Representability of Hom implies flatness

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Abstract. Let $X$ be a projective scheme over a noetherian base scheme $S$, and let $\mathcal{F}$ be a coherent sheaf on $X$. For any coherent sheaf $E$ on $X$, consider the set-valued contravariant functor $\text{hom}_{(E,\mathcal{F})}$ on $S$-schemes, defined by $\text{hom}_{(E,\mathcal{F})}(T) = \text{Hom}(E_T, F_T)$ where $E_T$ and $F_T$ are the pull-backs of $E$ and $F$ to $X_T = X \times_S T$. A basic result of Grothendieck ([EGA], III 7.7.8, 7.7.9) says that if $\mathcal{F}$ is flat over $S$ then $\text{hom}_{(E,\mathcal{F})}$ is representable for all $E$.

We prove the converse of the above, in fact, we show that if $L$ is a relatively ample line bundle on $X$ over $S$ such that the functor $G^r(T) = H^0(X_T, F_T \otimes L^{\otimes r})$ is representable for infinitely many positive integers $r$, then $\mathcal{F}$ is flat over $S$. As a corollary, taking $X = S$, it follows that if $\mathcal{F}$ is a coherent sheaf on $S$ then the functor $T \mapsto H^0(T, \mathcal{F}_T)$ on the category of $S$-schemes is representable if and only if $\mathcal{F}$ is locally free on $S$. This answers a question posed by Angelo Vistoli.

The techniques we use involve the proof of flattening stratification, together with the methods used in proving the author’s earlier result (see [N1]) that the automorphism group functor of a coherent sheaf on $S$ is representable if and only if the sheaf is locally free.

Keywords. Flattening stratification; Q-sheaf; group-scheme; base change.

Let $S$ be a noetherian scheme, and let $X$ be a projective scheme over $S$. If $E$ and $\mathcal{F}$ are coherent sheaves on $X$, consider the contravariant functor $\text{hom}_{(E,\mathcal{F})}$ from the category of schemes over $S$ to the category of sets which is defined by putting $\text{hom}_{(E,\mathcal{F})}(T) = \text{Hom}(E_T, F_T)$ for any $S$-scheme $T \to S$, where $X_T = X \times_S T$, and $E_T$ and $F_T$ denote the pull-backs of $E$ and $\mathcal{F}$ under the projection $X_T \to X$. This functor is clearly a sheaf in the fpqc topology on $\text{Sch}/S$. It was proved by Grothendieck that if $\mathcal{F}$ is flat over $S$ then the above functor is representable (see [EGA], III 7.7.8, 7.7.9).

Our main theorem is as follows, which is a converse to the above.

**Theorem 1.** Let $S$ be a noetherian scheme, $X$ a projective scheme over $S$, and $L$ a relatively ample line bundle on $X$ over $S$. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then the following three statements are equivalent:

1. The sheaf $\mathcal{F}$ is flat over $S$.
2. For any coherent sheaf $E$ on $X$, the set-valued contravariant functor $\text{hom}_{(E,\mathcal{F})}$ on $S$-schemes, defined by $\text{hom}_{(E,\mathcal{F})}(T) = \text{Hom}(E_T, F_T)$, is representable.
3. There exist infinitely many positive integers $r$ such that the set-valued contravariant functor $G^r(T) = H^0(X_T, F_T \otimes L^{\otimes r})$ is representable.
COROLLARY 2

Let $S$ be a noetherian scheme, and $F$ a coherent sheaf on $S$. Consider the contravariant functor $F$ from $S$-schemes to sets, which is defined by putting $F(T) = H^0(T, f^*\mathcal{F})$ for any $S$-scheme $f : T \to S$. This functor (which is a sheaf in the fpqc topology) is representable if and only if $F$ is locally free as an $\mathcal{O}_S$-module.

Note that the affine line $\mathbb{A}^1_S$ over a base $S$ admits a ring-scheme structure over $S$ in the obvious way. A linear scheme over a scheme $S$ will mean a module-scheme $V \to S$ under the ring-scheme $\mathbb{A}^1_S$. This means $V$ is a commutative group-scheme over $S$ together with a ‘scalar-multiplication’ morphism $\mu : \mathbb{A}^1_S \times_S V \to V$ over $S$, such that the module axioms (in diagrammatic terms) are satisfied.

A linear functor $F$ on $S$-schemes will mean a contravariant functor from $S$-schemes to sets together with the structure of an $H^0(T, \mathcal{O}_T)$-module on $F(T)$ for each $S$-scheme $T$, which is well-behaved under any morphism $f : U \to T$ of $S$-schemes in the following sense: $F(f) : F(T) \to F(U)$ is a homomorphism of the underlying additive groups, and $F(f)(a \cdot v) = f^*(a) \cdot (F(f)v)$ for any $a \in H^0(T, \mathcal{O}_T)$ and $v \in F(T)$. In particular note that the kernel of $F(f)$ will be an $H^0(T, \mathcal{O}_T)$-submodule of $F(T)$. The functor of points of a linear scheme is naturally a linear functor. Conversely, it follows by the Yoneda lemma that if a linear functor $F$ on $S$-schemes is representable, then the representing scheme $V$ is naturally a linear scheme over $S$.

For example, the linear functor $T \mapsto H^0(T, \mathcal{O}_T)^n$ (where $n \geq 0$) is represented by the affine space $\mathbb{A}^n_S$ over $\text{Spec} \, \mathbb{Z}$, with its usual linear-scheme structure. More generally, for any coherent sheaf $\mathcal{O}$ on $S$, the scheme $\text{Spec} \, \text{Sym}(\mathcal{O})$ is naturally a linear-scheme over $S$, where $\text{Sym}(\mathcal{O})$ denotes the symmetric algebra of $\mathcal{O}$ over $\mathcal{O}_S$. It represents the linear functor $F(T) = \text{Hom}(\mathcal{O}_T, \mathcal{O}_T)$ where $\mathcal{O}_T$ denotes the pull-back of $\mathcal{O}$ under $T \to S$.

With this terminology, the functor $G^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\text{det}})$ of Theorem 1(3) is a linear functor. Therefore, if a representing scheme $G^{(r)}$ exists, it will naturally be a linear scheme. Note that each $G^{(r)}$ is obviously a sheaf in the fpqc topology.

The proof of Theorem 1 is by a combination of the result of Grothendieck on the existence of a flattening stratification [TDTE IV] together with the techniques which were employed in [N1] to prove the following result.

**Theorem 3 (Representability of the functor $GL_E$).** Let $S$ be a noetherian scheme, and $E$ a coherent $\mathcal{O}_S$-module. Let $GL_E$ denote the contrafunctor on $S$-schemes which associates to any $S$-scheme $f : T \to S$ the group of all $\mathcal{O}_T$-linear automorphisms of the pull-back $E_T = f^*E$ (this functor is a sheaf in the fpqc topology). Then $GL_E$ is representable by a group scheme over $S$ if and only if $E$ is locally free.

We re-state Grothendieck’s result (see [TDTE IV]) on the existence of a flattening stratification in the following form, which emphasises the role of the direct images $\pi_*\mathcal{F}(r)$.

**Theorem 4 (Grothendieck).** Let $S$ be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}_S^n$ where $n \geq 0$. There exists an integer $m$, and a collection of locally closed subschemes $S_{i} \subset S$ indexed by polynomials $f \in \mathbb{Q}[\lambda]$, with the following properties.

(i) The underlying set of $S_i$ consists of all $s \in S$ such that the Hilbert polynomial of $\mathcal{F}_s$ is $f$, where $\mathcal{F}_s$ denotes the pull-back of $\mathcal{F}$ to the schematic fibre $\mathbb{P}_s^n$ over $s$ of the
A projection $\pi : P^a_S \to S$. All but finitely many $S_f$ are empty (only finitely many Hilbert polynomials occur). In particular, the $S_f$ are mutually disjoint, and their set-theoretic union is $S$.

(ii) For each $r \geq m$, the higher direct images $R^j\pi_* (F(r))$ are zero for $j \geq 1$ and the subschemes $S_f$ give the flattening stratification for the direct image $\pi_* (F(r))$, that is, the morphism $i : \bigsqcup S_f \to S$ induced by the locally closed embeddings $S_f \hookrightarrow S$ has the universal property that for any morphism $g : T \to S$, the sheaf $g^*\pi_* (F(r))$ is locally free on $T$ if and only if $g$ factors via $i : \bigsqcup S_f \to S$.

(iii) The subschemes $S_f$ give the flattening stratification for $F$, that is, for any morphism $g : T \to S$, the sheaf $g^* F = (1 \times g)^* F$ on $P^a_T$ is flat over $T$ if and only if $g$ factors via $i : \bigsqcup S_f \to S$. In particular, $F$ is flat over $S$ if and only if each $S_f$ is an open subscheme of $S$.

(iv) Let $\mathbb{Q}[\lambda]$ be totally ordered by putting $f_1 < f_2$ if $f_1(p) < f_2(p)$ for all $p \gg 0$. Then the closure of $S_f$ in $S$ is set-theoretically contained in $\bigcup_{g \geq f} S_g$. Moreover, whenever $S_f$ and $S_g$ are non-empty, we have $f < g$ if and only if $f(p) < g(p)$ for all $p \geq m$.

The following elementary lemma of Grothendieck on base-change does not need any flatness hypothesis. The price paid is that the integer $r_0$ may depend on $\phi$. (See [N2] for a cohomological proof.)

**Lemma 5.** Let $\phi : T \to S$ be a morphism of noetherian schemes, let $F$ be a coherent sheaf on $P^a_S$, and let $F_T$ denote its pull-back under the induced morphism $P^a_T \to P^a_S$. Let $\pi_S : P^a_S \to S$ and $\pi_T : P^a_T \to T$ denote the projections. Then there exists an integer $r_0$ such that the base-change homomorphism $\phi^*\pi_* S_f F(r) \to \pi_* F_T(r)$ is an isomorphism for all $r \geq r_0$.

**Proof of Theorem 1.** The implication $(1) \Rightarrow (2)$ follows by [EGA], III 7.7.8, 7.7.9, while the implication $(2) \Rightarrow (3)$ follows by taking $E = L^{\otimes -r}$. Therefore it now remains to show the implication $(3) \Rightarrow (1)$. This we do in a number of steps.

**Step 1:** **Reduction to $S$ = Spec $R$ with $R$ local, $X = P^a_S$ and $L = O_{P^a_S}(1)$**. Suppose that $F$ is not flat over $S$, but the linear scheme $G^{(r)}$ on $S$-schemes, defined by $G^{(r)}(T) = H^0(X_T, F_T \otimes L^{\otimes r})$, is representable by a linear scheme $G^{(r)}$ over $S$ for arbitrarily large integers $r$. As $F$ is not flat, by definition there exists some $x \in X$ such that the stalk $F_x$ is not a flat module over the local ring $O_{S, \pi(x)}$, where $\pi : X \to S$ is the projection. Let $U = \text{Spec } O_{S, \pi(x)}$, let $F_U$ be the pull-back of $F$ to $X_U = X \times_S U$ and let $G^{(r)}_U$ denote the pull-back of $G^{(r)}$ to $U$. Then $F_U$ is not flat over $U$ but given any integer $m$, there exists an integer $r \geq m$ such that the functor $G^{(r)}_U$ on $U$-schemes, defined by $G^{(r)}_U(T) = H^0(X_T, F_T \otimes L^{\otimes r})$, is representable by the $U$-scheme $G^{(r)}_U$.

Therefore, by replacing $S$ by $U$, we can assume that $S$ is of the form $\text{Spec } R$ where $R$ is a noetherian local ring. Let $i : X \hookrightarrow P^a_S$ be the embedding given by $L$. Then replacing $F$ by $i_* F$, we can further assume that $X = P^a_S$ and $L = O_{P^a_S}(1)$.

**Step 2:** **Flattening stratification of Spec $R$**. There exists an integer $m$ as asserted by Theorem 4, such that for any $r \geq m$, the flattening stratification of $S$ for the sheaf $\pi_* F(r)$ on $S$ is the same as the flattening stratification of $S$ for the sheaf $F$ on $P^a_S$. Let $r \geq m$ be any integer. As $F$ is not flat over $S = \text{Spec } R$, the sheaf $\pi_* F(r)$ is not flat. Let $M_r = H^0(S, \pi_* F(r))$, which is a finite $R$-module. Let $m \subset R$ be the maximal ideal, and let $k = R/m$ the residue
field. Let \( s \in S = \text{Spec } R \) be the closed point, and let \( d = \dim_k(M_r/mM_r) \). Then there exists a right-exact sequence of \( R \)-modules of the form

\[
R^8 \xrightarrow{\psi} R^d \rightarrow M_r \rightarrow 0.
\]

Let \( I \subset R \) be the ideal formed by the matrix entries of the \((d \times s)\)-matrix \( \psi \). Then \( I \) defines a closed subscheme \( S' \subset S \) which is the flattening stratification of \( S \) for \( M_r \). As \( M_r \) is not flat by assumption, \( I \) is a non-zero proper ideal in \( R \).

It follows from Theorem 4 that \( I \) is independent of \( r \) as long as \( r \geq m \).

**Step 3:** Reduction to Artin local case with principal \( I \) with \( mI = 0 \). Let \( I = (a_1, \ldots, a_t) \) where \( a_1, \ldots, a_t \) is a minimal set of generators of \( I \). Let \( J \subset R \) be the ideal defined by

\[
J = (a_2, \ldots, a_t) + mI.
\]

Then note that \( J \subset I \subset m \), and the quotient \( R' = R/J \) is an Artin local \( R \)-algebra with maximal ideal \( m' = m/J \), and \( I' = I/J \) is a non-zero principal ideal which satisfies \( m' I' = 0 \). For the base-change under \( f : \text{Spec } R' \rightarrow \text{Spec } R \), the flattening stratification for \( f^* \pi_a F(r) \) is defined by the ideal \( I' \) for \( r \geq m \). Let \( F' \) denote the pull-back of \( F \) to \( P_a^{n'} \), and let \( \pi' : P_a^{n'} \rightarrow \text{Spec } R' \) the projection. As \( f \) is a morphism of noetherian schemes, by Lemma 5 there exists some integer \( m' \) such that the base-change homomorphism \( f^* \pi_a F(r) \rightarrow \pi'_a F'(r) \) is an isomorphism whenever \( r \geq m' \). Choosing some \( m' \geq m \) with this property, and replacing \( R \) by \( R' \), \( F \) by \( F' \) and \( m \) by \( m' \), we can assume that \( R \) is Artin local, and \( I \) is a non-zero principal ideal with \( mI = 0 \), which defines the flattening stratification for \( \pi_a F(r) \) for all \( r \geq m \).

**Step 4:** Decomposition of \( \pi_a F(r) \) via lemma of Srinivas.

**Lemma (Srinivas).** Let \( R \) be an Artin local ring with maximal ideal \( m \), and let \( E \) be any finite \( R \)-module whose flattening stratification is defined by an ideal \( I \) which is a non-zero proper principal ideal with \( mI = 0 \). Then there exist integers \( i \geq 0 \) and \( j > 0 \) such that \( E \) is isomorphic to the direct sum \( R^i \oplus (R/I)^j \).

**Proof.** See Lemma 4 in [N1].

We apply the above lemma to the \( R \)-module \( M_r = H^0(S, \pi_a F(r)) \), which has flattening stratification defined by the principal ideal \( I \) with \( mI = 0 \), to conclude that (up to isomorphism) \( M_r \) has the form

\[
M_r = R^{i(r)} \oplus (R/I)^{j(r)}
\]

for non-negative integers \( i(r) \) and \( j(r) \) with \( j(r) > 0 \). Note that \( i(r) + j(r) = \Phi(r) \) where \( \Phi \) is the Hilbert polynomial of \( F \).

**Step 5:** Structure of the hypothetical representing scheme \( G^{(r)} \). Let \( \phi : \text{Spec } (R/I) \hookrightarrow \text{Spec } R \) denote the inclusion and \( F' \) denote the pull-back of \( F \) under \( P_a^{d} \). The sheaf \( F' \) is flat over \( R/I \), and the functor \( G^{(r)}_{R/I} \), which is the restriction of \( G^{(r)} \), is represented by the linear scheme \( A^{d}_{R/I} = \text{Spec } (R/I)[y_1, \ldots, y_d] \) over \( R/I \), where \( d = \Phi(r) \) where \( \Phi \) is the Hilbert polynomial of \( F \). Hence, the pull-back of the hypothetical representing scheme \( G^{(r)} \) to \( R/I \) is the linear scheme \( A^{d}_{R/I} \). We now use the following fact (see Lemmas 6 and 7 of [N1] for a proof).
Lemma. Let $R$ be a ring and $I$ a nilpotent ideal ($I^n = 0$ for some $n \geq 1$). Let $X$ be a scheme over $\text{Spec} \ R$, such that the closed subscheme $Y = X \otimes_R (R/I)$ is isomorphic over $R/I$ to $\text{Spec} \ B$ where $B$ is a finite-type $R/I$-algebra. Let $b_1, \ldots, b_d \in B$ be a set of $R/I$-algebra generators for $B$ over $R/I$. Then $X$ is isomorphic over $R$ with $\text{Spec} \ A$ where $A$ is a finite-type $R$-algebra. Moreover, there exists a set of $R$-algebra generators $a_1, \ldots, a_d$ for $A$, such that each $a_i$ restricts to the zero section of $\sigma_i = \text{Spec} \ (R/I)$.

From the above, using $\text{Rep} \text{Hom}$ implies flatness $11$, it follows that for any $(b_1, \ldots, b_d) \in I^d$, we have a well-defined $R$-algebra homomorphism

$$\Psi_{(b_1, \ldots, b_d)} : A \to R : a_i \mapsto b_i.$$ 

We now express the linear-scheme structure of $G^{(r)}$ in terms of the ring $A$, using the fact that each $a_i$ restricts to $y_i$ modulo $I$, and $G^{(r)}_{R/I}$ is the standard linear-scheme $A^d_{R/I}$ with $R/I$-algebra generators $y_i$. Note that the vector addition morphism $A^d_{R/I} \times_{R/I} A^d_{R/I} \to A^d_{R/I}$ corresponds to the $R/I$-algebra homomorphism

$$(R/I)[y_1, \ldots, y_d] \to (R/I)[y_1, \ldots, y_d] \otimes_{R/I} (R/I)[y_1, \ldots, y_d] : y_i \mapsto y_i \otimes 1 + 1 \otimes y_i.$$
while the scalar-multiplication morphism \( \mathbf{A}^1_{R/I} \times_{R/I} \mathbf{A}^d_{R/I} \to \mathbf{A}^d_{R/I} \) corresponds to the \( R/I \)-algebra homomorphism

\[
(R/I)[y_1, \ldots, y_d] \to (R/I)[t, y_1, \ldots, y_d]
\]

\[
= (R/I)[t] \otimes_{R/I} (R/I)[y_1, \ldots, y_d] : y_i \mapsto ty_i.
\]

It follows that the addition morphism \( \alpha : G^{(r)} \times G^{(r)} \to G^{(r)} \) corresponds to the \( R/I \)-algebra homomorphism \( \alpha^* : A \to A \otimes_R A \) which has the form

\[
a_i \mapsto a_i \otimes 1 + 1 \otimes a_i + u_i \text{ where } u_i \in I(A \otimes_R A).
\]

Let the element \( u_i \) in the above equation for \( \alpha^*(a_i) \) be written as a polynomial expression

\[
u_i = f_i(a_1 \otimes 1, \ldots, a_d \otimes 1, 1 \otimes a_1, \ldots, 1 \otimes a_d)
\]

with coefficients in \( I \). The additive identity \( 0 \) of \( G^{(r)}(R) \) corresponds to \( \sigma^* : A \to R \) with \( \sigma^*(a_i) = 0 \), and we have \( 0 + 0 = 0 \) in \( G^{(r)}(R) \). This implies that \( f_i(0, \ldots, 0) = 0 \), and so the constant term of \( f_i \) is zero. From this, using \( I^2 = 0 \), we get the important consequence that

\[
f_i(w_1, \ldots, w_{2d}) = 0 \text{ for all } w_1, \ldots, w_{2d} \in I.
\]

The scalar-multiplication morphism \( \mu : \mathbf{A}^1_{R} \times_R G^{(r)} \to G^{(r)} \) prolongs the standard scalar multiplication on \( \mathbf{A}^d_{R/I} \), and so \( \mu \) corresponds to an algebra homomorphism \( \mu^* : A \to A[t] = R[t] \otimes_R A \) which has the form

\[
a_i \mapsto ta_i + v_i \text{ where } v_i \in I A[t].
\]

Let \( v_i \) be expressed as a polynomial \( v_i = g_i(t, a_1, \ldots, a_d) \) with coefficients in \( I \). As multiplication by the scalar \( 0 \) is the zero morphism on \( G^{(r)} \), it follows by specialising under \( t \mapsto 0 \) that \( g_i(0, a_1, \ldots, a_d) = 0 \). This means \( v_i = g_i(t, a_1, \ldots, a_d) \) can be expanded as a finite sum

\[
v_i = \sum_{j \geq 1} t^j h_{i,j}(a_1, \ldots, a_d),
\]

where the \( h_{i,j}(a_1, \ldots, a_d) \) are polynomial expressions with coefficients in \( I \). As the zero vector times any scalar is zero, it follows by specialising under \( \sigma^* \) that \( g_i(t, 0, \ldots, 0) = 0 \). It follows that the constant term of each \( h_{i,j} \) is zero. From this, and the fact that \( I^2 = 0 \), we get the important consequence that

\[
g_i(t, b_1, \ldots, b_d) = 0 \text{ for all } b_1, \ldots, b_d \in I.
\]

**Step 6:** The kernel of the map \( G^{(r)}(R) \to G^{(r)}(R/I) \).

**Lemma.** Let \( \Psi_{(b_1, \ldots, b_d)} : A \to R \) be the \( R \)-algebra homomorphism defined in terms of the generators by \( \Psi_{(b_1, \ldots, b_d)}(a_k) = b_k \). Let \( \Psi : I^d \to \text{Hom}_{R \text{-alg}}(A, R) \) be the set-map defined by \( (b_1, \ldots, b_d) \mapsto (\Psi_{(b_1, \ldots, b_d)} : A \to R) \). Then \( \Psi \) is a homomorphism of \( R \)-modules, where the \( R \)-module structure on \( \text{Hom}_{R \text{-alg}}(A, R) \) is defined by its identification with the \( R \)-module \( G^{(r)}(R) \).

The map \( \Psi \) is injective, and its image is the \( R \)-submodule \( G^{(r)}(\phi) \subset G^{(r)}(R) \), where \( \phi : \text{Spec}(R/I) \leftrightarrow \text{Spec } R \) is the inclusion.
Proof. For any \((b_1, \ldots, b_d)\) and \((c_1, \ldots, c_d)\) in \(I^d\), we have
\[
(\Psi_{(b_1, \ldots, b_d)} + \Psi_{(c_1, \ldots, c_d)})(a_i) = (\Psi_{(b_1, \ldots, b_d)} \otimes \Psi_{(c_1, \ldots, c_d)})(\alpha^*(a_i)) = b_1 + c_1 + f_1(b_1, \ldots, b_d, c_1, \ldots, c_d)
\]
by substituting for \(\alpha^*(a_i)\)
\[
= b_1 + c_1 \text{ as } b_k, c_k \in I
\]
\[
= \Psi_{(b_1+c_1, \ldots, b_d+c_d)}(a_i).
\]
This shows the equality \(\Psi_{(b_1, \ldots, b_d)} + \Psi_{(c_1, \ldots, c_d)} = \Psi_{(b_1, \ldots, b_d) + (c_1, \ldots, c_d)}\), which means the map \(\Psi : I^d \rightarrow G^{(r)}(R)\) is additive.

For any \(\lambda \in R\), let \(f_\lambda : R[t] \rightarrow R\) be the \(R\)-algebra homomorphism defined by \(f_\lambda(t) = \lambda\). Then for any \((b_1, \ldots, b_d)\) in \(I^d\) we have
\[
(\lambda \cdot \Psi_{(b_1, \ldots, b_d)})(a_i) = (f_\lambda \otimes \Psi_{(b_1, \ldots, b_d)})(\mu^*(a_i)) = (f_\lambda \otimes \Psi_{(b_1, \ldots, b_d)})(ta_i + g(t, a_1, \ldots, a_d)) = \lambda b_1 + g_\lambda(t, b_1, \ldots, b_d) = \lambda b_1 \text{ as } b_k \in I
\]
\[
= \Psi_{(\lambda b_1, \ldots, \lambda b_d)}(a_i).
\]
This shows the equality \(\lambda \cdot \Psi_{(b_1, \ldots, b_d)} = \Psi_{\lambda b_1, \ldots, b_d}\), hence the map \(\Psi : I^d \rightarrow G^{(r)}(R)\) preserves scalar multiplication. This completes the proof that \(\Psi : I^d \rightarrow G^{(r)}(R)\) is a homomorphism of \(R\)-modules.

The map \(\Psi\) is clearly injective. The map \(G^{(r)}(\phi) : G^{(r)}(R) \rightarrow G^{(r)}(R/I)\) is in algebraic terms the map \(\text{Hom}_{R_{\text{alg}}}(A, R) \rightarrow \text{Hom}_{R_{\text{alg}}}(A, R/I)\) induced by the quotient \(R \rightarrow R/I\). An element \(g \in \text{Hom}_{R_{\text{alg}}}(A, R/I)\) represents the zero element of \(G^{(r)}(R/I)\) exactly when \(g(a_i) = 0 \in R/I\) for the generators \(a_i\) of \(A\). Therefore \(f \in \text{Hom}_{R_{\text{alg}}}(A, R)\) is in the kernel of \(G^{(r)}(\phi)\) precisely when \(f(a_i) \in I\) for the generators \(a_i\). Putting \(b_j = f(a_i)\), we see that such an \(f\) is the same as \(\Psi_{(b_1, \ldots, b_d)}\).

This completes the proof of the lemma that \(\ker G^{(r)}(\phi) = I^d\).

In particular, as \(mI = 0\), it follows from the above lemma that \(\ker G^{(r)}(\phi)\) is annihilated by \(m\), so it is a vector space over \(R/m\), and its dimension as a vector space over \(R/m\) is \(d = \Phi(r)\), as by assumption \(I\) is a non-zero principal ideal.

The above determination of the dimension over \(R/m\) of the kernel of \(G^{(r)}(\phi)\) will contradict a more direct functorial description, which is as follows.

Step 7: Functorial description of kernel of \(G^{(r)}(R) \rightarrow G^{(r)}(R/I)\). As \(\mathcal{F}_{R/I}(r)\) is flat over \(R/I\), and as for \(r \geq m\) all higher direct images of \(\mathcal{F}(r)\) vanish, \(G^{(r)}(R/I)\) is isomorphic to the \(R/I\)-module \((R/I)^{d'}\) where \(d = \Phi(r)\). By Lemma 5, there exists \(m'' \geq m\) such that for \(r \geq m''\) the inclusion \(\phi : \text{Spec}(R/I) \hookrightarrow \text{Spec} R\) induces an isomorphism \(\phi^* \pi_* \mathcal{F}(r) \rightarrow \pi'_* \mathcal{F}'(r)\) where \(\pi' : P_{R/I}^m \rightarrow \text{Spec}(R/I)\) is the projection and \(\mathcal{F}'\) is the pull-back of \(\mathcal{F}\) under \(P_{R/I}^m \hookrightarrow P_R^m\). Note that \(G^{(r)}(R) = R^{(r)} \oplus (R/I)^{j(r)}\), and so for \(r \geq m''\) we get an induced decomposition
\[
G^{(r)}(R/I) = (R/I)^{j(r)} \oplus (R/I)^{j(r)}\]
such that the map \( G^{(r)}(\phi) : G^{(r)}(R) \rightarrow G^{(r)}(R/I) \) is the map

\[
(q, 1) : R^{i(r)} \oplus (R/I)^{j(r)} \rightarrow (R/I)^{i(r)} \oplus (R/I)^{j(r)},
\]

where \( q \) is the quotient map modulo \( I \). It follows that the kernel of \( G^{(r)}(\phi) \) is the \( R \)-module \( I^{i(r)} \oplus 0 \subset R^{i(r)} \oplus (R/I)^{j(r)} = G^{(r)}(R) \). This is a vector space over \( R/m \) of dimension \( i(r) < i(r) + j(r) = \Phi(r) \).

We thus obtain two different values for the dimension of the same vector space \( \ker G^{(r)}(\phi) = \ker G^{(r)}(\phi) \), which shows that our assumption that \( G^{(r)} \) is representable for arbitrarily large values of \( r \) is false. This completes the proof of Theorem 1.

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References


