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Representability of Hom implies flatness

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Abstract. Let X be a projective scheme over a noetherian base scheme S, and let $\mathcal{F}$ be a coherent sheaf on X. For any coherent sheaf $\mathcal{E}$ on X, consider the set-valued contravariant functor $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}$on S-schemes, defined by $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}(T) = \mathrm{Hom}(\mathcal{E}_T, \mathcal{F}_T)$ where $\mathcal{E}_T$ and $\mathcal{F}_T$ are the pull-backs of $\mathcal{E}$ and $\mathcal{F}$ to $X_T = X \times_S T$. A basic result of Grothendieck ([EGA], III 7.7.8, 7.7.9) says that if $\mathcal{F}$ is flat over S then $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}$ is representable for all $\mathcal{E}$.

We prove the converse of the above, in fact, we show that if L is a relatively ample line bundle on X over S such that the functor $\mathfrak{hom}_{(L^{-n},\mathcal{F})}$ is representable for infinitely many positive integers n, then $\mathcal{F}$ is flat over S. As a corollary, taking $X = S$, it follows that if $\mathcal{F}$ is a coherent sheaf on S then the functor $T \mapsto H^0(T, \mathcal{F}_T)$ on the category of S-schemes is representable if and only if $\mathcal{F}$ is locally free on S. This answers a question posed by Angelo Vistoli.

The techniques we use involve the proof of flattening stratification, together with the methods used in proving the author's earlier result (see [N1]) that the automorphism group functor of a coherent sheaf on S is representable if and only if the sheaf is locally free.

Keywords. Flattening stratification; Q-sheaf; group-scheme; base change.

Let S be a noetherian scheme, and let X be a projective scheme over S. If $\mathcal{E}$ and $\mathcal{F}$ are coherent sheaves on X, consider the contravariant functor $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}$ from the category of schemes over S to the category of sets which is defined by putting

$$\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}(T) = \mathrm{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$$

for any S-scheme $T \to S$, where $X_T = X \times_S T$, and $\mathcal{E}_T$ and $\mathcal{F}_T$ denote the pull-backs of $\mathcal{E}$ and $\mathcal{F}$ under the projection $X_T \to X$. This functor is clearly a sheaf in the fpqc topology on Sch/S. It was proved by Grothendieck that if $\mathcal{F}$ is flat over S then the above functor is representable (see [EGA], III 7.7.8, 7.7.9).

Our main theorem is as follows, which is a converse to the above.

Theorem 1. *Let S be a noetherian scheme, X a projective scheme over S, and L a relatively very ample line bundle on X over S. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following three statements are equivalent*:

(1) *The sheaf $\mathcal{F}$ is flat over S.*
(2) *For any coherent sheaf $\mathcal{E}$ on X, the set-valued contravariant functor $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}$ on S-schemes, defined by $\mathfrak{hom}_{(\mathcal{E},\mathcal{F})}(T) = \mathrm{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$, is representable.*
(3) *There exist infinitely many positive integers r such that the set-valued contravariant functor $\mathcal{G}^{(r)}$ on S-schemes, defined by $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable.*

In particular, taking $X = S$ and $L = \mathcal{O}_X$, we get the following corollary.

COROLLARY 2

Let S be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on S. Consider the contravariant functor $\mathbf{F}$ *from S-schemes to sets, which is defined by putting* $\mathbf{F}(T) = H^0(T, f^*\mathcal{F})$ *for any S-scheme $f : T \to S$. This functor* (*which is a sheaf in the fpqc topology*) *is representable if and only if $\mathcal{F}$ is locally free as an $\mathcal{O}_S$-module.*

Note that the affine line $\mathbf{A}^1_S$ over a base S admits a ring-scheme structure over S in the obvious way. A *linear scheme* over a scheme S will mean a module-scheme $V \to S$ under the ring-scheme $\mathbf{A}^1_S$. This means V is a commutative group-scheme over S together with a 'scalar-multiplication' morphism $\mu : \mathbf{A}^1_S \times_S V \to V$ over S, such that the module axioms (in diagrammatic terms) are satisfied.

A *linear functor* $\mathbf{F}$ on S-schemes will mean a contravariant functor from S-schemes to sets together with the structure of an $H^0(T, \mathcal{O}_T)$-module on $\mathbf{F}(T)$ for each S-scheme T, which is well-behaved under any morphism $f : U \to T$ of S-schemes in the following sense: $\mathbf{F}(f) : \mathbf{F}(T) \to \mathbf{F}(U)$ is a homomorphism of the underlying additive groups, and $\mathbf{F}(f)(a \cdot v) = f^*(a) \cdot (\mathbf{F}(f)v)$ for any $a \in H^0(T, \mathcal{O}_T)$ and $v \in \mathbf{F}(T)$. In particular note that the kernel of $\mathbf{F}(f)$ will be an $H^0(T, \mathcal{O}_T)$-submodule of $\mathbf{F}(T)$. The functor of points of a linear scheme is naturally a linear functor. Conversely, it follows by the Yoneda lemma that if a linear functor $\mathbf{F}$ on S-schemes is representable, then the representing scheme V is naturally a linear scheme over S.

For example, the linear functor $T \mapsto H^0(T, \mathcal{O}_T)^n$ (where $n \geq 0$) is represented by the affine space $\mathbf{A}^n_{\mathbb{Z}}$ over $\operatorname{Spec} \mathbb{Z}$, with its usual linear-scheme structure. More generally, for any coherent sheaf $\mathfrak{Q}$ on S, the scheme $\operatorname{Spec} \operatorname{Sym}(\mathfrak{Q})$ is naturally a linear-scheme over S, where $\operatorname{Sym}(\mathfrak{Q})$ denotes the symmetric algebra of $\mathfrak{Q}$ over $\mathcal{O}_S$. It represents the linear functor $\mathbf{F}(T) = \operatorname{Hom}(\mathfrak{Q}_T, \mathcal{O}_T)$ where $\mathfrak{Q}_T$ denotes the pull-back of $\mathfrak{Q}$ under $T \to S$.

With this terminology, the functor $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$ of Theorem 1(3) is a linear functor. Therefore, if a representing scheme $G^{(r)}$ exists, it will naturally be a linear scheme. Note that each $\mathcal{G}^{(r)}$ is obviously a sheaf in the fpqc topology.

The proof of Theorem 1 is by a combination of the result of Grothendieck on the existence of a flattening stratification [TDTE IV] together with the techniques which were employed in [N1] to prove the following result.

Theorem 3 (Representability of the functor GL_E). *Let S be a noetherian scheme, and E a coherent $\mathcal{O}_S$-module. Let GL_E denote the contrafunctor on S-schemes which associates to any S-scheme $f : T \to S$ the group of all $\mathcal{O}_T$-linear automorphisms of the pull-back $E_T = f^*E$* (*this functor is a sheaf in the fpqc topology*)*. Then GL_E is representable by a group scheme over S if and only if E is locally free.*

We re-state Grothendieck's result (see [TDTE IV]) on the existence of a flattening stratification in the following form, which emphasises the role of the direct images $\pi_*(\mathcal{F}(r))$. For an exposition of flattening stratification, see [M] or [N2].

Theorem 4 (Grothendieck). *Let S be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_S$ where $n \geq 0$. There exists an integer m, and a collection of locally closed subschemes $S_f \subset S$ indexed by polynomials $f \in \mathbb{Q}[\lambda]$, with the following properties.*

(i) *The underlying set of S_f consists of all $s \in S$ such that the Hilbert polynomial of $\mathcal{F}_s$ is f, where $\mathcal{F}_s$ denotes the pull-back of $\mathcal{F}$ to the schematic fibre $\mathbf{P}^n_s$ over s of the* *projection $\pi : \mathbf{P}^n_S \to S$. All but finitely many S_f are empty (only finitely many Hilbert polynomials occur). In particular, the S_f are mutually disjoint, and their set-theoretic union is S.*

(ii) *For each $r \geq m$, the higher direct images $R^j\pi_*(\mathcal{F}(r))$ are zero for $j \geq 1$ and the subschemes S_f give the flattening stratification for the direct image $\pi_*(\mathcal{F}(r))$, that is, the morphism $i : \coprod_f S_f \to S$ induced by the locally closed embeddings $S_f \hookrightarrow S$ has the universal property that for any morphism $g : T \to S$, the sheaf $g^*\pi_*(\mathcal{F}(r))$ is locally free on T if and only if g factors via $i : \coprod_f S_f \to S$.*

(iii) *The subschemes S_f give the flattening stratification for $\mathcal{F}$, that is, for any morphism $g : T \to S$, the sheaf $\mathcal{F}_T = (1 \times g)^*\mathcal{F}$ on $\mathbf{P}^n_T$ is flat over T if and only if g factors via $i : \coprod_f S_f \to S$. In particular, $\mathcal{F}$ is flat over S if and only if each S_f is an open subscheme of S.*

(iv) *Let $\mathbb{Q}[\lambda]$ be totally ordered by putting $f_1 < f_2$ if $f_1(p) < f_2(p)$ for all $p \gg 0$. Then the closure of S_f in S is set-theoretically contained in $\bigcup_{g \geq f} S_g$. Moreover, whenever S_f and S_g are non-empty, we have $f < g$ if and only if $f(p) < g(p)$ for all $p \geq m$.*

The following elementary lemma of Grothendieck on base-change does not need any flatness hypothesis. The price paid is that the integer r_0 may depend on ϕ. (See [N2] for a cohomological proof.)

Lemma 5. *Let $\phi : T \to S$ be a morphism of noetherian schemes, let $\mathcal{F}$ a coherent sheaf on $\mathbf{P}^n_S$, and let $\mathcal{F}_T$ denote its pull-back under the induced morphism $\mathbf{P}^n_T \to \mathbf{P}^n_S$. Let $\pi_S : \mathbf{P}^n_S \to S$ and $\pi_T : \mathbf{P}^n_T \to T$ denote the projections. Then there exists an integer r_0 such that the base-change homomorphism $\phi^*\pi_{S*}\mathcal{F}(r) \to \pi_{T*}\mathcal{F}_T(r)$ is an isomorphism for all $r \geq r_0$.*

Proof of Theorem 1. The implication (1) $\Rightarrow$ (2) follows by [EGA], III 7.7.8, 7.7.9, while the implication (2) $\Rightarrow$ (3) follows by taking $\mathcal{E} = L^{\otimes -r}$. Therefore it now remains to show the implication (3) $\Rightarrow$ (1). This we do in a number of steps.

Step 1: *Reduction to $S = \operatorname{Spec} R$ with R local, $X = \mathbf{P}^n_S$ and $L = \mathcal{O}_{\mathbf{P}^n_S}(1)$.* Suppose that $\mathcal{F}$ is not flat over S, but the linear functor $\mathcal{G}^{(r)}$ on S-schemes, defined by $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by a linear scheme $G^{(r)}$ over S for arbitrarily large integers r. As $\mathcal{F}$ is not flat, by definition there exists some $x \in X$ such that the stalk $\mathcal{F}_x$ is not a flat module over the local ring $\mathcal{O}_{S,\pi(x)}$ where $\pi : X \to S$ is the projection. Let $U = \operatorname{Spec} \mathcal{O}_{S,\pi(x)}$, let $\mathcal{F}_U$ be the pull-back of $\mathcal{F}$ to $X_U = X \times_S U$ and let $G^{(r)}_U$ denote the pull-back of $G^{(r)}$ to U. Then $\mathcal{F}_U$ is not flat over U but given any integer m, there exists an integer $r \geq m$ such that the functor $\mathcal{G}^{(r)}_U$ on U-schemes, defined by $\mathcal{G}^{(r)}_U(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by the U-scheme $G^{(r)}_U$.

Therefore, by replacing S by U, we can assume that S is of the form $\operatorname{Spec} R$ where R is a noetherian local ring. Let $i : X \hookrightarrow \mathbf{P}^n_S$ be the embedding given by L. Then replacing $\mathcal{F}$ by $i_*\mathcal{F}$, we can further assume that $X = \mathbf{P}^n_S$ and $L = \mathcal{O}_{\mathbf{P}^n_S}(1)$.

Step 2: *Flattening stratification of* Spec R. There exists an integer m as asserted by Theorem 4, such that for any $r \geq m$, the flattening stratification of S for the sheaf $\pi_*\mathcal{F}(r)$ on S is the same as the flattening stratification of S for the sheaf $\mathcal{F}$ on $\mathbf{P}^n_S$. Let $r \geq m$ be any integer. As $\mathcal{F}$ is not flat over $S = \operatorname{Spec} R$, the sheaf $\pi_*\mathcal{F}(r)$ is not flat. Let $M_r = H^0(S, \pi_*\mathcal{F}(r))$, which is a finite R-module. Let $\mathfrak{m} \subset R$ be the maximal ideal, and let $k = R/\mathfrak{m}$ the residue field. Let $s \in S = \operatorname{Spec} R$ be the closed point, and let $d = \dim_k(M_r/\mathfrak{m}M_r)$. Then there exists a right-exact sequence of R-modules of the form

$$R^{\delta} \xrightarrow{\psi} R^d \to M_r \to 0.$$

Let $I \subset R$ be the ideal formed by the matrix entries of the $(d \times \delta)$-matrix ψ. Then I defines a closed subscheme $S' \subset S$ which is the flattening stratification of S for M_r. As M_r is not flat by assumption, I is a non-zero proper ideal in R.

It follows from Theorem 4 that I is independent of r as long as $r \geq m$.

Step 3: *Reduction to Artin local case with principal I with $\mathfrak{m}I = 0$.* Let $I = (a_1, \ldots, a_t)$ where $a_1, \ldots, a_t$ is a minimal set of generators of I. Let $J \subset R$ be the ideal defined by

$$J = (a_2, \ldots, a_t) + \mathfrak{m}I.$$

Then note that $J \subset I \subset \mathfrak{m}$, and the quotient $R' = R/J$ is an Artin local R-algebra with maximal ideal $\mathfrak{m}' = \mathfrak{m}/J$, and $I' = I/J$ is a non-zero principal ideal which satisfies $\mathfrak{m}'I' = 0$. For the base-change under $f : \operatorname{Spec} R' \to \operatorname{Spec} R$, the flattening stratification for $f^*\pi_*\mathcal{F}(r)$ is defined by the ideal I' for $r \geq m$. Let $\mathcal{F}'$ denote the pull-back of $\mathcal{F}$ to $\mathbf{P}^n_{R'}$, and let $\pi' : \mathbf{P}^n_{R'} \to \operatorname{Spec} R'$ the projection. As f is a morphism of noetherian schemes, by Lemma 5 there exists some integer m' such that the base-change homomorphism $f^*\pi_*\mathcal{F}(r) \to \pi'_*\mathcal{F}'(r)$ is an isomorphism whenever $r \geq m'$. Choosing some $m' \geq m$ with this property, and replacing R by R', $\mathcal{F}$ by $\mathcal{F}'$ and m by m', we can assume that R is Artin local, and I is a non-zero principal ideal with $\mathfrak{m}I = 0$, which defines the flattening stratification for $\pi_*\mathcal{F}(r)$ for all $r \geq m$.

Step 4: *Decomposition of $\pi_*\mathcal{F}(r)$ via lemma of Srinivas.*

Lemma (*Srinivas*). *Let R be an Artin local ring with maximal ideal $\mathfrak{m}$, and let E be any finite R module whose flattening stratification is defined by an ideal I which is a non-zero proper principal ideal with $\mathfrak{m}I = 0$. Then there exist integers $i \geq 0$ and $j > 0$ such that E is isomorphic to the direct sum $R^i \oplus (R/I)^j$.*

Proof. See Lemma 4 in [N1].

We apply the above lemma to the R-module $M_r = H^0(S, \pi_*\mathcal{F}(r))$, which has flattening stratification defined by the principal ideal I with $\mathfrak{m}I = 0$, to conclude that (up to isomorphism) M_r has the form

$$M_r = R^{i(r)} \oplus (R/I)^{j(r)}$$

for non-negative integers $i(r)$ and $j(r)$ with $j(r) > 0$. Note that $i(r) + j(r) = \Phi(r)$ where Φ is the Hilbert polynomial of $\mathcal{F}$.

Step 5: *Structure of the hypothetical representing scheme $G^{(r)}$.* Let $\phi : \operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R$ denote the inclusion and $\mathcal{F}'$ denote the pull-back of $\mathcal{F}$ under $\mathbf{P}^n_{R/I} \hookrightarrow \mathbf{P}^n_R$. The sheaf $\mathcal{F}'$ is flat over R/I, and the functor $\mathcal{G}^{(r)}_{R/I}$, which is the restriction of $\mathcal{G}^{(r)}$, is represented by the linear scheme $\mathbf{A}^d_{R/I} = \operatorname{Spec}(R/I)[y_1, \ldots, y_d]$ over R/I, where $d = \Phi(r)$ where Φ is the Hilbert polynomial of $\mathcal{F}$. Hence, the pull-back of the hypothetical representing scheme $G^{(r)}$ to R/I is the linear scheme $\mathbf{A}^d_{R/I}$. We now use the following fact (see Lemmas 6 and 7 of [N1] for a proof).

Lemma. Let R be a ring and I a nilpotent ideal ($I^n = 0$ for some $n \geq 1$). Let X be a scheme over Spec R*, such that the closed subscheme $Y = X \otimes_R (R/I)$ is isomorphic over R/I to* Spec B *where B is a finite-type R/I-algebra. Let $b_1, \dots, b_d \in B$ be a set of algebra generators for B over R/I. Then X is isomorphic over R with* Spec A *where A is a finite-type R-algebra. Moreover, there exists a set of R-algebra generators $a_1, \dots, a_d$ for A, such that each a_i restricts modulo I to $b_i \in B$ over R/I. Let $R[x_1, \dots, x_d]$ be a polynomial ring in d variables over R, and consider the surjective R-algebra homomorphism $R[x_1, \dots, x_d] \to A$ defined by sending each x_i to a_i, and let J be its kernel. Then $J \subset IR[x_1, \dots, x_d]$.*

It follows from the above lemma that $G^{(r)}$ is affine of finite type over R, and its co-ordinate ring A as an R algebra is of the form

$$A = R[a_1, \dots, a_d] = R[x_1, \dots, x_d]/J,$$

where a_i is the residue of x_i, and $a_1, \dots, a_d$ restrict over R/I to the linear coordinates $y_1, \dots, y_d$ on the linear scheme $\mathbf{A}^d_{R/I}$, and J is an ideal with $J \subset I \cdot R[x_1, \dots, x_d]$. Being an additive group-scheme, $G^{(r)}$ has its zero section $\sigma : \operatorname{Spec} R \to G^{(r)}$, and this corresponds to an R-algebra homomorphism $\sigma^* : A \to R$. Modulo I, the section σ restricts to the zero section of $\mathbf{A}^d_{R/I}$ over $\operatorname{Spec}(R/I)$, therefore $\sigma^*(a_i) \in I$ for all $i = 1, \dots, d$. Let $x'_i = x_i - \sigma^*(a_i) \in R[x_1, \dots, x_d]$ and $a'_i = a_i - \sigma^*(a_i) \in A$ be its residue modulo J. Then $R[x_1, \dots, x_d] = R[x'_1, \dots, x'_d]$, the elements $a'_1, \dots, a'_d$ generate A as an R-algebra, and moreover the a'_i restrict over R/I to the linear coordinates y_i on the linear scheme $\mathbf{A}^d_{R/I}$. Therefore, by replacing the x_i by the x'_i and the a_i by the a'_i, we can assume that for each i, we have

$$\sigma^*(a_i) = 0.$$

Next, consider any element $f(x_1, \dots, x_d) \in J$. Then $f(a_1, \dots a_d) = 0$ in A, so $\sigma^* f(a_1, \dots a_d) = 0 \in R$, which shows that the constant coefficient of f is zero, as $\sigma^*(a_i) = 0$. As we already know that $J \subset I \cdot R[x_1, \dots, x_d]$, the vanishing of the constant term of any element of J now establishes that

$$J \subset I \cdot (x_1, \dots, x_d).$$

From the above, using $I^2 = 0$, it follows that for any $(b_1, \dots, b_d) \in I^d$, we have a well-defined R-algebra homomorphism

$$\Psi_{(b_1, \dots, b_d)} : A \to R : a_i \mapsto b_i.$$

We now express the linear-scheme structure of $G^{(r)}$ in terms of the ring A, using the fact that each a_i restricts to y_i modulo I, and $G^{(r)}_{R/I}$ is the standard linear-scheme $\mathbf{A}^d_{R/I}$ with linear co-ordinates y_i. Note that the vector addition morphism $\mathbf{A}^d_{R/I} \times_{R/I} \mathbf{A}^d_{R/I} \to \mathbf{A}^d_{R/I}$ corresponds to the R/I-algebra homomorphism

$$(R/I)[y_1, \dots, y_d] \to (R/I)[y_1, \dots, y_d] \otimes_{R/I} (R/I)[y_1, \dots, y_d] : y_i \mapsto y_i \otimes 1 + 1 \otimes y_i$$

while the scalar-multiplication morphism $\mathbf{A}^1_{R/I} \times_{R/I} \mathbf{A}^d_{R/I} \to \mathbf{A}^d_{R/I}$ corresponds to the R/I-algebra homomorphism

$$(R/I)[y_1, \dots, y_d] \to (R/I)[t, y_1, \dots, y_d]$$
$$= (R/I)[t] \otimes_{R/I} (R/I)[y_1, \dots, y_d] : y_i \mapsto ty_i.$$

It follows that the addition morphism $\alpha : G^{(r)} \times_R G^{(r)} \to G^{(r)}$ corresponds to an algebra homomorphism $\alpha^* : A \to A \otimes_R A$ which has the form

$$a_i \mapsto a_i \otimes 1 + 1 \otimes a_i + u_i \text{ where } u_i \in I(A \otimes_R A).$$

Let the element u_i in the above equation for $\alpha^*(a_i)$ be written as a polynomial expression

$$u_i = f_i(a_1 \otimes 1, \dots, a_d \otimes 1, 1 \otimes a_1, \dots, 1 \otimes a_d)$$

with coefficients in I. The additive identity 0 of $G^{(r)}(R)$ corresponds to $\sigma^* : A \to R$ with $\sigma^*(a_i) = 0$, and we have $0 + 0 = 0$ in $G^{(r)}(R)$. This implies that $f_i(0, \dots, 0) = 0$, and so the constant term of f_i is zero. From this, using $I^2 = 0$, we get the important consequence that

$$f_i(w_1, \dots, w_{2d}) = 0 \text{ for all } w_1, \dots, w_{2d} \in I.$$

The scalar-multiplication morphism $\mu : \mathbf{A}^1_R \times_R G^{(r)} \to G^{(r)}$ prolongs the standard scalar multiplication on $\mathbf{A}^d_{R/I}$, and so μ corresponds to an algebra homomorphism $\mu^* : A \to A[t] = R[t] \otimes_R A$ which has the form

$$a_i \mapsto ta_i + v_i \text{ where } v_i \in IA[t].$$

Let v_i be expressed as a polynomial $v_i = g_i(t, a_1, \dots, a_d)$ with coefficients in I. As multiplication by the scalar 0 is the zero morphism on $G^{(r)}$, it follows by specialising under $t \mapsto 0$ that $g_i(0, a_1, \dots, a_d) = 0$. This means $v_i = g_i(t, a_1, \dots, a_d)$ can be expanded as a finite sum

$$v_i = \sum_{j \geq 1} t^j h_{i,j}(a_1, \dots, a_d),$$

where the $h_{i,j}(a_1, \dots, a_d)$ are polynomial expressions with coefficients in I. As the zero vector times any scalar is zero, it follows by specialising under σ^* that $g_i(t, 0, \dots, 0) = 0$. It follows that the constant term of each $h_{i,j}$ is zero. From this, and the fact that $I^2 = 0$, we get the important consequence that

$$g_i(t, b_1, \dots, b_d) = 0 \text{ for all } b_1, \dots, b_d \in I.$$

Step 6: *The kernel of the map* $G^{(r)}(R) \to G^{(r)}(R/I)$.

Lemma. Let $\Psi_{(b_1,\dots,b_d)} : A \to R$ *be the* R*-algebra homomorphism defined in terms of the generators by* $\Psi_{(b_1,\dots,b_d)}(a_k) = b_k$. *Let* $\Psi : I^d \to \mathrm{Hom}_{R-\mathrm{alg}}(A, R)$ *be the set-map defined by* $(b_1, \dots, b_d) \mapsto (\Psi_{(b_1,\dots,b_d)} : A \to R)$. *Then* Ψ *is a homomorphism of* R*-modules, where the* R*-module structure on* $\mathrm{Hom}_{R-\mathrm{alg}}(A, R)$ *is defined by its identification with the* R*-module* $G^{(r)}(R)$.

The map Ψ *is injective, and its image is the* R*-submodule* $\ker G^{(r)}(\phi) \subset G^{(r)}(R)$, *where* $\phi : \mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}\, R$ *is the inclusion.*

Proof. For any $(b_1, \ldots, b_d)$ and $(c_1, \ldots, c_d)$ in I^d, we have

$$
\begin{aligned}
(\Psi_{(b_1,\ldots,b_d)} + \Psi_{(c_1,\ldots,c_d)})(a_i) &= (\Psi_{(b_1,\ldots,b_d)} \otimes \Psi_{(c_1,\ldots,c_d)})(\alpha^*(a_i)) \\
&= b_i + c_i + f_i(b_1, \ldots, b_d, c_1, \ldots, c_d) \\
&\quad \text{by substituting for } \alpha^*(a_i) \\
&= b_i + c_i \text{ as } b_k, c_k \in I \\
&= \Psi_{(b_1+c_1,\ldots,b_d+c_d)}(a_i).
\end{aligned}
$$

This shows the equality $\Psi_{(b_1,\ldots,b_d)} + \Psi_{(c_1,\ldots,c_d)} = \Psi_{(b_1,\ldots,b_d)+(c_1,\ldots,c_d)}$, which means the map $\Psi : I^d \to G^{(r)}(R)$ is additive.

For any $\lambda \in R$, let $f_\lambda : R[t] \to R$ be the R-algebra homomorphism defined by $f_\lambda(t) = \lambda$. Then for any $(b_1, \ldots, b_d) \in I^d$ we have

$$
\begin{aligned}
(\lambda \cdot \Psi_{(b_1,\ldots,b_d)})(a_i) &= (f_\lambda \otimes \Psi_{(b_1,\ldots,b_d)})(\mu^*(a_i)) \\
&= (f_\lambda \otimes \Psi_{(b_1,\ldots,b_d)})(ta_i + g_i(t, a_1, \ldots, a_d)) \\
&= \lambda b_i + g_i(\lambda, b_1, \ldots, b_d) \\
&= \lambda b_i \text{ as } b_k \in I \\
&= \Psi_{(\lambda b_1,\ldots,\lambda b_d)}(a_i).
\end{aligned}
$$

This shows the equality $\lambda \cdot \Psi_{(b_1,\ldots,b_d)} = \Psi_{\lambda\cdot(b_1,\ldots,b_d)}$, hence the map $\Psi : I^d \to G^{(r)}(R)$ preserves scalar multiplication. This completes the proof that $\Psi : I^d \to G^{(r)}(R)$ is a homomorphism of R-modules.

The map Ψ is clearly injective. The map $G^{(r)}(\phi) : G^{(r)}(R) \to G^{(r)}(R/I)$ is in algebraic terms the map $\mathrm{Hom}_{R-\mathrm{alg}}(A, R) \to \mathrm{Hom}_{R-\mathrm{alg}}(A, R/I)$ induced by the quotient $R \to R/I$. An element $g \in \mathrm{Hom}_{R-\mathrm{alg}}(A, R/I)$ represents the zero element of $G^{(r)}(R/I)$ exactly when $g(a_i) = 0 \in R/I$ for the generators a_i of A. Therefore $f \in \mathrm{Hom}_{R-\mathrm{alg}}(A, R)$ is in the kernel of $G^{(r)}(\phi)$ precisely when $f(a_i) \in I$ for the generators a_i. Putting $b_i = f(a_i)$, we see that such an f is the same as $\Psi_{(b_1,\ldots,b_d)}$.

This completes the proof of the lemma that $\ker G^{(r)}(\phi) = I^d$.

In particular, as $\mathfrak{m}I = 0$, it follows from the above lemma that $\ker G^{(r)}(\phi)$ is annihilated by $\mathfrak{m}$, so it is a vector space over $R/\mathfrak{m}$, and its dimension as a vector space over $R/\mathfrak{m}$ is $d = \Phi(r)$, as by assumption I is a non-zero principal ideal.

The above determination of the dimension over $R/\mathfrak{m}$ of the kernel of $G^{(r)}(\phi)$ will contradict a more direct functorial description, which is as follows.

Step 7: Functorial description of kernel of $\mathcal{G}^{(r)}(R) \to \mathcal{G}^{(r)}(R/I)$. As $\mathcal{F}_{R/I}(r)$ is flat over R/I, and as for $r \geq m$ all higher direct images of $\mathcal{F}(r)$ vanish, $\mathcal{G}^{(r)}(R/I)$ is isomorphic to the R/I-module $(R/I)^d$ where $d = \Phi(r)$. By Lemma 5, there exists $m'' \geq m$ such that for $r \geq m''$ the inclusion $\phi : \mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}\, R$ induces an isomorphism $\phi^*\pi_*\mathcal{F}(r) \to \pi'_*\mathcal{F}'(r)$ where $\pi' : \mathbf{P}^n_{R/I} \to \mathrm{Spec}(R/I)$ is the projection and $\mathcal{F}'$ is the pull-back of $\mathcal{F}$ under $\mathbf{P}^n_{R/I} \hookrightarrow \mathbf{P}^n_R$. Note that $\mathcal{G}^{(r)}(R) = R^{i(r)} \oplus (R/I)^{j(r)}$, and so for $r \geq m''$ we get an induced decomposition

$$
\mathcal{G}^{(r)}(R/I) = (R/I)^{i(r)} \oplus (R/I)^{j(r)}
$$

such that the map $\mathcal{G}^{(r)}(\phi) : \mathcal{G}^{(r)}(R) \to \mathcal{G}^{(r)}(R/I)$ is the map

$$(q, 1) : R^{i(r)} \oplus (R/I)^{j(r)} \to (R/I)^{i(r)} \oplus (R/I)^{j(r)},$$

where q is the quotient map modulo I. It follows that the kernel of $\mathcal{G}^{(r)}(\phi)$ is the R-module $I^{i(r)} \oplus 0 \subset R^{i(r)} \oplus (R/I)^{j(r)} = \mathcal{G}^{(r)}(R)$. This is a vector space over $R/\mathfrak{m}$ of dimension $i(r) < i(r) + j(r) = \Phi(r)$.

We thus obtain two different values for the dimension of the same vector space $\ker G^{(r)}(\phi) = \ker \mathcal{G}^{(r)}(\phi)$, which shows that our assumption that $\mathcal{G}^{(r)}$ is representable for arbitrarily large values of r is false. This completes the proof of Theorem 1.

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