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On the fields and equations of motion of point particles

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The investigation covers point particles possessing a charge, dipole and higher multipole moments interacting with fields of any spin satisfying the generalized wave equation (8). It is shown that the radiation field defined as the retarded minus the advanced field and all its derivatives is always finite at all points including those on the world line of the point particles. The symmetric field, defined as half the sum of the retarded and advanced fields, is shown to contain a part expressible as an integral along the world line from minus to plus infinity, which is continuous and finite everywhere. This integral vanishes if $\chi = 0$. The modified symmetric field is defined as the symmetric field minus this integral. The actual field is expressed as a sum of the modified symmetric field plus the modified mean field defined as half the sum of the ingoing and outgoing fields plus the integral just mentioned. It is proved that the part of the stress tensor of the field quadratic in the modified symmetric field plays no part in determining the equations of motion of the point particle. Being conserved by itself, it can always be subtracted away, thus defining a new stress tensor which is free from all the highest singularities in the usual stress tensor. The equations of motion of the particle are shown to depend only on the usual 'mixed terms' in the inflow with the modified mean field substituted for the ingoing field. The formulation for several particles is given.

The different fields associated with point particles are investigated and certain general properties established from which results about the general form of the equations of motion of the point particles can be deduced. The investigation covers generalized wave fields* of all spin, the usual scalar and vector meson or generalized Maxwell fields being only particular cases. Similarly, the field producing properties of the point particle can also be treated with great generality, and the particle may possess a charge, dipole or higher multipole moment or any combination of these.

The retarded field is defined by the boundary condition that it vanishes at all points on and before an infinitely extended space-like surface. Usually this surface is taken to be in the infinitely distant past. The advanced field is correspondingly defined by the boundary condition that it vanishes everywhere on and after an infinitely extended space-like surface which is usually taken to be in the infinitely distant future. Both these fields are unsymmetrical with respect to the past and the future by definition. They are both singular on the world line of the particle. In general theory it is more convenient to use two other fields derived from these, namely, the radiation field, defined as the retarded minus the advanced field, and the symmetric field, defined as half the sum of the retarded and advanced fields.

* The expression generalized wave field is used to cover any field the components of which satisfy the generalized wave equation (8) in free space, and covers fields of all integral spin. The expression meson field, which has been used in the previous literature, is inaccurate, since the meson having a spin of 0 or 1 unit is but a particular case. The fields with $\chi = 0$ are included in the above as limiting cases, but when it is necessary to distinguish these from the more general fields for which $\chi \neq 0$, we shall refer to them as specialized wave fields.

The radiation field changes its sign when the direction of the time axis is reversed, while the symmetric field is left unchanged.

The symmetric and radiation fields have important properties near the world line. The symmetric field is made up of a part consisting of terms depending on conditions at the retarded and advanced points on the world line only, and a part composed of two integrals along the world line from minus infinity to the retarded point and from the advanced point to infinity respectively. The latter vanish for a specialized wave field ($\chi = 0$). It is convenient to combine the latter into one integral from minus infinity to plus infinity, and to add a compensating integral from the retarded to the advanced point to the first part of the symmetric field. The integral from minus infinity to plus infinity is now a finite, continuous and differentiable function of the field point even on the world line of the particle, and satisfies the homogeneous generalized wave equation at all points of space. It is important for the theory to be developed here to introduce the *modified symmetric field* defined as the symmetric field minus the integral from minus to plus infinity just mentioned. The modified symmetric field contains all the singularities of the symmetric field, and its value at any point is independent of portions of the world line lying in the past or future light cones of the point. This is an important feature of the modified symmetric field which differentiates it from the symmetric field. Moreover, if from any point in space near the world line we drop a perpendicular to the world line, and call its length ϵ, then it will be shown that the modified symmetric field can be expanded in an ascending series containing only *odd* powers of ϵ, the highest singularity depending on the highest multipole possessed by the particle. This result has an important consequence in the form of the equations of motion of the point particle.

Dirac (1938) has already shown that the radiation field satisfying the Maxwell equations produced by a point charge is finite on the world line. It will be proved in this paper quite generally that the radiation field and all its derivatives are always finite on the world line for every type of point particle and every spin of the field. The radiation field and each of its derivatives can always be written as the sum of two parts, one containing terms depending on conditions at the retarded and advanced points only, and the other containing two integrals along the world line. The former can always be expressed near the world line as an ascending series in *even* powers of ϵ only, starting with a term independent of ϵ. For specialized wave fields ($\chi = 0$) the two integrals vanish and the whole radiation field then becomes expressible as a series in even powers of ϵ.

The actual field at a point can be expressed, following Dirac, as follows:

$$\begin{aligned}\text{actual field} &= \text{retarded field} + \text{ingoing field}\\ &= \text{advanced field} + \text{outgoing field}\\ &= \text{symmetric field} + \text{mean field},\end{aligned}$$

where the mean field is defined as half the sum of the ingoing and outgoing fields. In view of the simple property of the modified symmetric field mentioned above, which is not possessed by the symmetric field, it is important to split up the actual field as follows: actual field = modified symmetric field + modified mean field, where the modified mean field is defined as the mean field plus the integral from minus to plus infinity which was subtracted from the symmetric field. Since this integral is finite and continuous everywhere and satisfies the homogeneous generalized wave equation at all points, so also does the modified mean field. The stress tensor, which is a homogeneous quadratic expression in the field quantities, now falls into three parts, the first containing only the modified symmetric field, the second containing the modified symmetric and mean fields, and the third part only the modified mean field. It follows from the property of the symmetric field stated above that the first part of the energy tensor must be expressible near the world line as a series in even powers of ϵ only. It will be shown to follow immediately from this that the contribution of this part to the flow of energy and momentum into a thin tube of radius ϵ surrounding the world line must be a series containing odd powers of ϵ only. Now the theorem established in a previous paper (Bhabha & Harish-Chandra 1944) states that if the rate of inflow* be expressed in the vicinity of the world line in a series in powers of the radius of the tube then all the terms except the one independent of the radius of the tube must be identically perfect differentials. For brevity we shall refer to it as the inflow theorem. It follows that the part of the stress tensor containing only the symmetric field contributes only perfect differentials to the rate of the inflow of energy and momentum, which therefore have no effect on the equations of motion of the point particle. The same can be proved of the part containing only the modified mean field, which can in any case have no effect on the equations since it is non-singular. The only part of the energy tensor which determines the equation of motion of the particle is the mixed part. Since the mixed terms in the inflow are formally the same as those that one would have if radiation reaction were neglected, with the only difference that the modified mean field is written in place of the ingoing field, and can in general be derived from a Lagrangian, the process of finding the general equations for a point particle becomes very simple. One has merely to obtain the usual mixed terms in the inflow, and then substitute the mean field in place of the ingoing field. It will be shown by one of us (H.C.) in the paper which follows this that the form of the mixed terms in the inflow is completely determined by certain general considerations, and they can be written down in any given case without any calculation.

Each of the three parts of the stress tensor mentioned above are conserved everywhere except on the world line, and therefore it is possible to take as the new stress tensor for the field the original tensor minus the first part containing only the modified symmetric field. This removes all the worst singularities in the stress tensor.

* This word is used in the same sense as in the paper mentioned.

1. The retarded, advanced, radiation and symmetric fields

We use as far as possible the same notation as in the previous papers. The coordinates of a point in space time are denoted by x^μ where a Greek index takes on all values from 0 to 3, and the metric tensor is defined by

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1.$$

A point on the world line is denoted by $z^\mu(\tau)$, τ being the proper time on the world line measured from some point on it. A dot denotes differentiation with respect to τ. The velocity of the particle is $v^\mu \equiv \dot{z}^\mu$. The following symbols are used with the same meaning as in the previous papers:

$$u^\mu \equiv x^\mu - z^\mu(\tau),$$

$$\kappa = u_\mu v^\mu, \quad \kappa' = u_\mu \dot{v}^\mu, \quad \kappa'' = u_\mu \ddot{v}^\mu, \text{ etc.} \tag{1}$$

For every point x^μ near the world line a point called the 'contemporary' point with the proper time τ_c can be defined on it by the equation

$$(\kappa)_c = (x^\mu - z^\mu(\tau_c))\, v_\mu(\tau_c) = 0. \tag{2}$$

The suffix c attached to a symbol will be used to denote that it refers to the contemporary point. Write

$$l^\mu \equiv x^\mu - z^\mu(\tau_c), \quad \epsilon^2 = -l_\mu l^\mu, \tag{3}$$

where ϵ is real and positive. Then

$$\left(\frac{\partial u^2}{\partial \tau}\right)_c = -2(u^\mu v_\mu)_c = 0, \tag{4}$$

so that the distance from the point x^μ to a point on the world line is stationary at the contemporary point. ∂_μ will be used to denote $\partial/\partial x^\mu$. We note the following relations which will be required later:

$$\partial_\mu \tau_c = \left(\frac{v_\mu}{1-\kappa'}\right)_c,$$

$$\partial_\nu l^\mu = \delta^\mu_\nu - \left(\frac{v^\mu v_\nu}{1-\kappa'}\right)_c,$$

$$\partial_\nu \epsilon^2 = -2l_\mu \partial_\nu l^\mu = -2l_\nu. \tag{5}$$

The right-hand sides of all three equations are finite and continuous on the world line, and all higher derivatives of them also remain finite and continuous. The same is true of any positive *even* power of ϵ, which can be differentiated an unlimited number of times without the higher derivatives becoming singular as $\epsilon \to 0$. On the contrary, the differentiation of an *odd* power of ϵ a sufficient number of times can certainly lead to a term singular as $\epsilon \to 0$. For example,

$$\partial^\mu \partial_\nu \epsilon = -\delta^\mu_\nu \epsilon^{-1} - l^\mu l_\nu \epsilon^{-3} + \left(\frac{v^\mu v_\nu}{1-\kappa'}\right)_c \epsilon^{-1}.$$

The retarded point on the world line associated with the point x^μ is defined as the point for which

$$(u_\mu u^\mu)_r \equiv (x_\mu - z_\mu(\tau_r))(x^\mu - z^\mu(\tau_r)) = 0 \tag{6}$$

with $x^0 - z^0(\tau_r) > 0$. The suffix r will be used to denote that the quantities refer to the retarded point. The advanced point is determined by the solution of the equation (6) for which $x^0 - z^0(\tau_a) < 0$. The suffix a will be used to denote quantities referring to this point.

Let $S_{...}$, where the dots stand for any arbitrary number of tensor indices, be a tensor defined at all points of the world line and finite and continuous on it. It may be considered as a continuous function of τ. Let τ_A be a fixed point on the world line. Then the field function defined by

$$O^{\text{ret.}}_{...}(x^\mu) \equiv \left(\frac{S_{...}}{\kappa}\right)_r - \chi \int_{\tau_A}^{\tau_r} S_{...} \frac{J_1(\chi u)}{u} d\tau \tag{7}$$

for $\tau_r \geqslant \tau_A$ and identically zero for $\tau_r < \tau_A$ satisfies the generalized wave equation

$$(\partial_\mu \partial^\mu + \chi^2) U = 0 \tag{8}$$

at all points of space not on the world line. χ is a constant characteristic of the field and J_1 is the first order Bessel function. For later use we note that the nth order Bessel function is defined by

$$J_n(u) = \left(\frac{u}{2}\right)^n \sum_{s=0}^{\infty} (-1)^s \frac{1}{s!(n+s)!} \left(\frac{u}{2}\right)^{2s}. \tag{9}$$

The dots affixed to O stand for the same tensor indices as appear in $S_{...}$.

Take a fixed point $\tau_c > \tau_A$ on the world line and draw the two-dimensional surface around it generated by all points satisfying (2) and (3) with constant ϵ. This surface forms a sphere around the point τ_c in the rest system of the point, that is, in the Lorentz frame in which the velocity v^μ at the point has the components 1, 0, 0, 0. If we denote by $d\Omega$ an element of solid angle subtended by an element of the two-dimensional surface of the sphere at its centre at τ_c, then an element of the surface with its normal directed outwards and perpendicular to the world line is $l^\nu \epsilon d\Omega$. Given an arbitrary continuous function ϕ of position inside the sphere expansible in a Taylor series about the point τ_c, then it can be shown easily that

$$\underset{\epsilon \to 0}{\text{Lt}} \int \phi(\partial_\nu O^{\text{ret.}}_{...}) l^\nu \epsilon d\Omega = \underset{\epsilon \to 0}{\text{Lt}} \int \phi \left(\frac{\partial}{\partial \epsilon} O^{\text{ret.}}_{...}\right) \epsilon^2 d\Omega = -4\pi\phi_c(S_{...})_c. \tag{10}$$

The expression (7) is completely and uniquely defined by the requirement that it satisfies (a) the equation (8) at all points not on the world line, (b) the limiting condition (10) at the world line, and (c) the condition that it vanishes at all points on and prior to an infinite space-like surface passing through the point τ_A. We shall call it the fundamental retarded solution. Usually the condition (c) is imposed on a surface in the infinitely distant past, i.e. $\tau_A \to -\infty$, and we shall henceforth consider only this case.

All retarded solutions can be derived from (7) by inserting the appropriate tensors for $S_{...}$, by differentiating with respect to the co-ordinates x^μ a certain number of times and by contraction of tensor indices with those of the differentiations. The tensors $S_{...}$ may have to satisfy different symmetry conditions in their indices in different cases, but this does not concern us here. For example, if the field under consideration is the scalar field, then the retarded potential for a point charge is simply given by

$$U^{\text{ret.}} = g_1 O^{\text{ret.}}, \tag{11a}$$

with S a scalar function on the world line and g_1 a constant. The retarded potential for a dipole is

$$U^{\text{ret.}} = g_2 \partial^\mu O^{\text{ret.}}_\mu, \tag{11b}$$

where g_2 is a constant and we have to write S_μ for $S_{...}$ in (7). S_μ gives the direction of the dipole moment.

If the field under consideration is a vector meson field then the retarded potentials for a point charge are given by

$$U^{\text{ret.}}_\mu = g_1 O^{\text{ret.}}_\mu, \tag{11c}$$

where we now have to write v_μ for $S_{...}$. The corresponding retarded potentials for a dipole are

$$U^{\text{ret.}}_\mu = g_2 \partial^\nu O^{\text{ret.}}_{\mu\nu}, \tag{11d}$$

and $S_{\mu\nu}$ has to be written for $S_{...}$. Every other tensor field and higher multipole moment for the particle can be treated correspondingly. The general retarded potentials for a point particle can therefore be written in the form

$$U^{\text{ret.}}_{..} = D[O^{\text{ret.}}_{...}], \tag{12}$$

where the operator D denotes a sum of terms each one of which consists of a different number of differentiations of the fundamental solution (7) with appropriate tensors written for $S_{...}$ and contraction of indices. The dots written as suffixes to U denote a number of tensor indices and determine the spin of the field. The number of uncontracted indices in all the terms must be the same as in $U_{..}$.

The function which is uniquely and completely defined by the three conditions that it (*a*) satisfies the equation (8) at all points of space not on the world line, (*b*) fulfils the condition (10) at the world line, and (*c*) vanishes at all points on and after an infinite space-like surface cutting the world line at the point τ_B is given by

$$O^{\text{adv.}}_{...} \equiv -\left(\frac{S_{...}}{\kappa}\right)_a - \chi \int_{\tau_a}^{\tau_B} S_{...} \frac{J_1(\chi u)}{u} d\tau \tag{13}$$

for $\tau_B \geqslant \tau_a$ and zero for $\tau_B < \tau_a$. This is the fundamental advanced solution and all advanced potentials can be expressed in terms of it by

$$U^{\text{adv.}}_{..} = D[O^{\text{adv.}}_{...}], \tag{14}$$

where D is the same operator as in (12). The radiation field introduced by Dirac (1938) is defined by

$$U^{\text{rad.}}_{..} = U^{\text{ret.}}_{..} - U^{\text{adv.}}_{..} = D[O^{\text{ret.}}_{...} - O^{\text{adv.}}_{...}]. \tag{15}$$

We therefore investigate first the fundamental radiation solution defined by

$$O^{\text{rad.}}_{...} = O^{\text{ret.}}_{...} - O^{\text{adv.}}_{...} = \left(\frac{S_{...}}{\kappa}\right)_r + \left(\frac{S_{...}}{\kappa}\right)_a - \chi\int_{-\infty}^{\tau_r} S_{...}\frac{J_1(\chi u)}{u}\,d\tau + \chi\int_{\tau_a}^{\infty} S_{...}\frac{J_1(\chi u)}{u}\,d\tau. \tag{16a}$$

It is convenient to introduce the modified fundamental radiation solution $O'^{\text{rad.}}_{...}$ defined by

$$O'^{\text{rad.}}_{...} = O^{\text{rad.}}_{...} - \chi\int_{-\infty}^{\infty} S_{...}\frac{J_1(\chi u)}{u}\,d\tau = \left(\frac{S_{...}}{\kappa}\right)_r + \left(\frac{S_{...}}{\kappa}\right)_a - \chi\int_{-\infty}^{\tau_r} S_{...}\frac{J_1(\chi u)}{u}\,d\tau - \int_{-\infty}^{\tau_a} S_{...}\frac{J_1(\chi u)}{u}\,d\tau. \tag{16b}$$

For later use we introduce explicitly the radiation solution for the specialized wave equation ($\chi = 0$)

$$\bar{O}^{\text{rad.}}_{...} = \left(\frac{S_{...}}{\kappa}\right)_r + \left(\frac{S_{...}}{\kappa}\right)_a. \tag{16c}$$

Let the field point x^μ be kept fixed for the moment. The value of u^2 at a point τ on the world line can be expressed by a Taylor series in powers of $\Delta\tau \equiv \tau - \tau_c$ for sufficiently small values of $\Delta\tau$,

$$u^2 = (u^2)_c + \left(\frac{\partial u^2}{\partial\tau}\right)_c \Delta\tau + \frac{1}{2}\left(\frac{\partial^2 u^2}{\partial\tau^2}\right)_c (\Delta\tau)^2 + \dots. \tag{17a}$$

It can be seen at once that none of the coefficients $(\partial^n u^2/\partial\tau^n)_c$ of this series is singular. Using (1)–(4) we get in particular

$$\left.\begin{aligned} &(u^2)_c = -\epsilon^2, \quad \left(\frac{\partial u^2}{\partial\tau}\right)_c = -2\kappa_c = 0, \quad \left(\frac{\partial^2 u^2}{\partial\tau^2}\right)_c = 2(1-\kappa')_c, \\ &\left(\frac{\partial^3 u^2}{\partial\tau^3}\right)_c = -2\kappa''_c, \quad \left(\frac{\partial^4 u^2}{\partial\tau^4}\right)_c = -2(\dot{v}^2 + \kappa''')_c. \end{aligned}\right\} \tag{17b}$$

Now at the advanced and retarded points $u^2 \equiv 0$ by definition, and $\Delta\tau$ takes on one of the two values $\Delta\tau_a \equiv \tau_a - \tau_c$ or $\Delta\tau_r \equiv \tau_r - \tau_c$ respectively which satisfy the equation obtained by putting (17b) into (17a)

$$\epsilon^2 = (1-\kappa')_c(\Delta\tau)^2 - \tfrac{1}{3}\kappa''_c(\Delta\tau)^3 - \tfrac{1}{12}(\dot{v}^2 + \kappa''')_c(\Delta\tau)^4 + \dots. \tag{18a}$$

The essential feature of equation (18a) is that the coefficient of $\Delta\tau$ on the right is zero since the distance u from the point x^μ to the world line is stationary at τ_c, and the coefficient of $(\Delta\tau)^2$ is rational and tends to 1 as x^μ approaches the world line. Taking the root of both sides of $(18a)$ we get

$$\pm\frac{\epsilon}{\sqrt{(1-\kappa')}} = \Delta\tau - \frac{1}{6}\frac{\kappa''}{1-\kappa'}(\Delta\tau)^2 - \frac{1}{24}\left\{\frac{\dot{v}^2+\kappa'''}{1-\kappa'}+\frac{1}{3}\frac{\kappa''^2}{(1-\kappa')^2}\right\}(\Delta\tau)^3, \tag{18b}$$

where we have omitted the suffix c for brevity. This clearly shows that for the advanced point $\Delta\tau_a$ is obtained by taking the positive sign on the left, while $\Delta\tau_r$ is got by taking the negative sign. Reverting equation $(18b)$ by successive approximation we obtain

$$\begin{aligned}\Delta\tau_a &= \frac{\epsilon}{\sqrt{(1-\kappa')}} + \frac{1}{6}\frac{\kappa''\epsilon^2}{(1-\kappa')^2} + \frac{1}{24}\frac{\dot{v}^2\epsilon^3}{(1-\kappa')^{\frac{5}{2}}} + O(\epsilon^4)\\ &= \epsilon + \tfrac{1}{2}\epsilon\kappa' + \tfrac{3}{8}\epsilon\kappa'^2 + \tfrac{1}{6}\kappa''\epsilon^2 + \tfrac{1}{24}\dot{v}^2\epsilon^3 + O(\epsilon^4).\end{aligned} \tag{19}$$

We get $\Delta\tau_r$ by reversing the sign of ϵ in (19), as follows directly from $(18b)$. Hence $(\Delta\tau_r)^n + (\Delta\tau_a)^n$ is expressible as a series in even powers of ϵ only, while $(\Delta\tau_r)^n - (\Delta\tau_a)^n$ is expressible as a series in odd powers of ϵ.

Similarly, expressing the value of κ at the point τ by a Taylor series in powers of $\Delta\tau$ we get, using the last three of the equations $(17b)$,

$$\kappa = -(1-\kappa')_c\Delta\tau + \tfrac{1}{2}\kappa''_c(\Delta\tau)^2 + \tfrac{1}{6}(\dot{v}^2+\kappa''')(\Delta\tau)^3 + \ldots. \tag{20}$$

The essential feature of this series is again that the coefficient of $\Delta\tau$ is rational and tends to 1 as $\epsilon \to 0$. It follows that κ^{-n}, where n is an integer, can be expanded as a series in ascending powers of $\Delta\tau$ starting with $(\Delta\tau)^{-n}$, the coefficients of the series being rational and non-singular functions of l^μ and the particle variables at the contemporary point.

Any function of position on the world line expressible as a sum of terms containing only positive integral powers of l^μ and the particle variables at τ, and positive or negative integral powers of κ can therefore be expressed as a series in ascending powers of $\Delta\tau$ the coefficients of which are rational and non-singular functions of l^μ and the particle variables at τ_c. If the highest negative power of κ be κ^{-m} then the series commences with $(\Delta\tau)^{-m}$, thus

$$f(u^\mu,\tau) = \sum_{n=-m}^{\infty} f_n(l^\mu,\tau_c)(\Delta\tau)^n, \tag{21}$$

where f_n is a rational function of l^μ and the particle variables at τ_c. f_n cannot therefore contain an odd power of $\epsilon \equiv \sqrt{(-l_\mu l^\mu)}$, but may contain an even power of ϵ. Denoting by $(f)_r$ and $(f)_a$ the values of f at the retarded and advanced points respectively, we get

$$\begin{aligned}(f)_r + (f)_a &= \sum_{n=-m}^{\infty} f_n(l^\mu,\tau_c)\{(\Delta\tau_r)^n + (\Delta\tau_a)^n\}\\ &= \text{ascending series in even powers of } \epsilon,\end{aligned} \tag{22}$$

and

$$\begin{aligned}(f)_r - (f)_a &= \sum_{n=-m}^{\infty} f_n(l^\mu,\tau_c)\{(\Delta\tau_r)^n - (\Delta\tau_a)^n\}\\ &= \text{ascending series in odd powers of } \epsilon.\end{aligned} \tag{23}$$

All functions with which we have to deal in this paper satisfy the conditions laid down for f.

The result (23) can be applied at once to (16c) to show it must be expressible as a series in even powers of ϵ only. For the Maxwell field of a point charge this fact has already been noted by Dirac (1938). The series can have no negative powers of ϵ since the greatest singularity in either of the terms on the right of (16c) is of the order ϵ, and this cannot appear in the series since it is an odd power. Substituting the value of τ_a given by (19) into (20) we get

$$\kappa_a = -\epsilon\sqrt{(1-\kappa')} + \frac{1}{3}\frac{\kappa''\epsilon^2}{1-\kappa'} - \frac{1}{24}\frac{\dot{v}^2\epsilon^3}{(1-\kappa')^{\frac{3}{2}}} + O(\epsilon^4). \tag{24}$$

Similarly, the Taylor expansion of $S_{\ldots}$ gives in conjunction with (19)

$$(S_{\ldots})_a = (S_{\ldots})_c + (\dot{S}_{\ldots})_c\frac{\epsilon}{\sqrt{(1-\kappa'_c)}} + \tfrac{1}{2}(\ddot{S}_{\ldots})_c\frac{\epsilon^2}{1-\kappa'_c} + O(\epsilon^3). \tag{25}$$

Whence

$$\left(\frac{S_{\ldots}}{\kappa}\right)_a = -\frac{S_{\ldots}}{\epsilon},\ -\dot{S}_{\ldots} - \tfrac{1}{2}S_{\ldots}\frac{\kappa'}{\epsilon},\ -S_{\ldots}\left(\frac{3}{8}\frac{\kappa'^2}{\epsilon} + \tfrac{1}{3}\kappa'' - \tfrac{1}{24}\dot{v}^2\epsilon\right) - \dot{S}_{\ldots}\kappa' - \tfrac{1}{2}\ddot{S}_{\ldots}\epsilon,\ +O(\epsilon^2), \tag{26}$$

where we have separated terms of different orders in ϵ by a comma. Reversing the sign of ϵ to obtain $(S_{\ldots}/\kappa)_r$ we get

$$\left(\frac{S_{\ldots}}{\kappa}\right)_r + \left(\frac{S_{\ldots}}{\kappa}\right)_a = -2\dot{S}_{\ldots},\ \ -2S_{\ldots}\kappa'' - 2\dot{S}\kappa',\ \ +O(\epsilon^2). \tag{27}$$

The specialized radiation solution (16c) is therefore finite as $\epsilon \to 0$.

The two integrals on the right of (16a) remain finite as the point x^μ approaches the world line since the integrands always remain finite on account of the well-known property of the Bessel function

$$\operatorname*{Lt}_{u\to 0}\frac{J_n(u)}{u^n} = \frac{1}{2^n n!} \tag{28}$$

which follows from (9). Hence the fundamental radiation solution given by (16a) of the generalized wave equation (8) is always finite on the world line. The same is also obviously true of the modified radiation solution (16b).

Since the right-hand side of (27) only contains even powers of ϵ starting from zero with coefficients which are rational integral functions of l^μ and the particle variables, it follows from the equations (5) that all successive derivatives of this series with respect to x^μ can be series containing only positive even powers of ϵ none of the coefficients of which are singular. All successive derivatives of the left-hand side of (27) therefore remain finite and unambiguous as the field point approaches the world line.

Further, by a well-known property of the Bessel function which can be deduced immediately from (9), we have, for fixed τ,

$$\partial_\mu\frac{J_n(\chi u)}{u^n} = \frac{u_\mu}{u}\frac{\partial}{\partial u}\left(\frac{J_n(\chi u)}{u^n}\right) = -\chi u_\mu\frac{J_{n+1}(\chi u)}{u^{n+1}}. \tag{29}$$

Hence $$\partial_\nu \partial_\rho \dots \frac{J_1(\chi u)}{u} \tag{30}$$

is always finite even on the world line, whatever the number of differentiations. Writing $\partial_{\nu\rho\dots}$ for this expression for brevity and remembering that

$$\partial_\mu \tau_a = \left(\frac{u_\mu}{\kappa}\right)_a, \quad \partial_\mu \tau_r = \left(\frac{u_\mu}{\kappa}\right)_r, \tag{31}$$

we get

$$\partial_\mu\left\{-\int_{-\infty}^{\tau_r} S_{\dots}\,\partial_{\nu\rho\dots}\,d\tau + \int_{\tau_a}^{\infty} S_{\dots}\,\partial_{\nu\rho\dots}\,d\tau\right\}$$
$$= -\left(\frac{u_\mu}{\kappa} S_{\dots}\,\partial_{\nu\rho\dots}\right)_r - \left(\frac{u_\mu}{\kappa} S_{\dots}\,\partial_{\nu\rho\dots}\right)_a - \int_{-\infty}^{\tau} S_{\dots}\,\partial_{\mu\nu\rho\dots}\,d\tau + \int_{\tau_a}^{\infty} S_{\dots}\,\partial_{\mu\nu\rho\dots}\,d\tau. \tag{32}$$

Both of the first two terms on the right are finite on the world line, and being of the general form of the right-hand side of (22), all their successive derivatives are also finite. The two integrals on the right are also finite by (30), so that it can be deduced by induction that all derivatives of the two integrals on the right of (16a) are finite on the world line. We have therefore proved that $O^{\text{rad.}}_{\dots}$ and all its derivatives are finite on the world line, and hence the same must be true for $U^{\text{rad.}}_{\cdot\cdot}$ and all its derivatives. *The radiation field with all its derivatives is always finite on the world line.* The same is true of the modified radiation field.

From (15) it follows that we can write

$$U^{\text{rad.}}_{\cdot\cdot} = \bar{U}^{\text{rad.}}_{\cdot\cdot} - \chi\int_{-\infty}^{\tau_r} D\left[S_{\dots}\frac{J_1(\chi u)}{u}\right] d\tau + \chi\int_{\tau_a}^{\infty} D\left[S_{\dots}\frac{J_1(\chi u)}{u}\right] d\tau, \tag{33}$$

where $\bar{U}^{\text{rad.}}_{\cdot\cdot}$ contains only the sum of terms taken at the retarded and the advanced points and is expressible as a series in even powers of ϵ. It should be noted that $\bar{U}^{\text{rad.}}_{\cdot\cdot} \neq D[\bar{O}^{\text{rad.}}_{\dots}]$ but contains additional terms like the first two on the right of (32).

Now consider the fundamental symmetric solution $O^{\text{sym.}}_{\dots}$ defined by

$$O^{\text{sym.}}_{\dots} \equiv \tfrac{1}{2}(O^{\text{ret.}}_{\dots} + O^{\text{adv.}}_{\dots})$$
$$= \frac{1}{2}\left(\frac{S_{\dots}}{\kappa}\right)_r - \frac{1}{2}\left(\frac{S_{\dots}}{\kappa}\right)_a - \frac{\chi}{2}\int_{-\infty}^{\tau} S_{\dots}\frac{J_1(\chi u)}{u}\,d\tau - \frac{\chi}{2}\int_{\tau_a}^{\infty} S_{\dots}\frac{J_1(\chi u)}{u}\,d\tau. \tag{34}$$

Writing $$O'^{\text{sym.}}_{\dots} \equiv \frac{1}{2}\left(\frac{S_{\dots}}{\kappa}\right)_r - \frac{1}{2}\left(\frac{S_{\dots}}{\kappa}\right)_a + \frac{\chi}{2}\int_{\tau_r}^{\tau_a} S_{\dots}\frac{J_1(\chi u)}{u}\,d\tau, \tag{35}$$

$$O^{\text{sym.}}_{\dots} = O'^{\text{sym.}}_{\dots} - \frac{\chi}{2}\int_{-\infty}^{\infty} S_{\dots}\frac{J_1(\chi u)}{u}\,d\tau. \tag{36}$$

Although u^2 is negative and u purely imaginary for points on the world line between τ_r and τ_a, since $J_n(\chi u)/u^n$ is a series in ascending powers of u^2 as shown by (9), the integrand of all the integrals is real at every point of the world line and varies continuously along it.

The first two terms on the right of (35) are of the general form (23) and must be expressible as a series in odd powers of ϵ with rational integral coefficients. All derivatives of this part must have the same form, though they may be singular to a higher degree.

Similarly the integral in (35) can be expressed as a series in odd powers of ϵ. For the value of the integrand at the point τ can be expanded in a Taylor series. Denoting the integrand by $i(\tau)$ for brevity, and writing $\Delta\tau = \tau - \tau_c$ as before,

$$i(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\partial^n i}{\partial \tau^n}\right)_c (\Delta\tau)^n, \tag{37}$$

(9) shows that i is an ascending series in positive powers of u^2. All derivatives of it with respect to τ remain series in positive powers of u^2, and hence every coefficient $(\partial^n i/\partial\tau^n)_c$ is a series in positive powers of ϵ^2. On integration we get

$$\int_{\tau_r}^{\tau_a} i(\tau)\,d\tau = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{\partial^n i}{\partial\tau^n}\right)_c \{(\Delta\tau_a)^{n+1} - (\Delta\tau_r)^{n+1}\}.$$

As proved earlier, each term in curly brackets is a series in odd powers of ϵ only. The series on the right of (37) must therefore be expressible as an ascending series in odd powers of ϵ the coefficients of which are non-singular and integral functions of the variables at the contemporary point. Hence, in the neighbourhood of the world line $O'^{\text{sym.}}_{\dots}$ can be expressed as an ascending series in *odd* powers of ϵ. (For the Maxwell field of a point charge this fact has also been noted by Dirac (1938).) It is singular on the world line. It follows by using (5) that $O'^{\text{sym.}}_{\dots}$ *and all its successive derivatives are also and always expressible as ascending series in odd powers of* ϵ, each having a higher singularity than the previous one.

The integrand of the integral in (36) is continuous along the world line. Hence it follows from (30) that the integral and all its derivatives are finite even on the world line. Further

$$(\partial_\mu\partial^\mu + \chi^2)\int_{-\infty}^{\infty} S_{\dots}\frac{J_1(\chi u)}{u}\,d\tau = \int_{-\infty}^{\infty} S_{\dots}(\partial_\mu\partial^\mu + \chi^2)\frac{J_1(\chi u)}{u}\,d\tau$$
$$= \int_{-\infty}^{\infty} S_{\dots}\left(\frac{\partial^2}{\partial u^2} + 3\frac{\partial}{u\,\partial u} + \chi^2\right)\frac{J_1(\chi u)}{u}\,d\tau = 0, \tag{38}$$

since the Bessel function satisfies just the equation obtained by putting the integrand equal to zero. Hence it follows from (36) that $O'^{\text{sym.}}_{\dots}$ satisfies equation (8) at all points of space not on the world line, and condition (10) on it. All its derivatives therefore also satisfy equation (8) at any point not on the world line.

Following Dirac we write for the actual potential $U^{\text{act.}}_{\dots}$ at a point

$$U^{\text{act.}}_{\dots} = U^{\text{ret.}}_{\dots} + U^{\text{in.}}_{\dots} = U^{\text{adv.}}_{\dots} + U^{\text{out.}}_{\dots}. \tag{39}$$

Introducing the mean field defined by

$$U^{\text{mean}}_{\dots} \equiv \tfrac{1}{2}(U^{\text{in.}}_{\dots} + U^{\text{out.}}_{\dots}) \tag{40}$$

we can also write it
$$U^{\text{act.}}_{\dots} = U^{\text{sym.}}_{\dots} + U^{\text{mean}}_{\dots} = U'^{\text{sym.}}_{\dots} + U'^{\text{mean}}_{\dots}, \tag{41}$$

where
$$U'^{\text{mean}}_{\dots} = U^{\text{mean}}_{\dots} - \frac{\chi}{2}\int_{-\infty}^{\infty} D\left[S_{\dots}\frac{J_1(\chi u)}{u}\right]d\tau, \tag{42}$$

the operator D being the same as in (12), (14) and (15). It is clear from (38) that $U'^{\text{mean}}_{\dots}$ satisfies the equation (8) at all points of space and is continuous and finite everywhere. We call it the modified mean field.

2. The equations of motion

Certain general results about the form of the equations of motion of the point particles can be deduced at once from the properties of the radiation and symmetric fields established above. The equations of motion of the point particle are obtained by using the stress tensor of the field to calculate the flow of energy and momentum into a thin tube surrounding the world line. The inflow along an infinitesimal length of the tube must then be put equal to a perfect differential for conservation of energy and momentum, and this provides the equation of motion of the point particle.

Let the tube be defined by $l_\mu l^\mu = -\epsilon^2$ with constant ϵ. The element of three dimensional surface of this tube with its normal directed outwards is given by

$$l^\nu \epsilon(1-\kappa'_c)\,d\Omega\,d\tau, \tag{43}$$

$d\Omega$ being the element of solid angle about the contemporary point in its rest system introduced in the previous section. The flow of energy and momentum into a portion of the tube of length $d\tau$ is

$$\left\{\int T_{\mu\nu} l^\nu \epsilon(1-\kappa'_c)\,d\Omega\right\}d\tau \equiv T_\mu\,d\tau. \tag{44}$$

The equations of translational motion are then given by putting

$$T_\mu = \dot{A}_\mu, \tag{45}$$

where A_μ is some function of the variables describing the state of the particle and the field at the point τ and their derivatives.

The stress tensor $T_{\mu\nu}$ is a homogeneous quadratic expression in the potentials and the field strengths which are linear derivatives of them. We denote by $T_{\mu\nu}(U^A_{\dots}, U^B_{\dots})$ the more general expression that is obtained from it containing two independent fields $U^A_{\dots}$ and $U^B_{\dots}$ by replacing one $U_{\dots}$ in each term by $U^A_{\dots}$ and the other by $U^B_{\dots}$ and then making the expression symmetric in $U^A_{\dots}$ and $U^B_{\dots}$. The resulting expression is thus linear in $U^A_{\dots}$ and $U^B_{\dots}$ separately and quadratic in the two. The original stress is just $T_{\mu\nu}(U_{\dots}, U_{\dots})$ in this notation.

For example, the stress tensor for the generalized Maxwell field is

$$4\pi T_{\mu\nu} = G_{\mu\rho}G^\rho_\nu + \tfrac{1}{4}g_{\mu\nu}G_{\rho\sigma}G^{\rho\sigma} + \chi^2(U_\mu U_\nu - \tfrac{1}{2}g_{\mu\nu}U_\rho U^\rho),$$

the field strengths $G_{\mu\nu}$ being derivatives of the potentials U_μ. Then,

$$4\pi T_{\mu\nu}(U^A_{\dots}, U^B_{\dots}) \equiv \tfrac{1}{2}G^A_{\mu\rho}G^{B\rho}_{\nu} + \tfrac{1}{2}G^B_{\mu\rho}G^{A\rho}_{\nu} + \tfrac{1}{4}g_{\mu\nu}G^A_{\rho\sigma}G^{B\rho\sigma} + \tfrac{1}{2}\chi^2(U^A_\mu U^B_\nu + U^B_\mu U^A_\nu - g_{\mu\nu}U^A_\rho U^{B\rho}),$$

$G^A_{\mu\nu}$ being the field strengths corresponding to the potentials U^A_μ.

Returning to the general tensor for any field, we have, using (41),

$$\begin{aligned} T_{\mu\nu} \equiv T_{\mu\nu}(U^{\text{act.}}_{\dots}, U^{\text{act.}}_{\dots}) &= T_{\mu\nu}(U'^{\text{sym.}}_{\dots} + U'^{\text{mean}}_{\dots}, U'^{\text{sym.}}_{\dots} + U'^{\text{mean}}_{\dots}) \\ &= T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots}) + 2T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots}) + T_{\mu\nu}(U'^{\text{mean}}_{\dots}, U'^{\text{mean}}_{\dots}). \end{aligned} \tag{46}$$

Corresponding to this splitting up we get for the inflow

$$T_\mu = T'^{\text{sym.}}_\mu + T'^{\text{mix.}}_\mu + T'^{\text{mean}}_\mu, \tag{47}$$

where

$$T'^{\text{sym.}}_\mu \equiv \int T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})\, l^\nu \epsilon(1-\kappa'_c)\, d\Omega, \tag{48}$$

$$T'^{\text{mix.}}_\mu = 2\int T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})\, l^\nu \epsilon(1-\kappa'_c)\, d\Omega \tag{49}$$

and

$$T'^{\text{mean}}_\mu = \int T_{\mu\nu}(U'^{\text{mean}}_{\dots}, U'^{\text{mean}}_{\dots})\, l^\nu \epsilon(1-\kappa'_c)\, d\Omega. \tag{50}$$

Since $U'^{\text{sym.}}$ and its derivatives can be expressed near the world line as series in odd powers of ϵ only, it follows that $T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ is a series in even powers of ϵ only. As far as the integration with respect to $d\Omega$ is concerned, ϵ is a constant, and the only quantities which vary are l^ν and terms containing it like $\kappa'_c \equiv l^\nu \dot{v}_\nu$, $\kappa''_c = l^\nu \ddot{v}_\nu$, etc. The integral of a product of say s factors $l^\nu l^\rho \dots$ vanishes from symmetry if s is odd, and gives ϵ^s multiplied by a constant if s is even. Hence, as a result of the factor ϵ which appears explicitly in (43) and (48), $T'^{\text{sym.}}_\mu$ is a series in *odd* powers of ϵ only, the coefficients of which are just functions of the particle variables on the world line at the point τ_c. The inflow theorem (Bhabha & Harish-Chandra 1944) states that if the rate of inflow be expanded in a series in powers of ϵ, the coefficients of all the terms except the one independent of ϵ must be perfect differentials. $T'^{\text{sym.}}_\mu$ being a series in odd powers of ϵ has no such term independent of ϵ, and hence must be identically a perfect differential. It has also been proved that the inflow through two tubes of different shape can only differ by a perfect differential. $T'^{\text{sym.}}_\mu$ therefore plays no part in the equations of motion since it can always be eliminated by adding an identical term in A_μ. *Thus the part of the stress tensor containing only the modified symmetric field contributes nothing to the equations of motion.*

Since $U'^{\text{mean}}_{\dots}$ is a continuous and non-singular function at all points, T'^{mean}_μ must tend to zero as $\epsilon \to 0$ and hence it cannot contain any term independent of ϵ. Obviously, it can play no part in the equations of motion. It also follows from the inflow theorem that T'^{mean}_μ is identically a perfect differential. This can be easily verified by direct calculation.

The only significant contribution to the right-hand side of (45) comes from the terms in $T'^{\text{mix.}}_{\mu}$ independent of ϵ, which we denote by $[T'^{\text{mix.}}_{\mu}]_0$. The equation of motion must therefore be of the form

$$\dot{A}_{\mu} = [T'^{\text{mix.}}_{\mu}]_0. \tag{51}$$

We note that the right-hand side of (51) can only contain $U'^{\text{mean}}_{\dots}$ and its derivatives multiplied by functions of the particle variables at the point τ, and is linear in them. Another proof of this result which does not depend on an expansion of the symmetric field in powers of ϵ is given below.

For a point charge and a dipole the highest singularity in $U'^{\text{sym.}}_{\dots}$ is of the order ϵ^{-2}. Since the integration over $d\Omega$ of a function multiplied by the factor (43) gives at least a factor ϵ^3, no term containing $U'^{\text{mean}}_{\dots}$ can appear in the equations. The equations only contain the field strength $G'^{\text{mean}}_{\dots}$ and its derivatives. This result is quite general and true also for higher multipoles, as is proved by one of us (H.C.) in the paper which follows this.

Since $U'^{\text{mean}}_{\dots}$ contains an integral from $-\infty$ to ∞ it might appear as if the motion of the particle at τ_c depended not only on the motion of the particle in the past, but also in the future. Using (42), (40), (39) and (33) we see, however, that on the world line

$$\begin{aligned}(U'^{\text{mean}}_{\dots})_c &= \tfrac{1}{2}(U^{\text{in.}}_{\dots} + U^{\text{out.}}_{\dots})_c - \frac{\chi}{2}\int_{-\infty}^{\infty} D\left[S_{\dots\dots}\frac{J_1(\chi u)}{u}\right] d\tau \\ &= (U^{\text{in.}}_{\dots})_c + \tfrac{1}{2}(U^{\text{ret.}}_{\dots} - U^{\text{adv.}}_{\dots})_c - \frac{\chi}{2}\int_{-\infty}^{\infty} D\left[S_{\dots\dots}\frac{J_1(\chi u)}{u}\right] d\tau \\ &= (U^{\text{in.}}_{\dots})_c + \tfrac{1}{2}(\overline{U}^{\text{rad.}}_{\dots})_c - \chi\int_{-\infty}^{\tau_c} D\left[S_{\dots\dots}\frac{J_1(\chi u)}{u}\right] d\tau \\ &= (U^{\text{in.}}_{\dots})_c + \tfrac{1}{2}(U'^{\text{rad.}}_{\dots})_c,\end{aligned} \tag{52}$$

which clearly shows that the motion only depends on the actual motion of the particle at τ_c and in the past. It is reasonable to interpret $\frac{1}{2}(\overline{U}^{\text{rad.}}_{\dots})_c$ as the field giving the effects of radiation reaction, and the last integral as an addition to the ingoing field due to the particle's own motion in the past. It has already been mentioned that $\overline{U}^{\text{rad.}}_{\dots}$ is not just $D[\overline{O}^{\text{rad.}}_{\dots}]$ but contains in addition the terms which result from the differentiation of τ_r and τ_a in (16a) with respect to x^{μ}, as, for example, the first two terms on the right of (32).

We also note that

$$\begin{aligned}U'^{\text{sym.}}_{\dots} &= U^{\text{sym.}}_{\dots} + \frac{\chi}{2}\int_{-\infty}^{\infty} D\left[S_{\dots\dots}\frac{J_1(\chi u)}{u}\right] d\tau \\ &= U^{\text{ret.}}_{\dots} - \tfrac{1}{2}U^{\text{rad.}}_{\dots} + \frac{\chi}{2}\int_{-\infty}^{\infty} D\left[S_{\dots\dots}\frac{J_1(\chi u)}{u}\right] d\tau \\ &= U^{\text{ret.}}_{\dots} - \tfrac{1}{2}U'^{\text{rad.}}_{\dots}.\end{aligned} \tag{53}$$

The last term is continuous everywhere. Hence

$$[T'^{\text{mix.}}]_0 = \left[2\int T_{\mu\nu}(U^{\text{ret.}}_{\dots}, U'^{\text{mean}}_{\dots})\, l^{\nu}\epsilon(1-\kappa'_c)\, d\Omega\right]_0. \tag{54}$$

This gives precisely the mixed terms calculated in the previous papers with $U'^{\text{mean}}_{\dots}$ substituted in place of the $U^{\text{in.}}_{\dots}$ which appeared in them. The process of finding the actual equations of motion therefore becomes very simple. We merely calculate the term independent of ϵ in the inflow using the mixed part of the stress tensor, and then substitute $U'^{\text{mean}}_{\dots}$ given by (52), in place of $U^{\text{in.}}_{\dots}$. The only part of the calculation which is at all cumbersome is the evaluation of $\overline{U}^{\text{rad.}}_{\dots}$ required in (52). A convenient way of calculating this part is given by Harish-Chandra in the paper which follows this.

The general results established above hold also for the rotational equations. To calculate these use has to be made of the angular momentum tensor $M_{\lambda\mu\nu}$ instead of the stress tensor. Since

$$M_{\lambda\mu\nu} = x_\lambda T_{\mu\nu} - x_\mu T_{\lambda\nu} = (l_\lambda T_{\mu\nu} - l_\mu T_{\lambda\nu}) + (z_\lambda T_{\mu\nu} - z_\mu T_{\lambda\nu}), \tag{55}$$

we see at once that precisely the same arguments hold as before and the same result follows. Using (51) we have

$$\left[2\int M_{\lambda\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})\, l^\nu \epsilon(1-\kappa_c')\, d\Omega\right]_0 = [M'^{\text{mix.}}_{\lambda\mu}]_0 + \{z_\lambda [T'^{\text{mix.}}_{\mu}]_0 - z_\mu [T'^{\text{mix.}}_{\lambda}]_0\}$$

$$= [M'^{\text{mix.}}_{\lambda\mu}]_0 + \frac{d}{d\tau}(z_\lambda A_\mu - z_\mu A_\lambda) - (v_\lambda A_\mu - v_\mu A_\lambda), \tag{56}$$

where

$$[M'^{\text{mix.}}_{\lambda\mu}]_0 = \left[2\int \{l_\lambda T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots}) - l_\mu T_{\lambda\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})\}\, l^\nu \epsilon(1-\kappa_c')\, d\Omega. \tag{57}$$

The rotational equation must therefore have the form

$$\dot{B}_{\lambda\mu} + v_\lambda A_\mu - v_\mu A_\lambda = [M^{\text{mix.}}_{\lambda\mu}]_0. \tag{58}$$

We give an alternative proof of (51) which depends upon the symmetry existing between the retarded and the advanced points. This method of proof avoids the necessity of expanding the various quantities involved in series and therefore has the advantage of compactness. The retarded and advanced points are defined by (6) and indeed τ_r is that solution of (6) for which $(U_0)_r > 0$ and τ_a that solution for which $(u_0)_a < 0$. It is clear therefore that in every equation which does not explicitly utilize the condition $(u_0)_r > 0$ or $(u_0)_a < 0$ we can always replace r by a and vice versa, because whatever holds for τ_r would also hold for τ_a and conversely. In particular this symmetry is exhibited in differentiation:

$$\partial_\mu \tau_r = \left(\frac{u_\mu}{\kappa}\right)_r, \quad \partial_\mu \tau_a = \left(\frac{u_\mu}{\kappa}\right)_a,$$

$$\partial_\mu (u^\nu)_r = \delta^\nu_\mu - \left(\frac{u_\mu v^\nu}{\kappa}\right)_r,$$

$$\partial_\mu (u^\nu)_a = \delta^\nu_\mu - \left(\frac{u_\mu v^\nu}{\kappa}\right)_a. \tag{59}$$

Now if we interchange a and r in (35) $O'^{\text{sym.}}_{\dots}$ changes sign. $O'^{\text{sym.}}_{\dots}$ is therefore antisymmetric in the indices a and r. From the symmetry shown by (59) in a and r it follows that

$$U'^{\text{sym.}}_{\dots} = D[O'^{\text{sym.}}_{\dots}]$$

is also antisymmetric in a and r. On the other hand, the modified radiation field (16b) remains unaltered by an interchange of a and r. Therefore $O'^{\text{rad.}}_{\dots}$ and consequently $U'^{\text{rad.}}_{\dots}$ is symmetric in a and r. The same holds for $U'^{\text{mean}}_{\dots}$ as is obvious from (42).

We calculate the inflow into two tubes surrounding the world line, called the retarded and the advanced tubes respectively, defined by the equations

$$\left.\begin{aligned} \kappa_r &= \quad \epsilon \equiv \epsilon_r \quad \text{(retarded tube)}, \\ \kappa_a &= -\epsilon \equiv \epsilon_a \quad \text{(advanced tube)}, \end{aligned}\right\} \tag{60}$$

where ϵ is a positive constant. The intersection of these two tubes with the future and the past light cones starting from $z^\mu(\tau)$ respectively are called similarly the 'retarded' and the 'advanced' spheres of radius ϵ at τ. The three dimensional surface elements of the two tubes with the normals outwards can be written as

$$\left.\begin{aligned} dS^\nu_r &= \partial^\nu \kappa_r d\Omega_r d\tau \quad \text{(for retarded tube)}, \\ dS^\nu_a &= -\partial^\nu \kappa_a d\Omega_a d\tau \quad \text{(for advanced tube)}, \end{aligned}\right\} \tag{61}$$

where $d\Omega_r$ is an element of surface of the two dimensional retarded sphere of radius ϵ at τ and similarly $d\Omega_a$ is an element of the surface of the corresponding advanced sphere. The difference in sign in the two equations in (61) is important and arises from a similar difference in (60), where increasing ϵ corresponds to *increasing* κ_r and *decreasing* κ_a.

For brevity put
$$\int T^{\mu\nu}(U^A_{\dots}, U^B_{\dots})\,\partial_\nu \kappa_r d\Omega_r \equiv T^\mu_r(U^A_{\dots}, U^B_{\dots}), \tag{62a}$$

$$\int T^{\mu\nu}(U^A_{\dots}, U^B_{\dots})\,\partial_\nu \kappa_a d\Omega_a \equiv T^\mu_a(U^A_{\dots}, U^B_{\dots}). \tag{62b}$$

Let I^μ_r denote the rate of inflow calculated on the retarded tube and I^μ_a that on the advanced tube. For the equations of motion the significant part of the inflow is the part of these independent of ϵ, which we denote by $[I^\mu_r]_0$ and $[I^\mu_a]_0$ respectively. Then by (46) and (62) we get

$$[I^\mu_r]_0 = [T^\mu_r(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]_0 + [2T^\mu_r(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})]_0 \tag{63}$$

and
$$[I^\mu_a]_0 = -[T^\mu_a(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]_0 - 2T^\mu_a(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})_0. \tag{64}$$

For calculating $T^\mu_r(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ and $T^\mu_a(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ we have to put $U^A_{\dots} = U^B_{\dots} = U'^{\text{sym.}}_{\dots}$ in (62). Since $U'^{\text{sym.}}_{\dots}$ is antisymmetric in a and r it follows that $T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ is symmetric in a and r. Thus it is easy to see from (62) that if we interchange* a and r in the calculation of $T^\mu_r(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ we get $T^\mu_a(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$.

* This interchange also implies interchange of ϵ_r and ϵ_a.

But $[T^{\mu}_{r}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]_0$ is independent of ϵ and is a function only of the particle variables at τ so that the interchange of a and r does not affect it. Therefore we must have

$$[T^{\mu}_{a}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]_0 = [T^{\mu}_{r}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]_0. \tag{65}$$

On the other hand, since $U'^{\text{mean}}_{\dots}$ is symmetric in a and r while $U'^{\text{sym.}}_{\dots}$ is antisymmetric, it follows that $T^{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})$ must be antisymmetric in a and r. Therefore by an analogous reasoning

$$[T^{\mu}_{a}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots})]_0 = -[T^{\mu}_{r}(U'^{\text{sym.}}_{\dots\dots}, U'^{\text{mean}}_{\dots})]_0. \tag{66}$$

Therefore on subtracting (64) from (63) we get

$$[I^{\mu}_{r}]_0 - [I^{\mu}_{a}]_0 = 2[T^{\mu}_{r}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})]. \tag{67}$$

The left side being the difference between the rates of inflow on two tubes is a perfect differential and therefore so also is the right side. From this (51) follows immediately.

Since $U'^{\text{sym.}}_{\dots}$ satisfies the wave equation (8) at all points not on the world line, it follows that

$$\partial^{\nu} T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots}) = 0,$$

except on the world line. We can therefore take for the modified stress tensor of the field,

$$\begin{aligned} T'_{\mu\nu} &= T_{\mu\nu} - T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots}) \\ &= 2T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{mean}}_{\dots}) + T_{\mu\nu}(U'^{\text{mean}}_{\dots}, U'^{\text{mean}}_{\dots}). \end{aligned} \tag{68}$$

This tensor satisfies the conservation equation except on the world line and leads to the same equations of motion as the original tensor. It has the advantage that the worst singularities of $T_{\mu\nu}$ which are contained in the part $T_{\mu\nu}(U'^{\text{sym.}}_{\dots}, U'^{\text{sym.}}_{\dots})$ are absent from it. It does not contain in particular, the infinite static field energy of the point particle. The new angular momentum tensor is then defined by (55) with $T'_{\mu\nu}$ in place of $T_{\mu\nu}$.

The theory given above can be generalized immediately to the case of several point particles. The modified symmetric field $U'^{\text{sym.}(s)}_{\dots}$ of the sth particle is defined precisely as in (35). The actual field can then be split into

$$U^{\text{act.}}_{\dots} = U'^{\text{mean}}_{\dots} + \sum_{s=1}^{n} U'^{\text{sym.}(s)}_{\dots}, \tag{69}$$

where
$$\begin{aligned} U'^{\text{mean}}_{\dots} &= \tfrac{1}{2}(U^{\text{in.}}_{\dots} + U^{\text{out.}}_{\dots}) - \sum_{s=1}^{n} \frac{\chi}{2} \int_{-\infty}^{\infty} D\left[S^{(s)}_{\dots\dots} \frac{J_1(\chi u^{(s)})}{u^{(s)}}\right] d\tau^{(s)} \\ &= U^{\text{in.}}_{\dots} + \sum_{1}^{n} \tfrac{1}{2} U'^{\text{rad.}(s)}_{\dots}. \end{aligned} \tag{70}$$

$u^{(s)}$ denotes the distance from the point x^{μ} to the point $z^{\mu(s)}$ on the world line of the sth particle. The equation of the sth particle is then

$$\dot{A}^{(s)}_{\mu} = [T_{\mu}(U'^{\text{sym.}(s)}_{\dots}, U'^{\text{mean}(s)}_{\dots})]_0, \tag{71}$$

where
$$U'^{\text{mean}(s)}_{\dots} \equiv U'^{\text{mean}}_{\dots} + \sum_{t \neq s} U'^{\text{sym.}(t)}_{\dots}$$
$$= U^{\text{act.}}_{\dots} - U'^{\text{sym.}(s)}_{\dots}$$
$$= U^{\text{in.}}_{\dots} + \sum_{t \neq s} U^{\text{ret.}(t)}_{\dots} + \tfrac{1}{2} U'^{\text{rad.}(s)}$$
$$= U^{\text{in.}}_{\dots} + \sum_{t \neq s} U^{\text{ret.}(t)}_{\dots} + \tfrac{1}{2} \overline{U}^{\text{rad.}(s)}_{\dots} - \chi \int_{-\infty}^{\tau} D\left[S^{(s)}_{\dots} \frac{J_1(\chi u)^{(s)}}{u^{(s)}}\right] d\tau^{(s)}. \tag{72}$$

It has already been shown that $T_{\mu\nu}(U'^{\text{sym.}(s)}_{\dots}, U'^{\text{sym.}(s)}_{\dots})$ has no effect in determining the motion of the sth particle. Since $U'^{\text{sym.}(s)}$ is certainly finite and continuous on the world line of the other particles, it is clear that this part of the stress tensor also has no effect in determining the motion of the other particles. Hence we can take as the modified stress tensor

$$T'_{\mu\nu} \equiv T_{\mu\nu} - \sum_{s=1}^{n} T_{\mu\nu}(U'^{\text{sym.}(s)}_{\dots}, U'^{\text{sym.}(s)}_{\dots})$$
$$= \sum_{s=1}^{n} T_{\mu\nu}(U'^{\text{sym.}(s)}_{\dots}, U'^{\text{mean}}_{\dots}) + \sum_{s>t=1}^{n}\sum T_{\mu\nu}(U'^{\text{sym.}(s)}_{\dots}, U'^{\text{sym.}(t)}_{\dots}). \tag{73}$$

This tensor is conserved and gives the same equations of motion for each particle as $T_{\mu\nu}$. It also has the advantage that it does not contain the worst singularities contributed by the static field energies of the individual particles.

Finally, it may be mentioned that a 'Wenzel' field can be introduced in the general theory treated here just as in the case of a point charge moving in a Maxwell field (Dirac 1939). We take the world line of the particle to extend from $-\infty$ to a point τ_B. Now define the fundamental Wenzel solution for the particle by

$$O^W_{\dots}(x^\mu) = \begin{cases} -\chi \int_{-\infty}^{\tau_B} S_{\dots} \frac{J_1(\chi u)}{u} d\tau & \text{for } \tau_B < \tau_r, \\ \left(\frac{S_{\dots}}{\kappa}\right)_r - \chi \int_{-\infty}^{\tau_r} S_{\dots} \frac{J_1(\chi u)}{u} d\tau & \text{for } \tau_r < \tau_B < \tau_a, \\ \left(\frac{S_{\dots}}{\kappa}\right)_r + \left(\frac{S_{\dots}}{\kappa}\right)_a - \chi \int_{-\infty}^{\tau_r} S_{\dots} \frac{J_1(\chi u)}{u} d\tau + \chi \int_{\tau_a}^{\tau_B} S_{\dots} \frac{J_1(\chi u)}{u} d\tau & \text{for } \tau_a < \tau_B. \end{cases} \tag{74}$$

The first condition is satisfied when the point x^μ lies in the future light cone of the point τ_B, the second when it lies outside the light cone, and the third when it lies in the past light cone. This field is clearly just $O^{\text{ret.}}_{\dots} - O^{\text{adv.}}_{\dots}$ as given by (7) and (13) for a world line that extends from $-\infty$ to τ_B only. With the help of the Green's function given by one of us in an earlier paper (Bhabha 1939), which is a generalization of the relativistic delta-function of Jordan and Pauli, $O^W_{\dots}$ can be written in the form

$$O^W_{\dots} = \int_{-\infty}^{\tau_B} S_{\dots} G^{\text{rad.}}(u^2)\, d\tau, \tag{75}$$

where

$$G^{\text{rad.}}(u^2) = \begin{cases} 2\delta(u^2) - \chi\dfrac{J_1(\chi u)}{u} & \text{for } u^2 > 0,\ u_0 > 0, \\ 0 & \text{for } u^2 < 0. \\ -2\delta(u^2) + \chi\dfrac{J_1(\chi u)}{u} & \text{for } u^2 > 0. \end{cases} \tag{76}$$

$G^{\text{rad.}}$ satisfies the generalized wave equation (8) at all points not excepting $u^\mu = 0$, and hence $O^W_{\dots}$ satisfies (8) at all points including those on the world line. Define the generalized Wenzel potential by

$$U^W_{\dots} \equiv U^{\text{in.}}_{\dots} + D[O^W_{\dots\cdot}]. \tag{77}$$

Now if λ^μ be a small time-like vector, then it is easily seen that

$$\underset{\lambda\to 0}{\text{Lt}}\ \tfrac{1}{2}\{U^W_{\dots}(z^\mu(\tau_B)+\lambda^\mu) + U^W_{\dots}(z^\mu(\tau_B)-\lambda^\mu)\} = U^{\text{in.}}(z^\mu) + \tfrac{1}{2}\overline{U}^{\text{rad.}}(z^\mu)$$
$$-\chi\int_{-\infty}^{\tau_B} D\left[S_{\dots\cdot}\frac{J_1(\chi u)}{u}\right]d\tau = U'^{\text{mean}}_{\dots}(z^\mu). \tag{78}$$

This is precisely the field (52) which enters into the equation of motion (51) of the point particle.

If n particles are present we have simply to define the generalized Wenzel potential by

$$U^W_{\dots} = U^{\text{in.}}_{\dots} + \sum_{s=1}^{n} U^{W(s)}_{\dots}, \tag{79}$$

where

$$U^{W(s)}_{\dots} = D[O^{W(s)}_{\dots\cdot}],$$

and it follows at once that

$$\underset{\lambda\to 0}{\text{Lt}}\ \tfrac{1}{2}\{U^W_{\dots}(z^{\mu(s)}+\lambda^\mu) + U^W_{\dots}(z^{\mu(s)}-\lambda^\mu)\} = U'^{\text{mean}(s)}_{\dots}, \tag{80}$$

which is the field that determines the motion of the sth particle.

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