The Unexpected Appearance of Pi in Diverse Problems

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Rajendra Bhatia is Professor at the Indian Statistical Institute, New Delhi. In his work on linear operators *p* has appeared often, as for example in the theorem: If *A, B* are Hermitian matrices such that all eigenvalues of *A* are at distance at least one from those of *B*, then ||*X*|| £ *p*/2 || *AX* – *XB* || for every matrix *X*.

There is a famous essay titled The Unreasonable E®ec-tiveness of Mathematics in the Natural Sciences by the renowned physicist Eugene P Wigner. The essay opens with the paragraph:

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. \How can you know that?" was his query. \And what is this symbol here?" \Oh," said the statistician, \this is ¼" \What is that?" \The ratio of the circumference of the circle to its diameter." \Well, now you are pushing your joke too far," said the classmate,\surely the population has nothing to do with the circumference of the circle."

1 A part of a talk given at the Jesus and Mary College on 27th November, 2002 under the Delhi Mathematics Awareness Program sponsored by the National Board for Higher Mathematics.

Wigner then goes on to discuss the surprisingly powerful role mathematics plays in the study of nature. I have quoted this para for making a small point. The number ¼ the ratio of the circumference of the circle to its diameter, appears in many contexts that seem to have no connection with diameters, areas, or volumes. One such problem that I discuss here concerns properties of natural numbers.

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Pi, Riemann zeta function, probability distribution, Fourier expansion.

Every student of calculus learns the Wallis product formula

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots \qquad (1)$$

On the right hand side there is an infinite product and this is to be interpreted as

$$\lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{3}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}. \qquad (2)$$

Perhaps, Wallis's formula is the first occurrence of an infinite product in mathematics; it also connects π with natural numbers.

This formula attributed to John Wallis (1616-1703) is remarkable for several reasons. It is, perhaps, the first occurence of an infinite product in mathematics. And it connects π with natural numbers. The formula has a simple proof. Let

$$I_n = \int_0^{\pi/2} (\sin x)^n \, dx.$$

Integrate by parts to get the recurrence formula

$$I_n = \frac{n-1}{n} I_{n-2}.$$

The sequence I_n is a monotonically decreasing sequence of positive numbers. This and the recurrence formula show that

$$1 < \frac{I_n}{I_{n+1}} < 1 + \frac{1}{n}.$$

So I_n/I_{n+1} tends to 1 as $n \to \infty$. Note that $I_0 = \pi/2$ and $I_1 = 1$. The recurrence formula can be used to get

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}\frac{2}{\pi}.$$

Taking the limit as $n \to \infty$ we get (1).

Many infinite sums involving natural numbers lead to π. One that we need for our discussion is a famous formula due to Leonhard Euler (1707-1783)

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \qquad (3)$$

It is reasonable to expect that a randomly picked number is a multiple of *k* with probability 1/*k*.

A (natural) number is said to be square-free if in its prime factoring no factor occurs more than once. Thus $70 = 2 \times 5 \times 7$ is a square-free number while $12 = 2 \times 2 \times 3$ is not.

Many problems in number theory are questions about the distribution of various special kinds of numbers among all numbers. Thus we may ask:

What is the proportion of square-free numbers among all numbers?

Or

If a number is picked at random what is the probability that it is square-free?

Now, randomness is a tricky notion and this question needs more careful formulation. However, let us ignore that for the time being. It is reasonable to believe that if we pick a number at random it is as likely to be odd as it is even. This is because in the list

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$$

every alternate number is even. In the same way every third numer is a multiple of 3; every fourth number is a multiple of 4; and so on. Thus the probability that a randomly picked number is a multiple of k is $1/k$; and the probability that it is not a multiple of k is $1 - 1/k$.

Let $p_1, p_2, p_3, \ldots$ be the sequence of prime numbers. Let n be a randomly chosen number. For each prime p the probability that p_j^2 is not a factor of n is $1 - 1/p_j^2$. Given two primes p_j and p_k, what is the probability that neither p_j^2 nor p_k^2 is a factor of n? Again from probabilistic reasoning we know that the probability of the simultaneous occurence of two independent events is the product of their individual probabilities. (Thus the probability of getting two consecutive heads when a coin is tossed twice is $1/4$.) Whether n has a factor p_j^2 has no bearing on its having p_k^2 as a factor. Thus the probability that neither p_j^2 nor p_k^2 is a factor of n is $\left(1-1/p_j^2\right)(1-1/p_k^2)$. Extending this reasoning one sees that the probability of n being square free is the infinite product

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^2}\right). \qquad (4)$$

There is a connection between this product and the series in (3). It is convenient to introduce here a famous object called the Riemann zeta function. This is defined by the series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \qquad (5)$$

This series surely converges for all real numbers $s>1$. Let us restrict ourselves to these values of s, though the zeta function can be defined meaningfully for other complex numbers. The formula (3) can be written as

$$\zeta(2)=\frac{\pi^2}{6}. \qquad (6)$$

The zeta function and prime numbers come together in the following theorem of Euler.

Finding the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ was one of the major triumphs of Euler.

Theorem. For all $s>1$

$$\zeta(s)=\prod_{n=1}^{\infty}\frac{1}{1-p_n^{-s}}. \qquad (7)$$

Proof. Fix an N, and use the geometric series expansion of $\frac{1}{1-x}$ to get

$$\prod_{n=1}^{N}\frac{1}{1-p_n^{-s}}=\prod_{n=1}^{N}\sum_{m=0}^{\infty}p_n^{-ms}. \qquad (8)$$

The last expression is equal to

$$\sum_{j=1}^{\infty}\frac{1}{n_j^s},$$

The probability that a number picked at random in square free is $6/\pi^2$.

where $n_1, n_2, \ldots$ is an enumeration of those numbers that have $p_1, p_2, \ldots, p_N$ as their only prime factors. As $N \to \infty$, the sequence $\{n_j\}$ expands to include all natural numbers. This proves the theorem.

As a consequence the product (4) has the value $6/\pi^2$. This is the probability that a number picked at random is square-free.

This is one more situation where the number π has made an appearance quite unexpectedly. Our main point has been made; several interesting side-lines remain.

First note that our argument shows that if we pick a number n at random, then the probability that it has no prime factor with multiplicity k is $1/\zeta(k)$.

With a little thinking one can see that the probability that two numbers picked at random are coprime is $6/\pi^2$. (This problem is equivalent to the one we have been discussing.)

There is another interesting way of looking at this problem. Let $\mathbb{Z}^2$ be the collection of all points in the plane whose coordinates are integers. This is called the integer lattice. If the line segment joining the origin $(0,0)$ to a point (m,n) does not pass through any other lattice point we say that the point (m,n) can be seen from the origin. For example, the point $(1,-1)$ can be seen from the origin but the point $(2,-2)$ can not be seen. Among all lattice points what is the proportion of those that can be seen from the origin? The answer, again, is $6/\pi^2$. The proof of this is left to the reader.

The probability that two numbers picked at random are coprime is $6/\pi^2$.

The argument used in proving the Theorem above can be modified to give a proof of the fact that there are infinitely many prime numbers. The probability that a randomly picked number from the set $\{1, 2, \ldots, N\}$ is 1 goes to zero as N becomes large. So the product $\prod_p (1 - 1/p)$ where p varies over all primes is smaller than any positive number. This would not be possible if there were only a finitely many factors in the product.

The number π entered the picture via the formula (3). How does one prove it? Several proofs are known. The daring 'proof' first given by Euler goes as follows.

Let $\alpha_1, \alpha_2, \ldots$ be the roots of the polynomial equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m = 0$. Then

$$\sum \frac{1}{\alpha_i} = \frac{-a_1}{a_0}.$$

We can write

$$\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} + \cdots.$$

This is a 'polynomial of infinite degree', and the roots of $\cos\sqrt{x} = 0$ are

$$\frac{(2n+1)^2\pi^2}{4}, \quad n = 0, 1, 2, \ldots.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \qquad (9)$$

The formula (3) follows from this easily.

Surely this argument has flaws. They can all be removed! With the notions of uniform convergence and $\varepsilon - \delta$ arguments, we can prove formulas like

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right), \qquad (10)$$

from which the formulas (1) and (3) can be derived by simple manipulations. Finding the sum of the series (3) was one of the early major triumphs of Euler. He was aware that the argument we have described above

Neither Euler, nor anyone else in three centuries after him, has found much about the values of $\zeta(k)$ when k is an odd integer. In 1978 R Apéry showed that $\zeta(3)$ is an irrational number. Even this much is not known about $\zeta(5)$.

is open to several criticisms. So he gave another proof that goes as follows.

$$
\begin{aligned}
\frac{\pi^2}{8} &= \frac{(\arcsin 1)^2}{2} = \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx \\
&= \int_0^1 \frac{1}{\sqrt{1-x^2}}\left[x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,\frac{x^{2n+1}}{2n+1}\right] dx \\
&= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n(2n+1)}\,\frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.
\end{aligned}
$$

Following the ideas of his first proof Euler showed that $\zeta(2m)$ is π^{2m} multiplied by a rational number. Thus for example,

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}. \tag{11}$$

Neither Euler, nor anyone else in three centuries after him, has found much about the values of $\zeta(k)$ when k is an odd integer. In 1978 R Apéry showed that $\zeta(3)$ is an irrational number. Even this much is not known about $\zeta(5)$.

Another general method for finding sums like (3) and (11) goes via Fourier series. If f is a continuous function on $[-\pi, \pi]$ and $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ its Fourier expansion, then

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{12}$$

Very recently, Rivoal has shown that infinitely many Zeta values at odd natural numbers are irrational.

The method depends on recognising the summands of a particular series as coefficient of the Fourier series of a particular function f and then computing the integral in (12).

Having seen expression like (10) and (12) one is no longer surprised that $\zeta(2m)$ involves π in some way.

Finally, let us briefly discuss some issues related to 'picking a natural number at random'.

Two standard examples of completely random phenomena are tossing of a coin and throwing of a dice. In the first case we have two, and in the second case six, equally likely outcomes. The 'sample space' in the first case is the set $\{1, 2\}$ (representing the two outcomes head and tail) and in the second case it is the set $\{1, 2, \ldots, 6\}$. One can imagine an experiment with N equally likely outcomes $\{1, 2, \ldots, N\}$.

The uniform probability distribution on the set $X = \{1, 2, \ldots, N\}$ is the function that assigns to each subset E of X values according to the following rules

$$\mu(\{j\}) = \mu(\{k\}) \quad \text{for all } j, k; \tag{13}$$

$$\mu(E) = \sum_{j \in E} \mu(\{j\}); \tag{14}$$

$$\mu(X) = 1. \tag{15}$$

Note that these three conditions imply that $\mu(\{j\}) = 1/N$ for all j. This is a model for a random phenomenon (like in some games of chance) with N equally likely outcomes.

It is clear that if X is replaced by the set $\mathbb{N}$ of all natural numbers, then no function satisfying the three conditions (13)-(15) exists. So, if 'picking an element of $\mathbb{N}$ at random' means assigning each of its elements j an equal 'probability' we run into a problem. However, there is a way to get around this.

Let $X = \{1, 2, \ldots, N\}$ and let E be the set of even numbers in X. If N is even, then $\mu(E) = 1/2$. But if $N = 2m + 1$ is odd, then $\mu(E) = m/(2m + 1)$. This is less than $1/2$, but gets very close to $1/2$ for large N. In this sense a number picked at random is as likely to be even as odd.

In the same spirit we can prove the following.

For every $\varepsilon > 0$, there exists a number N, such that if μ is the uniform probability distribution on the set $X = \{1, 2, \ldots, N\}$ and E is the set of square-free numbers in X, then

$$\frac{6}{\pi^2} < \mu(E) < \frac{6}{\pi^2} + \varepsilon.$$

The reader may prove this using the following observations. We know that

$$\prod_{j=1}^{\infty}\left(1 - \frac{1}{p_j^2}\right) = \frac{6}{\pi^2}.$$

The factors in this product are smaller than 1. So, the sequence

$$\prod_{j=1}^{M}\left(1 - \frac{1}{p_j^2}\right), \quad M = 1, 2, \ldots$$

decreases to its limit. Choose an M such that

$$\frac{6}{\pi^2} < \prod_{j=1}^{M}\left(1 - \frac{1}{p_j^2}\right) < \frac{6}{\pi^2} + \varepsilon.$$

and let $N = \prod_{j=1}^{M} p_j^2$.

A (non-uniform) probability distribution on X is a function μ that satisfies the conditions (14)-(15) but not (necessarily) the condition (13). There is nothing that prevents the existence of such a distribution on $\mathbb{N}$. Any series with non negative terms and with sum 1 gives such a distribution. In particular if we set

$$\mu(\{j\}) = \frac{6}{\pi^2}\frac{1}{j^2}, \quad j = 1, 2, \ldots, \tag{16}$$

then μ is a probability distribution on $\mathbb{N}$. This assigns different probabilities to different elements of $\mathbb{N}$. The reader may like to interpret and prove the following statement.

The probability that two natural numbers picked at random have j as their greatest common divisor is $\mu(\{j\})$ as defined by (16).

Suggested Reading

[1] G H Hardy and E M Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1959. See Chapter VIII, and in particular Theorems 332 and 333. The latter theorem attributed to Gegenbauer (1885) says that if $Q(x)$ is the number of square-free numbers not exceeding x; then

$$Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Here $O(\sqrt{x})$ represents a function whose absolute value is bounded by $A\sqrt{x}$ for some constant A.

Use this formula, with a computer program for testing whether a number is square-free, to obtain the value of π up to the third decimal place.

[2] P J Davis and R Hersch, The Mathematical Experience, Birkhauser, 1981. We have borrowed our main argument from the discussion on page 366 here. This occurs in a chapter titled The Riemann Hypothesis where the authors present an argument showing that this most famous open problem in mathematics has an affirmative solution with probability one.

[3] M Kac, Enigmas of Chance, University of California Press, 1987. See Chapters 3,4, and in particular pages 89-91 of this beautiful autobiography for deeper connections between number theory and probability found by its author. See also his book Statistical Independence in Probability, Analysis and Number Theory, Mathematical Association of America, 1959.

[4] W Dunham, Journey Through Genius, Penguin Books, 1991. See Chapter 9 titled The Extraordinary Sums of Leonhard Euler for an entertaining history of the formula (3).

[5] R Bhatia, Fourier Series, Hindustan Book Agency, Second Edition, 2003. See Chapter 3 for several series and products that lead to π

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