Monopole Topology and the Problem of Color

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(Received 29 November 1982)

Topological obstructions to the definition of unbroken-symmetry transformations in the presence of monopoles with non-Abelian flux are exposed. Consequences for color and electroweak SU(2) in the context of the SU(5) model are discussed.

PACS numbers: 14.80.Hv, 11.30.Qc, 12.10.En

Current grand unified theories of fundamental interactions envisage a gauge-invariant theory with a group G spontaneously broken to a subgroup H. All such theories exhibit monopole solutions with nontrivial topological properties. In monopole sectors with non-Abelian magnetic flux we show in this paper that there are topological obstructions to a global definition of some of the unbroken-symmetry transformations. For $G = \mathrm{SU}(5)$ and $H = \mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$ (locally), neither the full color group nor the electroweak SU(2) group transformations can be globally defined in the presence of such monopoles.

We shall illustrate these remarks in the context of the Georgi-Glashow model where SU(5) $\equiv G$ is broken by a Higgs field in $\underline{24}$. The asymptotic form for the Higgs field in the lowest monopole sector is known to be,[1] up to a constant,

$$\varphi_\infty(\vec{x}) = \tfrac{5}{4}\vec{\tau}\cdot\hat{x} + \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -\tfrac{1}{4} & & \\ & & & -\tfrac{1}{4} & \\ & & & & -\tfrac{3}{2} \end{pmatrix}, \qquad (1)$$

$\hat{x} = \vec{x}/|\vec{x}|$.

The $\tau_i/2$ are the SU(2) generators acting on the third and fourth members of the $\underline{5}^*$ multiplet $(d_1{}^c, d_2{}^c, d_3{}^c, e^-, \nu_e)_L$. This configuration has the little group $H_{\hat{x}} \subset G$ obtained from $H_{(0,0,1)} \equiv H$ by conjugation: $H_{\hat{x}} = u(\hat{x})Hu^{-1}(\hat{x})$. [Hence, for all $\hat{x}$, $H_{\hat{x}}$ is isomorphic to $H \equiv \mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$]. Here $u(\hat{x}) = \exp[i\vec{\xi}(\hat{x})\cdot\vec{\tau}]$ is such that

$$u(\hat{x})\tau_3 u^{-1}(\hat{x}) = \vec{\tau}\cdot\hat{x}. \qquad (2)$$

Global definition of the action of this H requires setting up an isomorphism $H \to H_{\hat{x}}$ varying smoothly with $\hat{x}$. This gives an association of an element $h(\hat{x}) \in H_{\hat{x}}$ for each $h \equiv h(0,0,1) \in H$ compatible with the group properties. With a suitable extension $g(\vec{x}) \in G$ of $h(\hat{x})$ from $|x| = \infty$ to all $\vec{x}$,[2] the action of an element $h \in H$ on the fields of the theory is realized by the gauge transformation $g(\vec{x})$. However, there are topological obstructions in setting up the isomorphism $h \to h(\hat{x})$. We see this as follows. A possible choice is

$$h(\hat{x}) = u(\hat{x})hu^{-1}(\hat{x}). \qquad (3)$$

Now it is well known that whatever be the choice of $u(\hat{x})$, it will fail to have a unique value at least at one point on the two-sphere, say at $(0,0,-1)$.[3] Thus let u_φ be the limit of $u(\hat{x})$ as $\hat{x} \to (0,0,-1)$ along azimuth φ. Then $u_\varphi = u_0 \exp[i\alpha(\varphi)\tau_3]$, where $\alpha(\varphi)$ changes by 2π as φ goes from 0 to 2π. Equation (3) therefore fails to give, for a general $h \in H$, a well defined image $h(0,0,-1)$ in $H_{(0,0,-1)}$.[4] The problem is absent for the commutant of τ_3. The corresponding subgroup, which in our case is $\mathrm{U}(2)\otimes\mathrm{U}(1)\otimes\mathrm{U}(1)$, is thus the only set of globally definable unbroken-symmetry *transformations*. Note, however, that the *set* $H_{\hat{x}}$ is well defined for all $\hat{x}$ since $\exp[i\alpha(\varphi)\tau_3] \in H$.

If in spite of this obstruction we attempt to use Eq. (3) to implement the action of a generic h, the result is to transform a finite-energy field configuration to one with infinite energy. Let W_i' be the gauge-transformed Yang-Mills potentials $g(\vec{x})(\partial_i + W_i)g^{-1}(\vec{x})$. Because of the multivaluedness of $u(\hat{x})$ at $(0,0,-1)$, W_i' receives contributions proportional to $\partial_i\alpha(\varphi)$ near the negative z axis. Since $\exp[i\alpha(\varphi)]$ is a nontrivial closed loop in U(1),[3] the above-mentioned term in W_i' leads to a term in the field strength $F_{ij}(W')$ proportional to $\theta(-z)\delta(x)\delta(y)$ and hence to infinite energy.

The preceding discussion also shows that fields which transform tensorially under an unbroken non-Abelian subgroup cannot in general be globally defined in the monopole sector. For if $\chi = \{\chi_a\}$ is such a multiplet, it should transform under H according to the rule $\chi(x) \to \chi'(x) = \mathfrak{D}(h(\hat{x}))\chi(x)$ where $\{\mathfrak{D}(h)\}$ is the appropriate representation of H. Since $h(\hat{x})$ gets undefined for some $\hat{x}$, we see that either χ or χ' will cease to be single valued in $\hat{x}$.[5] This problem exists in particular for the conserved currents $j_\mu{}^a$ of H which are supposed to transform under the adjoint representation. The consequences of these observations are serious as shown by the following: (a) The construction of Ward identities (which are of fundamental importance for the renormalization program) in a manner consistent with H invariance runs into difficulties in the non-Abelian monopole sector. (b) The construction of effective Lagrangians in such a sector which are invariant under H presents problems. Thus, for example, it is not clear how to describe the catalysis of proton decay by monopoles[6] with use of operator product expansions while maintaining H invariance.

Similar topological obstructions occur for test particles in such a monopole field. They are caused in quantum mechanics by the fact that the wave functions have to transform tensorially under H while we have seen that such single-valued tensorial fields do not usually exist. In classical mechanics, a similar global problem occurs with a spinlike variable which accounts for the internal degrees of freedom of the test particle. Briefly, the results are as follows. (A full account will be given elsewhere.) At the classical level the configuration space does not permit an appropriate action of a general $h \in H$. For a spinless, nonrelativistic, quantum mechanical particle (for instance) belonging to a representation of G, the domain of the Hamiltonian $\mathcal{H}$ consists of wave functions of the form

$$\sum_{l,m,\mu} f_{m,\mu}{}^l(|\vec{x}|) D_{m,-\mu}{}^l(\hat{x}) |\mu,k\rangle, \qquad \tfrac{1}{2}\vec{\tau}\cdot\hat{x}|\mu,k\rangle = \mu|\mu,k\rangle, \tag{4}$$

where $D_{m,-\mu}{}^l$ are the monopole harmonics,[7] k is a degeneracy index, and $|\mu,k\rangle$ is x dependent. These wave functions exhibit a coupling between the angular and the internal symmetry parts by the occurrence of the common μ. Application of a transformation $h(\hat{x})$ which does not commute with $\vec{\tau}\cdot\hat{x}$ will obviously produce terms of the form $D_{m,-\mu}{}^l|\rho,k\rangle$, $\rho\neq\mu$. These are not in the domain of $\mathcal{H}$. Further, they lead to infinite expectation value for the energy either when $m=\mu$ or when $m=-\mu$.

Locally, that is, in any contractible region of R^3 excluding the monopole, there is no obstruction in defining the action of H. Even such local actions lead to peculiar consequences. In particular an irreducible G or H multiplet may contain both bosons and fermions, implying that a generic H transformation does not commute with angular momentum. This is because the conserved angular momentum (ignoring intrinsic spin) is $-i(\vec{r}\times\nabla)+\vec{\tau}/2$. Thus, for instance for the $\underline{5}^*$ multiplet of SU(5), since $\vec{\tau}$ vanishes on the first, second and fifth members, each of these in combination with a monopole behaves as a boson, while the remaining two behave as fermions.[8] Such a problem does not arise for irreducible H multiplets when H is Abelian since all the irreducible representations of an Abelian group are one dimensional.

When an H singlet composed of H-nonsinglet constituents is scattered off a monopole with non-Abelian magnetic flux, in general the outgoing system will not be an H singlet. For example, in a color singlet formed out of the $d_{L,\alpha}{}^c$ taken from the $\underline{5}$ of SU(5), only $d_{L,3}{}^c$ interacts with such a monopole. Therefore, the system will no longer be a pure color singlet after scattering.[9] This condition cannot be altered by color-conserving final-state interactions such as QCD forces. It thus appears that the presence of a colored monopole implies the existence of other colored objects. An understanding of the confinement mechanism of such monopoles is for this reason of fundamental importance. We may note here that such a production of colored asymptotic states has a gauge-invariant meaning. For large $|x|$, at any point $\vec{x}$, the color group is defined to be that subgroup of G which leaves the Higgs field invariant; we can therefore unambiguously identify the color group and color multiplets at large $|x|$ from the asymptotic behavior of the Higgs field. The phenomenon is due to the fact that general color transformations (which are locally definable) do not leave the potentials W_i invariant even for large $|x|$. Thus roughly speaking, a colored monopole carries color and can act as a source or sink for color. The relationship of this phenomenon to the topological problems we have discussed previously is not, however, clear.

A more detailed paper amplifying the preceding remarks will be published elsewhere.

Since the submission of this paper, a paper with similar results has been published.[10]

This work has been supported by the U. S. Department of Energy under Contracts No. DE-AC02-76ER03533 and No. DE-AS05-76ER03992, by the Swedish Natural Research Council under Contract No. F310, and by Istituto Nazionale di Fisica Nucleare, Italy. One of us (A.P.B.) thanks S. M. Roy for discussions.

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[4]This result is independent of the chosen isomorphism (3).

[5]An exception happens if the analog of $\{\exp(i\varphi\tau_3)\}$ which defines a general monopole leaves χ invariant. But then this U(1) must be an invariant subgroup of H, H must locally have the form $H' \otimes \mathrm{U}(1)$, and the magnetic flux of the monopole must be Abelian.

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