The Work of the Fields Medallists: 1998 [1]

2. William T Gowers

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The subject Functional Analysis started around the beginning of this century, inspired by a desire to have a unified framework in which the two notions of *continuity* and *linearity* that arise in diverse contexts could be discussed abstractly. The basic objects of study in this subject are Banach spaces and the spaces of bounded (continuous) linear operators on them; the space $C[a, b]$ of continuous functions on an interval $[a, b]$ with the supremum norm, the L^p spaces arising in the theory of integration, the sequence spaces l_p, the Sobolev spaces arising in differential equations, are some of the well-known examples of Banach spaces. Thus there are many concrete examples of these spaces, enabling application of the theory to a variety of problems.

It is generally agreed that finite-dimensional spaces are well understood and thus the main interest lies in infinite-dimensional spaces. A Banach space is *separable* if it has a countable dense subset in it. From now on we will talk only of separable Banach spaces; the nonseparable Banach spaces are very unwieldy.

The simplest examples of infinite-dimensional Banach spaces are the sequence spaces l_p, $1 \le p < \infty$ consisting of sequences $x = (x_1, x_2, ...)$ for which $\sum_{j=1}^{\infty} |x_j|^p$ is finite; the pth root of the latter is taken as the norm of x. These spaces are separable. The space of all bounded sequences, equipped with the supremum norm, is called l_∞. It is not separable, but contains in it the space c_0 consisting of all convergent sequences, which is separable. The following was an open question for a long time: does every Banach space contain in it a subspace that is isomorphic to either c_0 or some l_p, $1 \le p < \infty$? It was answered in the negative by B Tsirelson in 1974.

It may be recalled that in the theory of finite-dimensional vector spaces bases play an important role. A *Schauder basis* (or a *topological basis*) for a Banach space X is a sequence $\{e_n\}$ in X such that every vector in X has a unique expansion $x = \sum_{n=1}^{\infty} a_n e_n$, where the infinite series is understood to converge in norm. Unlike in the finite-dimensional case, in general this notion depends on the order in which $\{e_n\}$ is enumerated. We say a Schauder basis $\{e_n\}$ is an *unconditional* basis if $\{e_{\pi(n)}\}$ is a Schauder basis for every permutation π of natural numbers.

It is easy to see that if a Banach space has a Schauder basis, then it is separable. There was a famous problem as to whether every separable Banach space has a Schauder basis. P Enflo showed in 1973 that the answer is no. It had been shown quite early by S Mazur that every (infinite-dimensional)

Banach space has an (infinite-dimensional) subspace with a Schauder basis. (The spaces $l_p, 1 \le p < \infty$ and c_0 do have Schauder bases.)

One of the major results proved by W T Gowers, and independently by B Maurey, in 1991 is that there exist Banach spaces that do not have any infinite-dimensional subspace with an unconditional basis.

In many contexts the interest lies more in operators on a Banach space than the space itself. Many of the everyday examples of Banach spaces do have lots of interesting operators defined on them. But it is not clear whether every Banach space has nontrivial operators acting on it. If the Banach space has a Schauder basis one can construct examples of operators by defining their action on the basis vectors. Shift operators that act by shifting the basis vectors to the left or the right have a very rich structure. Another interesting family of operators is the projections. In a Hilbert space every subspace has an orthogonal complement. So, there are lots

of orthogonal decompositions and lots of projections that have infinite rank and corank. In an arbitrary Banach space it is not necessary that any infinite-dimensional subspace must have a complementary subspace. Thus one is not able to construct nontrivial projections in an obvious way.

The construction of Gowers and Maurey was later modified to show that there exists a Banach space X in which every continuous projection has finite rank or corank, and further every subspace of X has the same property. This is equivalent to saying that *no* subspace Y of X can be written as a direct sum $W \oplus Z$ of two infinite-dimensional subspaces. A space with this property is called *hereditarily indecomposable*. In 1993 Gowers and Maurey showed that such a space cannot be isomorphic to *any* of its proper subspaces. This is in striking contrast to the fact that an infinite-dimensional Hilbert space is isomorphic to *each* of its infinite-dimensional subspaces (all

of them are isomorphic to l_2). A Banach space with this latter property is called *homogeneous*.

In 1996 Gowers proved a dichotomy theorem showing that every Banach space X contains either a subspace with an unconditional basis or a hereditarily indecomposable subspace. A corollary of this is that every homogeneous space must have an unconditional basis. Combined with another recent result of R Komorowsky and N Tomczak–Jaegermann this leads to another remarkable result: every homogeneous space is isomorphic to l_2.

Another natural question to which Gowers has found a surprising answer is the Schroeder–Bernstein problem for Banach spaces. If X and Y are two Banach spaces, and each is isomorphic to a subspace of the other, then must they be isomorphic? The answer to this question has long been known to be no. A stronger condition on X and Y would be that each is a *complemented* subspace of the other. (A subspace is complemented if there is a continuous projection onto it; we noted earlier that not every subspace has this property.) Gowers has shown that even under this condition, X and Y need not be isomorphic. Furthermore, he showed this by constructing a space Z that is isomorphic to $Z \oplus Z \oplus Z$ but not to $Z \oplus Z$.

All these arcane constructions are not easy to describe. In fact, the norms for these Banach spaces are not given by any explicit formula, they are defined by indirect inductive procedures. All this suggests a potential new development in Functional Analysis. The concept of a Banach space has encompassed many interesting concrete spaces mentioned at the beginning. However, it might be *too* general since it also admits such strange objects. It is being wondered now whether there is a new theory of spaces whose norms are easy to describe. These spaces may have a richer operator theory that general Banach spaces are unable to carry.

In his work Gowers has used techniques from many areas, specially from combinatorics whose methods and concerns are generally far away from those of Functional Analysis. For example, one of his proofs uses the idea of two-person games involving sequences of vectors and Ramsey Theory. Not just that, he has also made several important contributions to combinatorial analysis. We end this summary with an example of such a contribution.

A famous theorem of E Szemeredi (which solved an old problem of P Erdos) states that for every natural number k and positive real number δ there exists N such that every subset of $\{1, 2, \dots, N\}$ of size δN contains an arithmetic progression of length k. Gowers has found a new proof of this theorem based on Fourier analysis. This proof gives additional important information that the original proof, and some others that followed, could not. It leads to interesting bounds for N in terms of k and δ.

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