}_g{}^2)^{-1}$, and $\underline{K}$. In addition, we have so far assumed that $(\underline{\sigma}_I{}^2)^{-1}$ and $(\underline{\sigma}_g{}^2)^{-1}$ are *positive definite.* The field matrices G that then arise have been studied in previous sections.

There are interesting limiting cases where we permit one or the other of $(\underline{\sigma}_I{}^2)^{-1}$ and $(\underline{\sigma}_g{}^2)^{-1}$ to become *positive semidefinite,* while both of course remain finite. Thus, for example, among IGSM fields, when $\underline{u}-\underline{w}=\underline{v}=\underline{0}_{3\times 1}$, $\chi=0$, and $\kappa^2=u^2$, the limit $\kappa=1$ is the fully coherent case with $\underline{\sigma}_g^{-1}\rightarrow\underline{0}_{2\times 2}$ while $\underline{G}$ remains positive definite. On the other hand, $\kappa\rightarrow 0$ is the quasihomogeneous limit when $\underline{\sigma}_I{}^{-1} \to \underline{0}_{2\times 2}$ and $\underline{G}$ becomes positive semidefinite. We consider similar limiting cases of the AGSM fields classified in Secs. II and IV.

A. Coherent limits with G positive definite

Let us take the limit $\kappa \to 1$ in the family I of AGSM fields defined in (4.18)–(4.21). The representative matrices $\underline{G}^{(0)}$, $\overline{\underline{G}}^{(0)}$ assume the forms

$$\underline{G}^{(0)} = \overline{\underline{G}}^{(0)} = \mathbb{1}_{4\times 4} \,. \tag{6.1}$$

From (2.19) we read off $\underline{A} = \underline{B} = \mathbb{1}_{2\times 2}$, $\underline{C} = \underline{0}_{2\times 2}$; thus in this limit (2.20) shows that $(\underline{\sigma}_I{}^2)^{-1}$, $(\underline{\sigma}_g{}^2)^{-1}$, and $\underline{K}$ do remain finite, and have values

$$\tfrac{1}{2}(\underline{\sigma}_I{}^2)^{-1} = \mathbb{1}_{2\times 2} \,, \quad (\underline{\sigma}_g{}^2)^{-1} = K = \underline{0}_{2\times 2} \,. \tag{6.2}$$

We thus obtain the set of fully coherent IGSM fields and their transforms by all possible FOS $\underline{S} \in \mathrm{Sp}(4,\mathbf{R})$.

On the other hand, in the family II of AGSM fields defined in (4.22)–(4.25), we can only consider the limit $\kappa_1 \to 1$, while κ_2 must remain less than unity. Thus we arrive at the limiting cases ($\kappa_1 = 1$, $\kappa_2 < 1$) of family II. The representative matrices, in this limit, are, from (4.25),

$$\underline{G}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & 0 & 0 & \kappa_2 \end{bmatrix} ,$$
$$\overline{\underline{G}}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \kappa_2 \end{bmatrix} . \tag{6.3}$$

These are not IGSM fields. The submatrices $\underline{A}$ and $\underline{B}$ of $\overline{\underline{G}}^{(0)}$ are ($\underline{C}$ vanishes)

$$\underline{A} = \underline{B} = \begin{bmatrix} 1 & 0 \\ 0 & \kappa_2 \end{bmatrix} , \tag{6.4}$$

and so, by (2.20),

$$\tfrac{1}{2}(\underline{\sigma}_I{}^2)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \kappa_2 \end{bmatrix} ,$$
$$(\underline{\sigma}_g{}^2)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left[\dfrac{1}{\kappa_2} - \kappa_2\right] \end{bmatrix} , \tag{6.5}$$
$$\underline{K} = \underline{0}_{2\times 2} \,.$$

In contrast to (6.2), $(\underline{\sigma}_g{}^2)^{-1}$ is singular but does not vanish identically. The field we have obtained is "coherent along lines parallel to the y axis."

In both limiting cases so far considered, $\underline{G}$ remains positive definite. We prefer not to think of (6.1) and (6.2) as a limit of (6.3)–(6.5) as $\kappa_2 \to 1$ because the $\underline{\zeta}$ vector is quite different in character in the two cases. In the limiting field of (6.1) and (6.2) [and its transforms by all Sp(4, **R**) systems] we have

$$\xi^2 = 4 \,, \quad \det \underline{G} = 1 \,, \quad \underline{\zeta} = \underline{0}_{1\times 3} \,. \tag{6.6}$$

On the other hand, in the limiting fields (6.3)–(6.5) (and their transforms),

$$\xi^2 = 2(1+\kappa_2^2) \,, \quad \det \underline{G} = \kappa_2^2, \quad \zeta^2 = (1-\kappa_2^2)^2 \,, \tag{6.7}$$

and so $\underline{\zeta}$ does not vanish identically.

B. Quasihomogeneous limits: G semidefinite (Ref. 24)

According to the results described in Appendix C, orbits in the Lie algebra of SO(3,2) with $\underline{G}$ positive semidefinite can be classified as follows; there are two isolated orbits with representative elements given in (C5) and (C6), and in (C7), and (C8), respectively; and there are two one-parameter families of orbits, (m) and $(m)'$, with representative elements given in (C13a) and (C14a), and (C13b) and (C14b), respectively. Let us see in which cases we get physically allowed AGSM fields with finite matrices $(\underline{\sigma}_I{}^2)^{-1}$, $(\underline{\sigma}_g{}^2)^{-1}$, and $\underline{K}$.

For the representative element (C7) (C8), the matrix $\overline{\underline{G}}^{(0)}$ is

$$\overline{\underline{G}}^{(0)} = \tfrac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} . \tag{6.8}$$

Consequently,

$$\underline{A} = \underline{B} = -\underline{C} = \tfrac{1}{2}\mathbb{1}_{2\times 2} \,, \tag{6.9}$$

leading via (2.20) to

$$(\underline{\sigma}_I{}^2)^{-1} = \underline{0}_{2\times 2} \,, \quad (\underline{\sigma}_g{}^2)^{-1} = \underline{K} = \mathbb{1}_{2\times 2} \,. \tag{6.10}$$

This is just the quasihomogeneous limit of the IGSM field (see Ref. 10), so we have obtained in this way this field and all its Sp(4, **R**) transforms. The associated invariants are

$$\xi^2 = \det \underline{G} = 0 \,, \quad \underline{\zeta} = \underline{0}_{3\times 1} \,. \tag{6.11}$$

To analyze the orbit described by (C13b) and (C14b), let us take in place of $(\underline{G}^{(0)})'$ in (C14b) another element $(\underline{G}^{(0)})''$ which lies in the same orbit:

$$(\underline{G}^{(0)})'' = \begin{bmatrix} m/2 & 0 & 0 & 0 \\ 0 & m/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{6.12}$$

$(\underline{G}^{(0)})''$ is related to $(\underline{G}^{(0)})'$ through the following FOS:

$$\underline{S} \in \mathrm{Sp}(4,\mathbf{R}): \ \underline{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b+1 & -b \\ 0 & 0 & -1 & 1 \end{bmatrix} , \quad b \text{ real} \,. \tag{6.13}$$

We note that $\underline{S}$ represents a cylindrical lens when $b = 0$. From (6.12) one obtains

$$\underline{A}=\begin{bmatrix} m/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{B}=\begin{bmatrix} m/2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{C}=\underline{0}_{2\times 2}, \qquad (6.14)$$

which leads via (2.20) to

$$(\underline{\sigma}_I{}^2)^{-1}=\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix},$$

$$(\underline{\sigma}_g{}^2)^{-1}=\frac{1}{2}\begin{bmatrix} \frac{2}{m}-\frac{m}{2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad (6.15)$$

$$\underline{K}=\underline{0}_{2\times 2}.$$

Evidently, this field is GSM in its x dependence (with $m/2$ playing the role of κ), and quasihomogeneous in its y dependence. It follows that the orbit represented by (C13b) and (C14b) consists of this field and all its $\mathrm{Sp}(4,R)$ transforms. The invariants over this orbit are given in (C12), and m must obey $0<m<2$.

The nature of the fields represented by the orbits (C5) and (C6), and (C13a) and (C14a) is quite different. The $\overline{\underline{G}}^{(0)}$ matrices are, respectively,

$$\overline{\underline{G}}^{(0)}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (6.16a)$$

$$\overline{\underline{G}}^{(0)}=\frac{m}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (6.16b)$$

Since the submatrix $\underline{B}$ is singular in both cases we see from (2.20) that the condition of finiteness of $(\underline{\sigma}_g{}^2)^{-1}$ is violated. Further computing $W(\underline{\rho},\underline{p})$ through (2.17) and inverse Fourier transforming it with respect to $\underline{p}$ to obtain Γ one finds that (6.16a) represents a field which is quasihomogeneous in x and incoherent homogeneous in y whereas (6.16b) represents a field which is GSM in x and incoherent homogeneous in y. The incoherence agrees in both cases with our conclusion that $(\underline{\sigma}_g{}^2)^{-1}$ ceases to be finite. Incoherent fields *are not beams* and we discard these two orbits.

VII. CONCLUDING REMARKS

We have analyzed AGSM fields and their transformation under action of FOS's using the method of generalized rays. This method together with the geometrical picture developed in the de Sitter space helps one find complete answers to questions related to this class of problems. Thus we found that the ten-parameter AGSM family is closed under action of all FOS's forming $\mathrm{Sp}(4,\mathbb{R})$. Further, this action divides the AGSM field into two qualitatively distinct families of orbits. The invariants over each one of these orbits have been worked out.

As a fallout of our analysis we have proved that every real 4×4 matrix obeying (2.18) and (2.22) can be diagonalized using a $\mathrm{Sp}(4,\mathbb{R})$ transformation (3.8). Clearly, the diagonal matrix $\underline{G}$ corresponds to an AGSM field with vanishing phase curvature. If such a field is left to propagate freely its intensity width will necessarily increase. Thus, borrowing the terminology from coherent Gaussian beams, diagonal $\underline{G}$'s can be associated with the "waist" of AGSM beams. Now we can restate our conclusion in Sec. IV as follows: *Every AGSM beam can be brought to its waist by a suitable FOS* $\underline{S}\in\mathrm{Sp}(4,\mathbb{R})$. While coherent Gaussian beams, and also IGSM beams, can be brought to the waist by free propagation, to do the same thing with AGSM beams one needs more complicated FOS's.

Our analysis in Sec. V shows that one can also represent an AGSM field as three vectors and one scalar in a $(2+1)$-dimensional Minkowski space. Then the effects of a given FOS $\underline{S}\in\mathrm{SL}(2,\mathbb{R})\subset\mathrm{Sp}(4,\mathbb{R})$ is to "rotate" all the three vectors in the same way (leaving, of course, the scalar invariant). In Ref. 10 we have shown that such a representation naturally leads to the Kogelnik "*abcd* law." Thus our result of Sec. V can be viewed as a generalization of the Kogelnik "*abcd* law" to the AGSM fields. Given an AGSM field, in general there exists no FOS in $\mathrm{SL}(2,\mathbb{R})$ which will leave it invariant. However, if the three vectors representing the field are parallel, then there exists a one-parameter subgroup of FOS in $\mathrm{SL}(2,\mathbb{R})$ which will leave it invariant. This subgroup is the group of Lorentz transformations about these parallel vectors. Clearly, such fields form a six-parameter family: two parameters to specify the orientation of the vectors, three to specify their norms, and the invariant scalar.

It is desirable at this point to recognize that the curious condition (2.22) expresses, in the context of AGSM fields, a general physical requirement on pencils of light rays in wave optics. In classical radiative transfer theory any positive-definite ray density function $W(\underline{\rho};\underline{p})$ is permitted. However, in wave optics the fact that $W(\underline{\rho};\underline{p})$ is the Wolf function formed from the two-point correlation function automatically leads to new restrictions. Namely, the spreads in x and p_x must obey an uncertainty principle reflecting their "noncommutability" in wave optics, and similarly for y and p_y. For the AGSM field with

$$W(\underline{\rho};\underline{p})=(A/\pi^2)(\det\underline{G})^{1/2}\exp(-\underline{q}^T\underline{G}\underline{q}),$$

the ray density is obviously positive definite, and the total irradiance is finite because $\underline{G}$ is positive definite. But the identification of W as a Wolf function demands not only that $\det\underline{G}<1$, but also that if we obtain a marginal distribution for x and p_x by integrating W with respect to y and p_y, we should get a distribution of the form

$$A'\exp\left[-(x \;\; p_x)\underline{g}\begin{bmatrix} x \\ p_x \end{bmatrix}\right]$$

with the 2×2 matrix $\underline{g}$ obeying $\det\underline{g}<1$. This should also be true for the y-p_y marginal distribution and indeed for *any* marginal distribution for any direction in the x-y plane. It is these requirements that are expressed by (2.22). By integrating $W(\underline{\rho};\underline{p})$ in turn with respect to y,p_y and x,p_x and imposing the above conditions we see that in the description (5.1) of the AGSM field, the condition (2.22) appears in the explicitly $\mathrm{SL}(2,\mathbb{R})$ invariant form

$$\det(\underline{U}-\underline{V}\underline{W}^{-1}\underline{V}^T)<1 ,$$
$$\det(\underline{W}-\underline{V}^T\underline{U}^{-1}\underline{V})<1 .$$

For IGSM fields these conditions are automatically satisfied once we are given $\det\underline{G}<1$, because this inequality implies $\det\underline{g}<1$ for any marginal distribution.

Our analysis also shows the way to realize physically the ten-parameter AGSM fields from much simpler fields. All the fields belonging to family I can be generated from equiphase IGSM fields through transformation by appropriate FOS's in Sp(4, $\mathbb{R}$). Several procedures for generating IGSM fields (also called Collett-Wolf sources) are already known.[25] On the other hand, fields belonging to family II can be generated from the Li Wolf type of fields again through transformation by FOS's in Sp(4, $\mathbb{R}$).

Finally, our analysis can be simply extended to fields which are incoherent superpositions (convex combinations) of AGSM fields.[26] For such fields it is clear from the geometrical picture presented in Sec. IV that in addition to the invariants associated with the individual fields there will be new invariants corresponding to the inner products of $\underline{\xi}$'s and $\underline{\zeta}$'s.

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APPENDIX A: Sp(4, $\mathbb{R}$)—INVARIANCE OF CONDITION (2.22)

Let $\psi(\underline{\rho})$ denote the complex field amplitude over a transverse plane. The cross-spectral density is the ensemble average

$$\Gamma(\underline{\rho}_1;\underline{\rho}_2)=\langle\psi^*(\underline{\rho}_1)\psi(\underline{\rho}_2)\rangle . \tag{A1}$$

It is convenient to introduce a Hilbert space $\mathscr{H}$ of complex functions that are square integrable over the transverse plane:

$$\mathscr{H}=\{\psi(\underline{\rho}):\ \int d^2\rho\,|\psi(\underline{\rho})|^2<\infty\} . \tag{A2}$$

Let $|\underline{\rho}\rangle$ and $\langle\underline{\rho}|$ denote, in the usual quantum-mechanical sense, a basis of idealized ket and bra vectors for $\mathscr{H}$, obeying

$$\langle\underline{\rho}'|\underline{\rho}\rangle=\delta^{(2)}(\underline{\rho}'-\underline{\rho}) . \tag{A3}$$

Then any given cross-spectral density $\Gamma(\underline{\rho}_1;\underline{\rho}_2)$ can be formally associated with an operator Γ on $\mathscr{H}$ by writing

$$\Gamma(\underline{\rho}_1;\underline{\rho}_2)\equiv\langle\underline{\rho}_1|\Gamma|\underline{\rho}_2\rangle . \tag{A4}$$

The most important physical properties of $\Gamma(\underline{\rho}_1;\underline{\rho}_2)$ can then be expressed by saying that Γ is a Hermitian, positive-definite operator on $\mathscr{H}$:

$$\Gamma^\dagger=\Gamma , \tag{A5a}$$
$$\Gamma>0 . \tag{A5b}$$

In fact this is a complete physical characterization of all possible cross-spectral densities, and given the Hermiticity property, the positive definiteness of Γ is precisely equivalent to the statement that the (modulus of the) normalized degree of coherence is bounded above by unity.

The group Sp(4, $\mathbb{R}$) of FOS's acts via (3.2) on the generalized ray density distribution function $W(\underline{q})$. As is well known, this action results from or is induced by, a unitary operator action on the field amplitude ψ. Let us introduce the four Hermitian operators on $\mathscr{H}$,

$$x_a,\ \ P_a=-i\frac{\partial}{\partial x_a} ,\ \ a=1,2 \tag{A6}$$

obeying the canonical commutation relations

$$\begin{aligned}&[x_a,x_b]=[P_a,P_b]=0 ,\\ &[x_a,P_b]=i\delta_{ab} .\end{aligned} \tag{A7}$$

If we form the ten Hermitian quadratic expressions

$$\begin{aligned}&x_a x_b ,\\ &\tfrac{1}{2}(x_aP_b+P_bx_a) ,\\ &P_aP_b ,\end{aligned} \tag{A8}$$

we find that they are closed under commutation, and in fact give us a Hermitian representation of the Lie algebra of Sp(4, $\mathbb{R}$). By exponentiation, we then find a unitary representation of Sp(4, $\mathbb{R}$) acting on $\mathscr{H}$. Thus to each $\underline{S}\in$ Sp(4, $\mathbb{R}$), we have a corresponding unitary operator $U_{\rm op}(\underline{S})$ on $\mathscr{H}$, such that

$$U_{\rm op}(\underline{S}')U_{\rm op}(\underline{S})=U_{\rm op}(\underline{S}'\underline{S}) . \tag{A9}$$

Moreover, if we arrange the four operators (A6) into a column

$$\underline{Q}=\begin{bmatrix}x_1\\P_1\\x_2\\P_2\end{bmatrix} , \tag{A10}$$

[the operator form of (2.9)], we have

$$U_{\rm op}(\underline{S})\underline{Q}[U_{\rm op}(\underline{S})]^{-1}=\underline{S}^{-1}\underline{Q} . \tag{A11}$$

The action of a FOS $\underline{S}\in$ Sp(4, $\mathbb{R}$) on an incoming field whose cross-spectral density defines an operator Γ via (A4) is to map this operator onto a new Γ' according to the rule

$$\underline{S}\in\mathrm{Sp}(4,R):\Gamma\to\Gamma'=U_{\rm op}(\underline{S})\Gamma[U_{\rm op}(\underline{S})]^{-1} . \tag{A12}$$

We now see that *because* Sp(4, $\mathbb{R}$) *is unitarily represented on* $\mathscr{H}$, Γ' *obeys the conditions* (A5) *whenever* Γ *does.*

Let us now suppose Γ corresponds to some AGSM field with associated matrix $\underline{G}$ obeying (2.18) and (2.22). When this field passes through a FOS $\underline{S}\in$ Sp(4, $\mathbb{R}$), on the one hand we know that $\underline{G}$ changes according to (3.8), while on the other hand this same change can be described by (A12) above. Thus the emergent AGSM field is surely one for which the normalized degree of coherence remains bounded by unity from above. This means that the outgoing matrix $\underline{\sigma}'_g$ is positive definite, or that $\underline{G}'$ also obeys the condition (2.22). This proves that the somewhat difficult to handle matrix condition (2.22) is an Sp(4,$\mathbb{R}$) invariant condition.

Now from an analysis of the orbits in the Lie algebra of Sp(4,$\mathbb{R}$) and choice of representative elements, we have seen in Sec. IV that every AGSM matrix $\underline{G}$ can be diagonalized with a suitable $\underline{S}\in$Sp(4,$\mathbb{R}$). It therefore suffices to impose (2.22) on diagonal representative elements $\underline{G}^{(0)}$, which is what we have done in determining the two allowed families of AGSM fields and the ranges of their parameters.

While the Sp(4,$\mathbb{R}$) group is unitarily represented on $\mathscr{H}$, it is important to recognize that the group SL(4,$\mathbb{R}$) cannot be so represented. If we define an SL(4,$\mathbb{R}$) action on a general cross-spectral density by (3.2), it may happen that a physical Γ is mapped into a unphysical one. In fact this is always so for elements of SL(4,$\mathbb{R}$) outside Sp(4,$\mathbb{R}$). It is interesting that simple instances of this can be constructed using suitable AGSM fields.

Analogous to the Sp(4,$\mathbb{R}$)—SO(3,2) relationship, SL(4,$\mathbb{R}$) is the twofold covering group of SO(3,3). The extra SL(4,$\mathbb{R}$) generators are those real traceless 4×4 matrices that are outside the set of Sp(4,$\mathbb{R}$) generators $\underline{\Sigma}_{AB}$ given in (4.7). Introduce indices $R,S\ldots$ to run over 0,1,2,3,5,6 and extend the diagonal SO(3,2) metric η_{AB} to a diagonal SO(3,3) metric η_{RS} with $\eta_{66}=-1$. Then the extra SL(4,$\mathbb{R}$) generators $\underline{\Sigma}_{A6}=-\underline{\Sigma}_{6A}$ are

$$\underline{\Sigma}_{\mu6}=\tfrac{1}{2}\underline{\gamma}_{\mu}\underline{\gamma}_{5}\ ,\quad \underline{\Sigma}_{56}=\tfrac{1}{2}\underline{\gamma}_{5}\ ,$$
$$\underline{\gamma}_{5}=\underline{\gamma}_{0}\underline{\gamma}_{1}\underline{\gamma}_{2}\underline{\gamma}_{3}=i\underline{\sigma}_{2}\times\underline{\sigma}_{1}\ . \tag{A13}$$

The 15 real independent traceless $\underline{\Sigma}_{RS}$ generating SL(4,$\mathbb{R}$) obey the extension of the commutation rules (4.9) to six dimensions which is the SO(3,3) algebra. The extension Sp(4,$\mathbb{R}$)$\to$SL(4,$\mathbb{R}$) so defined is appropriate for action on matrices $\underline{G}$ representing AGSM fields.

It is evident that the extra generators of SL(4,$\mathbb{R}$) fall into three distinct categories: elliptic ones of the type $\underline{\Sigma}_{06}$, parabolic ones of the type $\underline{\Sigma}_{26}+\underline{\Sigma}_{56}$, and hyperbolic ones of the type $\underline{\Sigma}_{26}$. Thus any linear combination of $\underline{\Sigma}_{A6}$ can be transformed, via a suitable SO(3,2) transformation, into a multiple of either $\underline{\Sigma}_{06}$ or $\underline{\Sigma}_{26}+\underline{\Sigma}_{56}$ or $\underline{\Sigma}_{26}$: These cases are, of course, mutually exclusive. For each of these three characteristically different one-parameter subgroups in SL(4,$\mathbb{R}$), respectively, generated by $\underline{\Sigma}_{06}$, $\underline{\Sigma}_{26}+\underline{\Sigma}_{56}$, and $\underline{\Sigma}_{26}$, we can give examples of AGSM fields that are mapped into unphysical fields under their action. Since the crucial condition (2.22) is stated in terms of submatrices of $\overline{\underline{G}}$, we shall use SP(4,$\mathbb{R}$) and SL(4,$\mathbb{R}$) in the form appropriate to action on $\overline{\underline{G}}$ rather than on $\underline{G}$: this only means that the Pauli matrices in the Kronecker product must be interchanged. We shall thus write $\overline{\underline{\Sigma}}_{RS}$ for the SL(4,$\mathbb{R}$) generators appropriate for action directly on $\overline{\underline{G}}$.

The most general diagonal matrices $\overline{\underline{G}}$ obeying all the conditions (2.18) and (2.22) are of the form

$$\overline{\underline{G}}=\begin{bmatrix} a&0&0&0\\ 0&b&0&0\\ 0&0&c&0\\ 0&0&0&d\end{bmatrix}\ ,\quad a,b,c,d>0\ ,\quad ac<1\ ,\quad bd<1\ . \tag{A14}$$

The explicit examples we give are taken from the limiting case $ac=bd=1$. In the case of $\overline{\underline{\Sigma}}_{06}$, the SL(4,$\mathbb{R}$) elements are

$$\begin{aligned}\underline{S}(\theta)&=\exp(2\theta\overline{\underline{\Sigma}}_{06})\\ &=(\cos\theta)\mathbb{1}_{4\times4}+(i\sin\theta)\underline{\sigma}_{3}\otimes\underline{\sigma}_{2}\ .\end{aligned} \tag{A15}$$

Now take $\overline{\underline{G}}$ to be

$$\overline{\underline{G}}=\begin{bmatrix} \lambda&0&0&0\\ 0&\lambda^{-1}&0&0\\ 0&0&\lambda^{-1}&0\\ 0&0&0&\lambda\end{bmatrix}\ ,\quad \lambda>0\ . \tag{A16}$$

Transforming this by $\underline{S}(\theta)$ we get $\overline{\underline{G}}'=[\underline{S}(\theta)]^T\overline{\underline{G}}\underline{S}(\theta)$ whose submatrices $\underline{A}$, $\underline{B}$, and $\underline{C}$ are

$$\begin{aligned}\underline{A}&=\begin{bmatrix}\lambda\cos^2\theta+\lambda^{-1}\sin^2\theta & (\lambda-\lambda^{-1})\cos\theta\sin\theta\\ (\lambda-\lambda^{-1})\cos\theta\sin\theta & \lambda\sin^2\theta+\lambda^{-1}\cos^2\theta\end{bmatrix}\ ,\\ \underline{B}&=\begin{bmatrix}\lambda\sin^2\theta+\lambda^{-1}\cos^2\theta & (\lambda-\lambda^{-1})\cos\theta\sin\theta\\ (\lambda-\lambda^{-1})\cos\theta\sin\theta & \lambda\cos^2\theta+\lambda^{-1}\sin^2\theta\end{bmatrix}\ ,\\ \underline{C}&=\underline{0}_{2\times2}\ .\end{aligned} \tag{A17}$$

For any nonzero θ, it is easily seen that (2.22) is violated. In the case of $\overline{\underline{\Sigma}}_{26}+\overline{\underline{\Sigma}}_{56}$, the SL(4,$\mathbb{R}$) matrices are

$$\begin{aligned}\underline{S}(u)&=\exp[2u(\overline{\underline{\Sigma}}_{26}+\overline{\underline{\Sigma}}_{56})]\\ &=\mathbb{1}_{4\times4}-u(\mathbb{1}_{2\times2}\otimes\underline{\sigma}_{3}-i\underline{\sigma}_{1}\otimes\underline{\sigma}_{2})\ .\end{aligned} \tag{A18}$$

and in the case of $\overline{\underline{\Sigma}}_{26}$, we have

$$\begin{aligned}\underline{S}(v)&=\exp(2v\overline{\underline{\Sigma}}_{26})\\ &=(\cosh v)\mathbb{1}_{4\times4}-(\sinh v)\mathbb{1}_{2\times2}\otimes\underline{\sigma}_{3}\ .\end{aligned} \tag{A19}$$

For both these cases, if we start with $\overline{\underline{G}}=\mathbb{1}$, we immediately find that $[\underline{S}(u)]^T\underline{S}(u)$ and $[\underline{S}(v)]^T\underline{S}(v)$ both violate (2.22) for nonzero u and v.

APPENDIX B: RELATION BETWEEN PARAMETRIZATIONS of $\underline{G}$

Two different parametrizations of the AGSM field matrix $\underline{G}$ have been used, one in terms of ξ^{AB} in (4.11) and another in terms of $\underline{u},\underline{v},\underline{w},\chi$ in (5.1) and (5.3). Here we give the connection between them. The former expressed in terms of the latter are

$$\begin{aligned}&\xi^{01}=(\underline{u}-\underline{w})_0\ ,\quad \xi^{02}=2v_0\ ,\\ &\xi^{03}=(\underline{u}+\underline{w})_5\ ,\quad \xi^{05}=-(\underline{u}+\underline{w})_3\ ;\\ &\xi^{12}=\chi\ ,\quad \xi^{13}=(\underline{u}-\underline{w})_3\ ,\quad \xi^{15}=(\underline{w}-\underline{u})_5\ ;\\ &\xi^{23}=2v_3\ ,\quad \xi^{25}=-2v_5\ ;\\ &\xi^{35}=(\underline{u}+\underline{w})_0\ .\end{aligned} \tag{B1}$$

The reverse relationships are best expressed using the SO(2,1) notation of Sec. V. Indices $a,b,\ldots$ run over 3 5 0, the symbol ϵ_{abc} is normalized to $\epsilon_{350}=+1$, and these indices are raised and lowered with the diagonal metric $\eta_{33}=1$, $\eta_{55}=\eta_{00}=-1$. Then,

$$\begin{aligned} u_a &= \tfrac{1}{2}\xi_{1a}+\tfrac{1}{4}\epsilon_{abc}\xi^{bc}\ , \\ v_a &= \tfrac{1}{2}\xi_{2a}\ , \\ w_a &= -\tfrac{1}{2}\xi_{1a}+\tfrac{1}{4}\epsilon_{abc}\xi^{bc}\ , \\ \chi &= \xi_{12}\ . \end{aligned} \tag{B2}$$

Finally, the Pauli-Lubanski vector $\underline{\zeta}$ is given by

$$\begin{aligned} \zeta_a &= \chi(\underline{u}+\underline{w})_a - 2\epsilon_{abc}(\underline{u}-\underline{w})^b v^c\ , \\ \zeta_1 &= 2(\underline{u}+\underline{w})^a v_a\ , \\ \zeta_2 &= w^a w_a - u^a u_a\ . \end{aligned} \tag{B3}$$

These expressions make it easy to see why for an IGSM field, when $\underline{v}=\underline{0}_{3\times 1}$ and $\chi=0$, and $\underline{w}=\underline{u}$, $\underline{\zeta}=\underline{0}_{5\times 1}$ identically.

APPENDIX C: ORBITS IN THE SO(3,2) LIE ALGEBRA

Let us denote by $\underline{e}_A$, $A=0,1,2,3,5$ the five mutually orthogonal unit vectors along the coordinate axes in the five-dimensional de Sitter space. $\underline{e}_0$ and $\underline{e}_5$ are timelike with negative squared norm, while $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ are spacelike with positive squared norm. Let J_{AB} be a basis for the abstract Lie algebra of SO(3,2); in the 4-dimensional [Sp(4,$\mathbb{R}$)] representation, the $\underline{J}_{AB}$ are realized as the 4×4 matrices $\underline{\Sigma}_{AB}$ of Sec. IV.

For some ξ^{AB}, let the generator

$$\underline{J}=\tfrac{1}{2}\xi^{AB}\underline{J}_{AB} \tag{C1}$$

of SO(3,2) be given. With it is associated an infinitesimal pseudorotation in 3+2 space, under which the (contravariant) components Z^A of a vector change by the amounts

$$\delta Z^A=\epsilon\xi^A{}_B Z^B\ ,\quad |\epsilon| \ll 1. \tag{C2}$$

If one associates with $\underline{J}$ the vector $\underline{\zeta}$ defined by (4.14), this vector is invariant under the rotation generated by $\underline{J}$:

$$\delta\zeta^A=\epsilon\xi^A{}_B\zeta^B=0\ . \tag{C3}$$

This fact expresses the special relationship between $\underline{\xi}$ and $\underline{\zeta}$.

More generally one can ask how many independent vectors are left invariant under the rotation (C2) corresponding to a given $\underline{J}$. Because we are dealing with a five-dimensional space, this number is restricted to be either 3 or 1: the rank of a 5×5 antisymmetric matrix has to be either 2 or 4, leaving out the case $\underline{J}=\underline{0}_{4\times 4}$.

We shall say that $\underline{J}$ belongs to class I if it leaves three independent vectors invariant, and to class II if it leaves just one nonzero vector invariant. Detailed analysis shows that for $\underline{J}$ of class I, $\underline{\zeta}=\underline{0}_{5\times 1}$ identically; while for $\underline{J}$ of class II, $\underline{\zeta}$ does not vanish identically, and is in fact the nonzero invariant vector.

1. Class I generators

Without loss of generality, we can assume that the three independent invariant vectors are mutually orthogonal. Let the symbols $\underline{t}$, $\underline{l}$, and $\underline{s}$ stand for "timelike," "lightlike," and "spacelike," vectors in de Sitter space: $Z^A Z_A$ is less than, equal to, and greater than zero, respectively. Then one can imagine the following ten configurations for the three mutually orthogonal invariant vectors: $\underline{ttt}$, $\underline{lll}$, $\underline{sss}$; $\underline{ttl}$, $\underline{tts}$, $\underline{tll}$, $\underline{lls}$, $\underline{tss}$, $\underline{lss}$; $\underline{tls}$. However, the 3+2 space with signature $-+++-$ cannot accommodate the configurations $\underline{ttt}$, $\underline{lll}$, $\underline{ttl}$, and $\underline{tll}$. Therefore, in class I we have to consider the six distinct possibilities $\underline{sss}$; $\underline{tts}$, $\underline{lls}$, $\underline{tss}$, $\underline{lss}$; $\underline{tls}$.

Consider the situation $\underline{sss}$. If $\underline{J}$ leaves each of three mutually orthogonal spacelike vectors invariant, so does every $\underline{J}'$ on the same orbit as $\underline{J}$. However, the invariant triad varies as $\underline{J}$ varies over the orbit. Starting with $\underline{J}$, one can move to another point $\underline{J}^{(0)}$ on the orbit of $\underline{J}$ at which the invariant vectors are $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$. This $\underline{J}^{(0)}$ can be taken as the orbit representative; it is necessarily a multiple of $\underline{J}_{50}$:

$$\underline{J}^{(0)}=m\underline{J}_{50}\ ,\quad m\neq 0\ . \tag{C4}$$

We thus arrive at a one-parameter family of orbits, actually consisting of two disconnected subfamilies for $m>0$ and $m<0$, respectively. Out of these, only for $0<m<2$ do the matrices $\underline{G}^{(0)}=\beta\underline{J}^{(0)}$ obey all conditions (2.18) and (2.22), so we are led to the family I of AGSM fields described in Sec. IV.

In all the other five possibilities arising in class I, $\underline{G}$ is never positive definite. It is in some cases at best positive semidefinite. For the situations $\underline{tts}$, $\underline{tss}$, and $\underline{tls}$, we find that $\underline{G}$ always has both positive and negative eigenvalues. In the situation $\underline{lls}$, we find that there are two distinct orbits (*not* families of orbits) and only over one of them is $\underline{G}$ positive semidefinite. A representative $\underline{J}^{(0)}$ for this orbit is

$$\underline{J}^{(0)}=\underline{J}_{30}+\underline{J}_{32}+\underline{J}_{50}+\underline{J}_{52}\ . \tag{C5}$$

The corresponding matrix $\underline{G}^{(0)}$ is

$$\underline{G}^{(0)}=\tfrac{1}{2}(\mathbb{1}_{2\times 2}-\underline{\sigma}_1)\otimes(\mathbb{1}_{2\times 2}+\underline{\sigma}_3) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{C6}$$

and has eigenvalues (2,0,0,0). In the situation $\underline{lss}$, we again find just two distinct orbits, and only over one of them is $\underline{G}$ positive semidefinite. A representative $\underline{J}^{(0)}$ for this orbit is

$$\underline{J}^{(0)}=\underline{J}_{50}+\underline{J}_{53}\ . \tag{C7}$$

The corresponding matrix $\underline{G}^{(0)}$ is

$$\underline{G}^{(0)}=\tfrac{1}{2}\mathbb{1}_{2\times2}\otimes(\mathbb{1}_{2\times2}-\underline{\sigma}_1)$$
$$=\frac{1}{2}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{bmatrix} \tag{C8}$$

and has eigenvalues (1,0,1,0).

In all the situations encountered under class I, the generator $\underline{J}^{(0)}$ is determined up to a multiplicative factor, or even uniquely, because it leaves three independent vectors invariant. The results for class II are quite different.

2. Class II generators

Here we have just one invariant vector, and that is $\underline{\zeta}$. We may immediately divide the discussion into three distinct subclasses: II(a) when $\underline{\zeta}$ is $\underline{t}$: II(b) when $\underline{\zeta}$ is $\underline{l}$; and II(c) when $\underline{\zeta}$ is $\underline{s}$.

II(a): Timelike ζ. Given $\underline{J}$, one can pass to those generators on the orbit of $\underline{J}$ for which the invariant vector is, say, $\underline{e}_5$. In other words, without loss of generality we may assume to begin with that $\zeta_\mu=0$, $\zeta_5\neq0$. Then $\underline{J}$ belongs to the Lie algebra of the subgroup SO(3,1)⊂SO(3,2) acting on the dimensions 0 1 2 3; moreover, $\underline{J}$ *must not* leave invariant any linear combination of $\underline{e}_0$, $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$. Analyzing the situation one finds that in this subclass there is a two-parameter continuous family of distinct orbits, but over every one of them $\underline{G}$ has both positive and negative eigenvalues.

II(b): Lightlike ζ. Here we may assume without loss of generality that $\zeta_0=\zeta_1=\zeta_3=0$, $\zeta_2=-\zeta_5=1$ to begin with. Then $\underline{J}$ belongs to the Lie algebra of an E(2,1) subgroup of SO(3,2), and must not leave invariant any vector independent of $\underline{e}_2+\underline{e}_5$. One then finds that in this subclass there are four separate one-parameter continuous families of orbits, plus two distinct ones. But again $\underline{G}$ has both positive and negative eigenvalues throughout this subclass.

II(c): Spacelike ζ. We now assume that ζ_2 is the only nonzero component of $\underline{\zeta}$. Then $\underline{J}$ belongs to the Lie algebra of the subgroup SO(2,2)⊂SO(3,2) acting on the dimensions 0 1 3 5; and it *does not* leave invariant any linear combination of $\underline{e}_0$, $\underline{e}_1$, $\underline{e}_3$, and $\underline{e}_5$. Since locally SO(2,2)≃SO(2,1)×SO(2,1), one can analyze the situation completely and list all possible essentially distinct forms of $\underline{J}$. In this way one finds three two-parameter families of orbits and four one-parameter families, some made up of several disconnected components. Over most orbits $\underline{G}$ is neither positive definite nor semidefinite. We now list those orbits with positive definite or semidefinite $\underline{G}$.

$\underline{G}$ *positive definite.* There is a two-parameter family of such orbits, labeled by m,n obeying $m>n>0$. Over the orbit (m,n) we have

$$\xi^2=\tfrac{1}{2}(m^2+n^2)\ ,\quad \zeta^2=\tfrac{1}{16}(n^2-m^2)^2\ . \tag{C9}$$

A representative for this orbit is

$$\underline{J}^{(0)}=\tfrac{1}{2}(m+n)\underline{J}_{50}+\tfrac{1}{2}(m-n)\underline{J}_{13}\ , \tag{C10}$$

for which

$$\zeta_0^{(0)}=\zeta_1^{(0)}=\zeta_3^{(0)}=\zeta_5^{(0)}=0\ ,$$
$$\zeta_2^{(0)}=\tfrac{1}{4}(n^2-m^2)\ . \tag{C11}$$

The eigenvalues of $\underline{G}^{(0)}=\underline{\beta}\underline{J}^{(0)}$ are $(\tfrac{1}{2}m,\tfrac{1}{2}m,\tfrac{1}{2}n,\tfrac{1}{2}n)$. However, only for $0<n<m<2$ are all conditions (2.18) and (2.22) obeyed by $\underline{G}^{(0)}$, and so we are led to the family II of AGSM fields described in Sec. IV.

$\underline{G}$ *positive semidefinite.* There are two one-parameter families of such orbits, each labeled by $m>0$. We denote the orbits $(m),(m)'$. Over each, the invariants depend on m in the same way:

$$\xi^2=\frac{m^2}{2}\ ,\quad \zeta^2=\frac{m^4}{16}\ . \tag{C12}$$

Orbit representatives for (m) and for $(m)'$ can be chosen so that the only nonzero component of $\zeta^{(0)}$ is $\zeta_2^{(0)}=-m^2/4$. The representatives then are

$$\underline{J}^{(0)}=\frac{m}{2}(\underline{J}_{50}+\underline{J}_{13})\ , \tag{C13a}$$

$$(\underline{J}^{(0)})'=\tfrac{1}{2}(m+1)\underline{J}_{50}+\tfrac{1}{2}(m-1)\underline{J}_{13}+\tfrac{1}{2}(\underline{J}_{30}-\underline{J}_{15})\ . \tag{C13b}$$

The corresponding matrices $\underline{G}^{(0)}$ and $(\underline{G}^{(0)})'$ are

$$\underline{G}^{(0)}=\underline{\beta}\underline{J}^{(0)}=\frac{m}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \tag{C14a}$$

$$(\underline{G}^{(0)})'=\underline{\beta}(\underline{J}^{(0)})'=\begin{bmatrix}\frac{m}{2}&0&0&0\\0&\frac{m}{2}&0&0\\0&0&\frac{1}{2}&-\frac{1}{2}\\0&0&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}. \tag{C14b}$$

Summarizing the essential results, orbits with positive definite G are those with representative elements (C4) for $m>0$; and (C10) for $m>n>0$. Orbits with positive semidefinite $\underline{G}$ are those with representative elements (C5), (C7), (C13a), and (C13b) for $m>0$. *On all other orbits in the SO(3,2) Lie algebra, $\underline{G}$ always has both positive and negative eigenvalues.*

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