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Moduli of pre-$\mathscr{D}$-modules, perverse sheaves and the Riemann-Hilbert morphism – I

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Contents

1 Introduction

This paper is devoted to the moduli problem for regular holonomic $\mathscr{D}$-modules and perverse sheaves on a complex projective variety X. It treats the case where the singular locus of the $\mathscr{D}$-module is a smooth divisor S and the characteristic variety is contained in the union of the zero section T_X^*X of the cotangent bundle of X and the conormal bundle $N_{S,X}^*$ of S in X (also denoted T_S^*X). The sequel (part II) will treat the general case of arbitrary singularities.

A moduli space for $\mathscr{O}$-coherent $\mathscr{D}$-modules on a smooth projective variety was constructed by Simpson [S]. These are vector bundles with integrable connections, and they are the simplest case of $\mathscr{D}$-modules. In this moduli construction, the requirement of semistability is automatically fulfilled by all the objects.

Next in order of complexity are the so called 'regular meromorphic connections'. These $\mathcal{D}$-modules can be generated by vector bundles with connections which have logarithmic singularities on divisors with normal crossing. These $\mathcal{D}$-modules are not $\mathcal{O}$-coherent, but are torsion free as $\mathcal{O}$-modules. A moduli scheme does not exist for these $\mathcal{D}$-modules themselves (see Sect. 1 of [N]), but it is possible to define a notion of stability and construct a moduli for vector bundles with logarithmic connections. This was done in [N]. Though many logarithmic connections give rise to the same meromorphic connection, the choice of a logarithmic connection is infinitesimally rigid if its residual eigenvalues do not differ by nonzero integers (see Sect. 5 of [N]).

In the present paper and its sequel, we deal with general regular holonomic $\mathcal{D}$-modules. Such modules are in general neither $\mathcal{O}$-coherent, nor $\mathcal{O}$-torsion free or pure dimensional. We define objects called pre-$\mathcal{D}$-modules, which play the same role for regular holonomic $\mathcal{D}$-modules that logarithmic connections played for regular meromorphic connections. We define a notion of (semi-)stability, and construct a moduli scheme for (semi-)stable pre-$\mathcal{D}$-modules with prescribed singularity stratification and other numerical data. We also construct a moduli scheme for perverse sheaves with prescribed singularity stratification and other numerical data on a nonsingular variety, and show that the Riemann-Hilbert correspondence defines an analytic morphism between (an open set of) the moduli of pre-$\mathcal{D}$-modules and the moduli of perverse sheaves.

The contents of this paper are as follows. Let X be a smooth projective variety, and let S be a smooth hypersurface on X. In Sect. 2, we define pre-$\mathcal{D}$-modules on (X,S) which may be regarded as $\mathcal{O}_X$-coherent descriptions of those regular holonomic $\mathcal{D}_X$-modules whose characteristic variety is contained in $T_X^*X \cup T_S^*X$. The pre-$\mathcal{D}$-modules form an algebraic stack in the sense of Artin, which is a property that does not hold for the corresponding $\mathcal{D}$-modules.

In Sect. 3, we define a functor from the pre-$\mathcal{D}$-modules to $\mathcal{D}$-modules (in fact we mainly use the presentation of holonomic $\mathcal{D}$-modules given by Malgrange [Mal], that we call Malgrange objects). This is a surjective functor, and though not injective, it has an infinitesimal rigidity property (see Proposition 3.9) which generalizes the corresponding result for meromorphic connections.

In Sect. 4, we introduce a notion of (semi-)stability for pre-$\mathcal{D}$-modules, and show that semistable pre-$\mathcal{D}$-modules with fixed numerical data form a moduli scheme.

Next, we consider perverse sheaves on X which are constructible with respect to the stratification $(X - S) \cup S$. These have finite descriptions through the work Verdier, recalled in Sect. 5.

We observe that these finite descriptions are objects which naturally form an Artin algebraic stack. Moreover, we show in Sect. 6 that S-equivalence classes (Jordan-Hölder classes) of finite descriptions with given numerical data form a coarse moduli space which is an affine scheme. Here, no hypothesis about stability is necessary.

In Sect. 7, we consider the Riemann-Hilbert correspondence. When a pre-$\mathcal{D}$-module has an underlying logarithmic connection for which residual eigenvalues do not differ by nonzero integers, we functorially associate to it a finite description, which is the finite description of the perverse sheaf associated to the corresponding $\mathscr{D}$-module by the Riemann-Hilbert correspondence from regular holonomic $\mathscr{D}$-modules to perverse sheaves. We show that this gives an analytic morphism of stacks from the analytic open subset of the stack (or moduli) of pre-$\mathscr{D}$-modules on (X,S) where the 'residual eigenvalues are good', to the stack (or moduli) of finite descriptions on (X,S).

In Sect. 8, we show that the above morphism of analytic stacks is in fact a spread (surjective local isomorphism) in the analytic category. We also show that it has removable singularities in codimension 1, that is, is can be defined outside codimension two on any parameter space which is smooth in codimension 1.

2 Pre-$\mathscr{D}$-modules

Let X be a nonsingular variety and let $S \subset X$ be a smooth divisor (reduced). Let $\mathscr{I}_S \subset \mathscr{O}_X$ be the ideal sheaf of S, and let $T_X[\log S] \subset T_X$ be the sheaf of all tangent vector fields on X which preserve $\mathscr{I}_S$. Let $\mathscr{D}_X[\log S] \subset \mathscr{D}_X$ be the algebra of all partial differential operators which preserve I_S; it is generated as an $\mathscr{O}_X$ algebra by $T_X[\log S]$.

The $\mathscr{I}_S$-adic filtration on $\mathscr{O}_X$ gives rise to a (decreasing) filtration of $\mathscr{D}_X$ as follows: for $k \in \mathbb{Z}$ define $V^k\mathscr{D}_X$ as the subsheaf of $\mathscr{D}_X$ whose local sections consist of operators P which satisfy $P \cdot \mathscr{I}_S^j \subset \mathscr{I}_S^{k+j}$ for all j. By construction, one has $\mathscr{D}_X[\log S] = V^0\mathscr{D}_X$ and every $V^k(\mathscr{D}_X)$ is a coherent $\mathscr{D}_X[\log S]$-module.

Let $p : N_{S,X} \to S$ denote the normal bundle of S in X. The graded ring $\mathrm{gr}_V \mathscr{D}_X$ is naturally identified with $p_*\mathscr{D}_{N_{S,X}}$. Its V-filtration (corresponding to the inclusion of S in $N_{S,X}$ as the zero section) is then split.

There exists a canonical section θ of the quotient ring $\mathscr{D}_X[\log S]/\mathscr{I}_S\mathscr{D}_X[\log S] = \mathrm{gr}_V^0 \mathscr{D}_X$, which is locally induced by $x\partial_x$, where x is a local equation for S. It is a central element in $\mathrm{gr}_V^0 \mathscr{D}_X$. This ring contains $\mathscr{O}_S$ as a subring and $\mathscr{D}_S$ as a quotient (one has $\mathscr{D}_S = \mathrm{gr}_V^0 \mathscr{D}_X/\theta\,\mathrm{gr}_V^0 \mathscr{D}_X$). One can identify locally on S the ring $\mathrm{gr}_V^0 \mathscr{D}_X$ with $\mathscr{D}_S[\theta]$.

A coherent $\mathrm{gr}_V^0 \mathscr{D}_X$-module on which θ acts by 0 is a coherent $\mathscr{D}_S$-module. The locally free rank one $\mathscr{O}_S$-module $\mathscr{N}_{S,X} = \mathscr{O}_X(S)/\mathscr{O}_X$ is a $\mathrm{gr}_V^0 \mathscr{D}_X$-module on which θ acts by -1.

Definition 2.1 *A* **logarithmic module** *on (X,S) will mean a sheaf of $\mathscr{D}_X[\log S]$-modules, which is coherent as an $\mathscr{O}_X$-module. A* **logarithmic connection** *on (X,S) will mean a logarithmic module which is coherent and torsion-free as an $\mathscr{O}_X$-module.*

Remark 2.2 It is known that when S is nonsingular, any logarithmic connection on (X,S) is locally free as an $\mathscr{O}_X$-module.

Definition 2.3 (Family of logarithmic modules) *Let $f : Z \to T$ be a smooth morphism of schemes. Let $Y \subset Z$ be a divisor such that $Y \to T$ is smooth. Let*

$T_{Z/T}[\log Y] \subset T_{X/Y}$ be the sheaf of germs of vertical vector fields which preserve the ideal sheaf of Y in $\mathcal{O}_Z$. This generates the algebra $\mathcal{D}_{Z/T}[\log Y]$. A family of logarithmic modules on Z/T is a $\mathcal{D}_{Z/T}[\log Y]$-module which is coherent as an $\mathcal{O}_Z$-module, and is flat over $\mathcal{O}_T$. When $f : Z \to T$ is the projection $X \times T \to T$, and $Y = S \times T$, we get a **family of logarithmic modules on (X, S) parametrized by T**.

Remark 2.4 The restriction to S of a logarithmic module is acted on by θ: for a logarithmic connection, this is the action of the residue of the connection, which is an $\mathcal{O}_S$-linear morphism.

Remark 2.5 There is an equivalence (restriction to S) between logarithmic modules supported on the reduced scheme S and $\mathrm{gr}_V^0 \mathcal{D}_X$-modules which are $\mathcal{O}_S$-coherent, (hence locally free $\mathcal{O}_S$-modules, since they are locally $\mathcal{D}_S$-modules). In the following, we shall not make any difference between the corresponding objects.

We give two definitions of pre-$\mathcal{D}$-modules. The two definitions are 'equivalent' in the sense that they give not only equivalent objects, but also equivalent families, or more precisely, the two definitions give rise to isomorphic algebraic stacks. To give a familier example of such an equivalence, this is the way how vector bundles and locally free sheaves are 'equivalent'. Note also that mere equivalence of objects is not enough to give equivalence of families — for example, the category of flat vector bundles is equivalent to the category of π_1 representations, but an algebraic family of flat bundles gives in general only a holomorphic (not algebraic) family of π_1 representations.

In their first version, pre-$\mathcal{D}$-modules are objects that live on X, and the functor from pre-$\mathcal{D}$-modules to $\mathcal{D}$-modules has a direct description in their terms. The second version of pre-$\mathcal{D}$-modules is more closely related to the Malgrange description of $\mathcal{D}$-modules and the Verdier description of perverse sheaves, and the Riemann-Hilbert morphism to the stack of perverse sheaves has direct description in its terms.

Definition 2.6 (Pre-$\mathcal{D}$-module of first kind on (X, S)) *Let X be a nonsingular variety, and $S \subset X$ a smooth divisor. A pre-$\mathcal{D}$-module $\mathbf{E} = (E, F, t, s)$ on (X, S) consists of the following data*

(1) E is a logarithmic connection on (X, S).

(2) F is a logarithmic module on (X, S) supported on the reduced scheme S (hence a flat connection on S).

(3) $t : (E|S) \to F$ and $s : F \to (E|S)$ are $\mathcal{D}_X[\log S]$ linear maps, which satisfies the following conditions:

(4) On $E|S$, we have $st = R$ where $R \in End(E|S)$ is the residue of E.

(5) On F, we have $ts = \theta_F$ where $\theta_F : F \to F$ is the $\mathcal{D}_X[\log S]$ linear endomorphism induced by any Eulerian vector field $x\partial/\partial x$.

If (E, F, t, s) and (E', F', t', s') are two pre-$\mathcal{D}$-modules, a morphism between them consists of $\mathcal{D}_X[\log S]$ linear morphisms $f_0 : E \to E'$ and $f_1 : F \to F'$ which commute with t, t' and with s, s'.

Remark 2.7 It follows from the definition of a pre-$\mathscr{D}$-module (E,F,t,s) that E and F are locally free on X and S respectively, and the vector bundle morphisms R, s and t all have constant ranks on irreducible components of S.

Example Let E be a logarithmic connection on (X,S). We can associate functorially to E the following three pre-$\mathscr{D}$-modules. Take F_1 to be the restriction of E to S as an $\mathscr{O}$-module. Let $t_1 = R$ (the residue) and $s_1 = 1_F$. Then $\mathbf{E}_1 = (E, F_1, t_1, s_1)$ is a pre-$\mathscr{D}$-module, which under the functor from pre-$\mathscr{D}$-modules to $\mathscr{D}$-modules defined later will give rise to the meromorphic connection corresponding to E. For another choice, take $F_2 = E|S$, $t_2 = 1_F$ and $s_2 = R$. This gives a pre-$\mathscr{D}$-module $\mathbf{E}_2 = (E, F_2, t_2, s_2)$ which will give rise to a $\mathscr{D}$-module which has nonzero torsion part when R is not invertible. For the third choice (which is in some precise sense the minimal choice), take F_3 to be the image vector bundle of R. Let $t_3 = R : (E|S) \to F_3$, and let $s_3 : F_3 \hookrightarrow (E|S)$. This gives a pre-$\mathscr{D}$-module $\mathbf{E}_3 = (E, F_3, t_3, s_3)$. We have functorial morphisms $\mathbf{E}_3 \to \mathbf{E}_2 \to \mathbf{E}_1$ of pre-$\mathscr{D}$-modules.

Definition 2.8 (Families of pre-$\mathscr{D}$-modules) *Let T be a complex scheme. A family $\mathbf{E}_T$ of pre-$\mathscr{D}$-modules on (X,S) parametrized by the scheme T, a morphism between two such families, and pullback of a family under a base change $T' \to T$ have obvious definitions (starting from definition of families of $\mathscr{D}_X[\log S]$-modules), which we leave to the reader. This gives us a fibered category PD of pre-$\mathscr{D}$-modules over the base category of $\mathbb{C}$ schemes. Let $\mathscr{P}\mathscr{D}$ be the largest (nonfull) subcategory of PD in which all morphisms are isomorphisms. This is a groupoid over $\mathbb{C}$ schemes.*

Proposition 2.9 *The groupoid $\mathscr{P}\mathscr{D}$ is an algebraic stack in the sense of Artin.*

Proof It can be directly checked that $\mathscr{P}\mathscr{D}$ is a sheaf, that is, descent and effective descent are valid for faithfully flat morphisms of parameter schemes of families of pre-$\mathscr{D}$-modules. Let Bun_X be the stack of vector bundles on X, and let Bun_S be the stack of vector bundles on S. Then $\mathscr{P}\mathscr{D}$ has a forgetful morphism to the product stack $Bun_X \times_{\mathbb{C}} Bun_S$. The later stack is algebraic and the forgetful morphism is representable, hence the desired conclusion follows.

Before giving the definition of pre-$\mathscr{D}$-modules of the second kind, we observe the following.

Remark 2.10 Let N be any line bundle on a smooth variety S, and let $\overline{N} = P(N \oplus \mathscr{O}_S)$ be its projective completion, with projection $\pi : \overline{N} \to S$. Let $S^\infty = P(N)$ be the divisor at infinity. For any logarithmic connection E on $(\overline{N}, S \cup S^\infty)$, the restriction $E|S$ is of course a $\mathscr{D}_{\overline{N}}[\log S \cup S^\infty]$-module. But conversely, for any $\mathscr{O}$-coherent $\mathscr{D}_{\overline{N}}[\log S \cup S^\infty]$-module F scheme theoretically supported on S, there is a natural structure of a logarithmic connection on $(\overline{N}, S \cup S^\infty)$ on its pullup $\pi^*(F)$ to $\overline{N}$. The above correspondence is well behaved in families, giving an isomorphism between the algebraic stack of $\mathscr{D}_{\overline{N}}[\log S \cup S^\infty]$-modules F supported on S and the algebraic stack of logarithmic connections E on $(\overline{N}, S \cup$

S^{∞}) such that the vector bundle E is trivial on the fibers of $\pi : \overline{N} \to S$. The functors $\pi^*(-)$ and $(-)|S$ are fully faithful.

Remark 2.11 Let $S \subset X$ be a smooth divisor, and let $N = N_{S,X}$ be its normal bundle. Then the following are equivalent in the sense that we have fully faithful functors between the corresponding categories, which give naturally isomorphic stacks.

(1) $\mathscr{D}_X[\log S]$-modules which are scheme theoretically supported on S.

(2) $\mathscr{D}_N[\log S]$-modules which are scheme theoretically supported on S.

(3) $\mathscr{D}_{\overline{N}}[\log S \cup S^{\infty}]$-modules which are scheme theoretically supported on S.

The equivalence between (2) and (3) is obvious, while the equivalence between (1) and (2) is obtained as follows. The Poincaré residue map *res* : $\Omega^1_X[\log S] \to \mathscr{O}_S$ gives the following short exact sequence of $\mathscr{O}_S$-modules.

$$0 \to \Omega^1_S \to \Omega^1_X[\log S]|S \to \mathscr{O}_S \to 0$$

By taking duals, this gives

$$0 \to \mathscr{O}_S \to T_X[\log S]|S \to T_S \to 0.$$

It can be shown that there exists a canonical isomorphism $T_X[\log S]|S \to T_N[\log S]|S$ which makes the following diagram commute, where the rows are exact.

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{O}_S & \to & T_N[\log S]|S & \to & T_S & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & \mathscr{O}_S & \to & T_X[\log S]|S & \to & T_S & \to & 0 \end{array}$$

Remarks 2.12 (1) Observe that the element θ is just the image of 1 under the map $\mathscr{O}_S \to T_X[\log S]|S$.

(2) Using the notations of the beginning of this section, one can identify the ring $\pi_*\mathscr{D}_{\overline{N}}[\log S \cup S^{\infty}]$ with $\mathrm{gr}^0_V \mathscr{D}_X$. Hence θ is a global section of $\mathscr{D}_{\overline{N}}[\log S \cup S^{\infty}]$.

We now make the following important definition.

Definition 2.13 (Specialization of a logarithmic module) *Let E be a logarithmic module on (X, S). Then the specialization $\mathrm{sp}_S E$ will mean the logarithmic connection $\pi^*(E|S)$ on $(\overline{N_{S,X}}, S \cup S^{\infty})$.*

Now we are ready to define the second version of pre-$\mathscr{D}$-modules.

Definition 2.14 (Pre-$\mathscr{D}$-modules of the second kind on (X, S)) *A pre-$\mathscr{D}$-module (of the second kind) $\mathbf{E} = (E_0, E_1, c, v)$ on (X, S) consists of the following data*

(1) E_0 is a logarithmic connection on (X, S),

(2) E_1 is a logarithmic connection on $(\overline{N_{S,X}}, S \cup S^{\infty})$,

(3) $c : \mathrm{sp}_S E_0 \to E_1$ and $v : E_1 \to \mathrm{sp}_S E_0$ are $\mathscr{D}_{\overline{N_{S,X}}}[\log S \cup S^{\infty}]$-linear maps,

which satisfies the following conditions:

(4) on $\mathrm{sp}_S E_0$, *we have* $cv = \theta_{\mathrm{sp}_S E_0}$,

(5) on E_1, *we have* $vc = \theta_{E_1}$,

(6) the vector bundle underlying E_1 *is trivial in the fibers of* $\pi : \overline{N_{S,X}} \to S$ *(that is,* E_1 *is locally over* S *isomorphic to* $\pi^*(E_1|S)$*).*

If (E_0, E_1, c, v) *and* (E_0', E_1', c', v') *are two pre-*$\mathscr{D}$*-modules, a* **morphism** *between them consists of* $\mathscr{D}_X[\log S]$ *linear morphisms* $f_0 : E_0 \to E_0'$ *and* $f_1 : E_1 \to E_1'$ *such that* $\mathrm{sp}_S f_0$ *and* f_1 *commute with* v, v' *and with* c, c'.

Definition 2.15 (Families of pre-$\mathscr{D}$-modules of the second kind) *Let* T *be a complex scheme. A family* $\mathbf{E}_T$ *of pre-*$\mathscr{D}$*-modules on* (X, S) *parametrized by the scheme* T*, a morphism between two such families, and pullback of a family under a base change* $T' \to T$ *have obvious definitions which we leave to the reader. This gives us a fibered category* PM *of pre-*$\mathscr{D}$*-modules of second kind over the base category of* $\mathbb{C}$ *schemes.*

Proposition 2.16 *The functor which associates to each family of pre-*$\mathscr{D}$*-module* (E_0, E_1, c, v) *of the second kind parametrized by* T *the family of pre-*$\mathscr{D}$*-module of the first kind* $(E_0, E_1|S, c|S, v|S)$ *is an equivalence of fibered categories.*

Proof This follows from Remarks 2.10 and 2.11 above.

3 From pre-$\mathscr{D}$-modules to $\mathscr{D}$-modules

In this section we first recall the description of regular holonomic $\mathscr{D}$-modules due to Malgrange [Mal] and we associate a 'Malgrange object' to a pre-$\mathscr{D}$-module of the second kind (Proposition 3.6), which has good residual eigenvalues (definition 3.4), each component of S do not differ by positive integers. Having such a direct description of the Malgrange object enables us to prove that every regular holonomic $\mathscr{D}$-module with characteristic variety contained in $T_X^*X \cup T_S^*X$ arises from a pre-$\mathscr{D}$-module (Corollary 3.8), and also helps us to prove an infinitesimal rigidity property for the pre-$\mathscr{D}$-modules over a given $\mathscr{D}$-module (Proposition 3.9).

Malgrange objects

Regular holonomic $\mathscr{D}$-modules on X whose characteristic variety is contained in $T_X^*X \cup T_S^*X$ have an equivalent presentation due to Malgrange and Verdier, which we now describe.

Let us recall the definition of the **specialization** $\mathrm{sp}_S(M)$ of a regular holonomic $\mathscr{D}_X$-module M: fix a section σ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ and denote A its image; every such module admits a unique (decreasing) filtration $V^k M$ $(k \in \mathbb{Z})$ by $\mathscr{D}_X[\log S]$-submodules which is good with respect to $V\mathscr{D}_X$ and satisfies the following property: on $\mathrm{gr}_V^k M$, the action of θ admits a minimal polynomial all of whose roots are in $A + k$. Then by definition one puts

$\mathrm{sp}_S M = \oplus_{k\in\mathbf{Z}} \mathrm{gr}_V^k M$. One has $(\mathrm{sp}_S M)[*S] = \mathrm{sp}_S(M[*S]) = (\mathrm{gr}_V^{\geq k} M)[*S]$ for all $k \geq 1$, if we put $\mathrm{gr}_V^{\geq k} M = \oplus_{\ell\geq k} \mathrm{gr}_V^\ell M$. The $p_*\mathscr{D}_{N_S X}$-module $\mathrm{sp}_S M$ does not depend on the choice of σ (if one forgets its gradation).

If θ acts in a locally finite way on a $\mathrm{gr}_V^0 \mathscr{D}_X$ or a $p_*\mathscr{D}_{N_{S,X}}$-module, we denote Θ the action of $\exp(-2i\pi\theta)$.

Given a regular holonomic $\mathscr{D}_X$-module, we can functorially associate to it the following modules:

(1) $M[*S] = \mathscr{O}_X[*S] \otimes_{\mathscr{O}_X} M$ is the S-localized $\mathscr{D}_X$-module; it is also regular holonomic;

(2) $\mathrm{sp}_S M$ is the specialized module; this is a regular holonomic $p_*\mathscr{D}_{N_S X}$-module, which is also **monodromic**, i.e. the action of θ on each local section is locally (on S) finite.

The particular case that we shall use of the result proved in [Mal] is then the following:

Theorem 3.1 *There is an equivalence between the category of regular holonomic $\mathscr{D}_X$-modules and the category which objects are triples $(\mathscr{M}, \overline{M}, \alpha)$, where $\mathscr{M}$ is a S-localized regular holonomic $\mathscr{D}_X$-module, $\overline{M}$ is a monodromic regular holonomic $p_*\mathscr{D}_{N_S X}$-module and α is an isomorphism (of localized $p_*\mathscr{D}_{N_S X}$-modules) between* $\mathrm{sp}_S \mathscr{M}[*S]$ *and* $\overline{M}[*S]$.

In fact, the result of [Mal] does mention neither holonomicity nor regularity. Nevertheless, using standard facts of the theory, one obtains the previous proposition. Regularity includes here regularity at infinity, i.e. along S^∞. This statement can be simplified when restricted to the category of regular holonomic $\mathscr{D}$-modules which characteristic variety is contained in the union $T_X^*X \cup T_S^*X$.

Definition 3.2 *A* **Malgrange object** *on (X,S) is a tuple (M_0, M_1, C, V) where*

(1) M_0 is an S-localized regular holonomic $\mathscr{D}_X$-module which is a regular meromorphic connection on X with poles on S,

(2) M_1 is a S-localized monodromic regular holonomic $p_*\mathscr{D}_{N_S X}$-module which is a regular meromorphic connection on $N_{S,X}$ (or $\overline{N_{S,X}}$) with poles on S (or on $S \cup S^\infty$),

(3) C, V are morphisms (of $p_*\mathscr{D}_{N_{S,X}}$-modules) between $\mathrm{sp}_S M_0$ *and M_1 satisfying $VC = \Theta - \mathrm{id}$ on* $\mathrm{sp}_S M_0$ *and $CV = \Theta - \mathrm{id}$ on M_1.*

The morphisms between two Malgrange objects are defined in an obvious way, making them an abelian category.

The previous result can be translated in the following way, using [Ve]:

Corollary 3.3 *There is an equivalence between the category of regular holonomic $\mathscr{D}$-modules which characteristic variety is contained in $T_X^*X \cup T_S^*X$ and the category of Malgrange objects on (X,S).*

From pre-$\mathscr{D}$-modules to Malgrange objects

Definition 3.4 *(1) We say that a logarithmic connection F on (X,S) has* **good residual eigenvalues** *if for each connected component S_a of the divisor S, the residual eigenvalues $(\lambda_{a,k})$ of F along S_a do not include a pair $\lambda_{a,i}, \lambda_{a,j}$ such that $\lambda_{a,i} - \lambda_{a,j}$ is a nonzero integer.*

(2) We say that a pre-$\mathscr{D}$-module $\mathbf{E} = (E_0, E_1, s, t)$ has **good residual eigenvalues** *if the logarithmic connection E_0 has good residual eigenvalues as defined above.*

We now functorially associate a Malgrange object $\mathbf{M} = \eta(\mathbf{E}) = (M_0, M_1, C, V)$ to each pre-$\mathscr{D}$-module $\mathbf{E} = (E_0, E_1, c, v)$ on (X, S) with E_0 having good residual eigenvalues.

Remark 3.5 By definition of a pre-$\mathscr{D}$-module it follows that the nonzero eigenvalues of θ_a on $E_0|S_a$ (the residue along S_a) are the same as the nonzero eigenvalues of θ_a on $E_{1,a}$.

Proposition 3.6 (The Malgrange object associated to a pre-$\mathscr{D}$-module with good residual eigenvalues) *Let $\mathbf{E} = (E_0, E_1, c, v)$ be a pre-$\mathscr{D}$-module on (X,S) of the second kind (definition 2.14), such that E_0 has good residual eigenvalues. Let $\eta(\mathbf{E}) = (M_0, M_1, C, V)$ where*

(1) $M_0 = E_0[*S]$,

(2) $M_1 = E_1[*S]$,

(3) $C = c \circ \dfrac{e^{-2i\pi\theta_{E_0}} - 1}{\theta_{E_0}}$.

(4) $V = v$

Then $\eta(\mathbf{E})$ is a Malgrange object, and η is functorial in an obvious way.

Proof Because E_0 has good residual eigenvalues, one can use the filtration $V^k E_0[*S] = I_S^k E_0 \subset E_0[*S]$ in order to compute $\mathrm{sp}_S E_0[*S]$. It follows that the specialization of $E_0[*S]$ when restricted to $N_{S,X} - S$ is canonically isomorphic to the restriction of $\mathrm{sp}_S E_0 = \pi^*(E_0|S)$ to $N_{X,S} - S$.

Essential surjectivity

Proposition 3.7 *Every Malgrange object (M_0, M_1, C, V) on (X, S) can be obtained in this way from a pre-$\mathscr{D}$-module.*

Proof This follows from [Ve]: one chooses Deligne lattices in M_0 and M_1. One uses the fact that every $\mathscr{D}$-linear map between holonomic $\mathscr{D}$-modules is compatible with the V-filtration, so sends the specialized Deligne lattice of M_0 to the one of M_1. Moreover, the map v can be obtained from V because the only integral eigenvalue of θ on the Deligne lattice is 0, so $\dfrac{e^{-2i\pi\theta} - 1}{\theta}$ is invertible on it.

The previous two propositions give the following.

Corollary 3.8 *The functor from pre-$\mathscr{D}$-modules on (X, S) to regular holonomic $\mathscr{D}$-modules on X with characteristic variety contained in $T_X^*X \cup T_S^*X$ is essentially surjective.*

Infinitesimal rigidity

For a regular holonomic $\mathscr{D}$-module $\mathbf{M}$ with characteristic variety $T_X^*X \cup T_S^*X$, there exist several nonisomorphic pre-$\mathscr{D}$-modules $\mathbf{E}$ which give rise to the Malgrange object associated to $\mathbf{M}$. However, we have the following infinitesimal rigidity result, which generalizes the corresponding results in [N].

Proposition 3.9 (Infinitesimal rigidity) *Let $T = \operatorname{Spec} \dfrac{\mathbb{C}[\epsilon]}{(\epsilon^2)}$. Let $\mathbf{E}_T$ be a family of pre-$\mathscr{D}$-modules on (X, S) parametrized by T. Let the associated family $\mathbf{M}_T$ of $\mathscr{D}$-modules on X be constant (pulled back from X). Let $\mathbf{E}$ (which is the specialization at $\epsilon = 0$) be of the form $\mathbf{E} = (E, F, s, t)$ where along any component of S, no two distinct eigenvalues of the residue of the logarithmic connection E differ by an integer. Then the family $\mathbf{E}_T$ is also constant.*

Proof By [N], the family E_T is constant, as well as the specialization $\mathrm{sp}_S\, E_T$. As a consequence, the residue θ_{E_T} is constant. Let us now prove that the family F_T is constant.

Let S_a be a component of S along which the only possible integral eigenvalue of θ_E is 0. Then it is also the only possible integral eigenvalue of θ_F along S_a because the generalized eigenspaces of θ_E and θ_F corresponding to a nonzero eigenvalue are isomorphic by s and t (see remark 3.5). We also deduce from [N] that F_T is constant as a logarithmic module along this component.

Assume now that 0 is not an eigenvalue of θ_E along S_a but is an eigenvalue of θ_F along this component. Then θ_F may have two distinct integral eigenvalues, one of which is 0. Note that, in this case, θ_E is an isomorphism (along S_a), as well as θ_{E_T} which is obtained by pullback from θ_E. It follows that on S_a we have an isomorphism $F_T \simeq E_T|S_a \oplus \operatorname{Ker} \theta_{F_T}$. Consequently $\operatorname{Ker} \theta_{F_T}$ is itself a family. It is enough to show that this family is constant. But the corresponding meromorphic connection on $N_{S,X} - S$ is constant, being the cokernel of the constant map $C_T : M_{0T} \to M_{1T}$. We can then apply Proposition 5.3 of [N] because the only eigenvalue on $\operatorname{Ker} \theta_F$ is 0.

The maps s_T and t_T are constant if and only if for each component S_a of S and for some point $x_a \in S_a$ their restriction to $F_T|x_a \times T$ and $E_T|x_a \times T$ are constant. This fact is a consequence of the following lemma.

Lemma 3.10 *Let E and F be finite dimensional complex vector spaces, and let $\theta_E \in End(E)$ and $\theta_F \in End(F)$ be given. Let $V \subset W = Hom(F, E) \times Hom(E, F)$ be the closed subscheme consisting of (s, t) with $st = \theta_E$ and $ts = \theta_F$. Let $\phi : W \to W$ be the holomorphic map defined by*

$$\phi(s,t) = (s, t\frac{e^{st}-1}{st}).$$

Then the differential $d\phi$ is injective on the Zariski tangent space to V at any closed point (s,t).

Proof Let (a,b) be a tangent vector to V at (s,t). Then by definition of V, we must have $at + sb = 0$ and $ta + bs = 0$. Using $at + sb = 0$, we can see that $d\phi(a,b) = (a, bf(st))$ where f is the entire function on $End(E)$ defined by the power series $(e^x - 1)/x$. Suppose $(a, bf(st)) = 0$. Then $a = 0$ and so the condition $ta + bs = 0$ implies $bs = 0$. As the constant term of the power series f is 1 and as $bs = 0$, we have $bf(st) = b$. Hence $b = 0$, and so $d\phi$ is injective.

4 Semistability and moduli for pre-$\mathscr{D}$-modules.

In order to construct a moduli scheme for pre-$\mathscr{D}$-modules, one needs a notion of semistability. This can be defined in more than one way. What we have chosen below is a particularly simple and canonical definition of semistability. (In an earlier version of this paper, we had employed a definition of semistability in terms of parabolic structures, in which we had to fix the ranks of $s : E_1 \to E_0|S$ and $t : E_0|S \to E_1$ and a set of parabolic weights.)

Let S_a be the irreducible components of the smooth divisor $S \subset X$. For a pre-$\mathscr{D}$-module $\mathbf{E} = (E_0, E_1, s, t)$, we denote by E_a the restriction of E_1 to S_a, and we denote by s_a and t_a the restrictions of s and t.

Definition of semistability

We fix an ample line bundle on X, and denote the resulting Hilbert polynomial of a coherent sheaf F by $p(F,n)$. For constructing a moduli, we fix the Hilbert polynomials of E_0 and E_a, which we denote by $p_0(n)$ and $p_a(n)$. Recall (see [S]) that an $\mathcal{O}_X$-coherent $\mathscr{D}_X[\log S]$-module F is by definition **semistable** if it is pure dimensional, and for each $\mathcal{O}_X$ coherent $\mathscr{D}_X[\log S]$ submodule F', we have the inequality $p(F',n)/rank(F') \leq p(F,n)/rank(F)$ for large enough n. We call $p(F,n)/rank(F)$ the **normalized Hilbert polynomial** of F.

Definition 4.1 *We say that the pre-$\mathscr{D}$-module* $\mathbf{E}$ *is* **semistable** *if the $\mathscr{D}_X[\log S]$-modules E_0 and E_a are semistable.*

Remarks 4.2 (1) It is easy to prove that the semistability of the $\mathscr{D}_X[\log S]$-module E_a is equivalent to the semistability of the logarithmic connection $\pi_a^*(E_a)$ on $P(N_{S_a,X} \oplus 1)$ with respect to a natural choice of polarization.

(2) When X is a curve, a pre-$\mathscr{D}$-module $\mathbf{E}$ is semistable if and only if the logarithmic connection E_0 on (X,S) is semistable, for then E_1 is always semistable.

(3) Let the dimension of X be more than one. Then even when a pre-$\mathscr{D}$-module $\mathbf{E}$ is a pre meromorphic connection (equivalently, when $s : E_1 \to E_0|S$

is an isomorphism), the definition of semistability of $\mathbf{E}$ does not reduce to the semistability of the underlying logarithmic connection E_0 on (X, S). This is to be expected because we do not fix the rank of s (or t) when we consider families of pre-$\mathscr{D}$-modules. Also note that even in dimension one, meromorphic connections are not a good subcategory of the abelian category of all regular holonomic $\mathscr{D}$-modules with characteristic variety contained in $T_X^*X \cup T_S^*X$, in the sense that a submodule or a quotient module of a meromorphic connection is not necessarily a meromorphic connection.

Boundedness and local universal family

We let the index i vary over 0 and over the a.

Proposition 4.3 (Boundedness) *Semistable pre-$\mathscr{D}$-modules with given Hilbert polynomials p_i form a bounded set, that is, there exists a family of pre-$\mathscr{D}$-modules parametrized by a noetherian scheme of finite type over $\mathbb{C}$ in which each semistable pre-$\mathscr{D}$-module with given Hilbert polynomials occurs.*

Proof This is obvious as each E_i (where $i = 0, a$) being semistable with fixed Hilbert polynomial, is bounded.

Next, we construct a local universal family. By boundedness, there exists a positive integer N such that for all $n \geq N$, the sheaves $E_0(N)$ and $E_1(N)$ are generated by global sections and have vanishing higher cohomology. Let $\Lambda = D_X[\log S]$. Let $\mathscr{O}_X = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda$ be the increasing filtration of Λ by the order of the differential operators. Note that each Λ_k is an $\mathscr{O}_X$ bimodule, coherent on each side. Let r be a positive integer larger than the ranks of the E_i. Let Q_i be the quot scheme of quotients $q_i : \Lambda_r \otimes \mathscr{O}_X(-N)^{p_i(N)} \twoheadrightarrow E_i$ where the right $\mathscr{O}_X$-module structure on Λ_r is used for making the tensor product. Note that $G_i = PGL(p_i(N))$ has a natural action on Q_i. Simpson defines a locally closed subscheme $C_i \subset Q_i$ which is invariant under G_i, and a local universal family E of Λ-modules parametrized by C_i with the property that for two morphisms $T \to C_i$, the pull back families are isomorphic over an open cover $T' \to T$ if and only if the two morphisms define T' valued points of C_i which are in a common orbit of $G_i(T')$.

On the product $C_0 \times C_a$, consider the linear schemes A_a and B_a which respectively correspond to $Hom_\Lambda(E_1, E_0)$ and $Hom_\Lambda(E_0, E_1)$ (see Lemma 2.7 in [N] for the existence and universal property of such linear schemes). Let F_a be the fibered product of A_a and B_a over $C_0 \times C_a$. Let H_a be the closed subscheme of F_a where the tuples (q_0, q_1, t, s) satisfy $st = \theta$ and $ts = \theta$. Finally let H be the fibered product of the pullbacks of the H_a to $C = C_0 \times \prod_a C_a$. Note that H parametrizes a tautological family of pre-$\mathscr{D}$-modules on (X, S) in which every semistable pre-$\mathscr{D}$-module with given Hilbert polynomials occurs.

The group

$$\mathscr{G} = G_0 \times \prod_a (G_a \times GL(1))$$

has a natural action on H, with

$$(q_0, q_a, t_a, s_a) \cdot (g_0, g_a, \lambda_a) = (q_0 g_0, q_a g_a, (1/\lambda_a) t_a, \lambda_a s_a)$$

It is clear from the definitions of H and this action that two points of H parametrise isomorphic pre-$\mathscr{D}$-modules if and only if they lie in the same G orbit.

The morphism $H \to C \times \prod_a C_a$ is an affine morphism which is $\mathscr{G}$-equivariant, and by Simpson's construction of moduli for Λ-modules, the action of $\mathscr{G}$ on $C \times \prod_a C_a$ admits a good quotient in the sense of geometric invariant theory. Hence a good quotient $H//\mathscr{G}$ exists by Ramanathan's lemma (see Proposition 3.12 in [Ne]), which by construction and universal properties of good quotients is the coarse moduli scheme of semistable pre-$\mathscr{D}$-modules with given Hilbert polynomials.

Note that under a good quotient in the sense of geometric invariant theory, two different orbits can in some cases get mapped to the same point (get identified in the quotient). In the rest of this section, we determine what are the closed points of the quotient $H//\mathscr{G}$.

Remark 4.4 For simplicity of notation, we assume in the rest of this section that S has only one connected component. It will be clear to the reader how to generalize the discussion when S has more components.

Reduced modules

Assuming for simplicity that S has only one connected component, so that $\mathscr{G} = \mathscr{H} \times GL(1)$ where $H = G_0 \times G_1$, we can construct the quotient $H//\mathscr{G}$ in two steps: first we go modulo the factor $GL(1)$, and then take the quotient of $R = H//GL(1)$ by the remaining factor $\mathscr{H}$. The following lemma is obvious.

Lemma 4.5 *Let T be a scheme of finite type over k, and let $V \to T$ and $W \to T$ be linear schemes over T. Let $V \times W$ be their fibered product (direct sum) over T, and let $V \otimes W$ be their tensor product. Let $\phi : V \times W \to V \otimes W$ be the tensor product morphism. Then its schematic image $D \subset V \otimes W$ is a closed subscheme which (i) parametrizes all decomposable tensors, and (ii) base changes correctly. Let $GL(1,k)$ act on $V \times W$ by the formula $\lambda \cdot (v, w) = (\lambda v, (1/\lambda) w)$. Then $\phi : V \times W \to D$ is a good quotient for this action.*

Proof The statement is local on the base, so we can assume that (i) the base T is an affine scheme, and (ii) both the linear schemes are closed linear subschemes of trivial vector bundles on the base, that is, $V \subset A^m_T$ and $W \subset A^n_T$ are subschemes defined respectively by homogeneous linear equations $f_p(x_i) = 0$ and $g_q(y_j) = 0$ in the coordinates x_i on A^m_T and y_j on A^n_T. Let $z_{i,j}$ be the coordinates on A^{mn}_T, so that the map $\otimes : A^m_T \times_T A^n_T \to A^{mn}$ sends $(x_i, y_j) \mapsto (z_{i,j})$ where $z_{i,j} = x_i y_j$. Its schematic image is the subscheme C of A^{mn}_T defined by the equations $z_{a,b} z_{c,d} - z_{a,d} z_{b,c} = 0$, that is, the matrix $(z_{i,j})$ should have rank < 2. Take

D to be the subscheme of C defined by the equations $f_p(z_{1,j}, \ldots, z_{m,j}) = 0$ and $g_q(z_{i,1}, \ldots, z_{i,n}) = 0$. Now the lemma 4.5 follows trivially from this local coordinate description.

The above lemma implies the following. To get the quotient $H//GL(1)$, we just have to replace the fibered product $A \times B$ over $C_0 \times C_1$ by the closed subscheme $Z \subset D \subset A \otimes B$, where D is the closed subscheme consisting of decomposable tensors u, and Z is the closed subscheme of D defined as follows. Let μ_0 and μ_1 be the canonical morphisms from $A \otimes B$ to the linear schemes representing $End_\Lambda(E_0|S)$ and $End_\Lambda(E_1)$ respectively. Then Z is defined to consist of all u such that $\mu_0(u) = \theta \in End_\Lambda(E_0|S)$ and $\mu_1(u) = \theta \in End_\Lambda(E_1)$. There is a canonical $GL(1)$ quotient morphism $A \times B \to D$ over $C_0 \times C_1$, which sends $(s,t) \mapsto u = s \otimes t$. These give the $GL(1)$ quotient map $H \to Z$. Note that the map $H \to C_0 \times C_1$ is $\mathscr{G}$ equivariant, and the action of $GL(1)$ on $C_0 \times C_1$ is trivial, so we get a $\mathscr{H}$-equivariant map $Z \to C_0 \times C_1$.

In order to describe the identifications brought about by the above quotient, we make the following definition.

Definition 4.6 *A* **reduced module** *is a tuple (E_0, E_1, u) where E_0 and E_1 are as in a pre-$\mathscr{D}$-module, and $u \in Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1)$ is a decomposable tensor, such that the canonical maps $\mu_0 : Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1) \to End_\Lambda(E_0|S)$ and $\mu_1 : Hom_\Lambda(E_1, E_0|S) \otimes Hom_\Lambda(E_0, E_1) \to End_\Lambda(E_1)$, both map u to the endomorphism θ of the appropriate module. In other words, there exist s and t such that (E_0, E_1, s, t) is a pre-$\mathscr{D}$-module, and $u = s \otimes t$. We say that the reduced module $(E_0, E_1, s \otimes t)$ is the* **associated reduced module** *of the pre-$\mathscr{D}$-module (E_0, E_1, s, t). Moreover, we say that a reduced module is* **semistable** *if it is associated to a semistable pre-$\mathscr{D}$-module.*

Lemma 4.7 *Let V and W are two vector spaces, $v, v' \in V$ and $w, w' \in W$, then*

(1) If $v \otimes w = 0$ then $v = 0$ or $w = 0$.

(2) If $v \otimes w = v' \otimes w' \neq 0$, then there exists a scalar $\lambda \neq 0$ such that $v = \lambda v'$ and $w = (1/\lambda) w'$.

Remark 4.8 The above lemma shows that if $\mathbf{E}$ and $\mathbf{E}'$ are two non-isomorphic pre-$\mathscr{D}$-modules whose associated reduced modules are isomorphic, then we must have $s \otimes t = s' \otimes t' = 0$. In particular, θ will act by zero on $E_0|S$ and also on E_1 for such pre-$\mathscr{D}$-modules as $st = 0$ and $ts = 0$.

S-equivalence and stability

Definition 4.9 *By a* **filtration** *of a logarithmic connection E we shall mean an increasing filtration E_p indexed by $\mathbb{Z}$ by subvector bundles which are logarithmic connections. Similarly, a filtration on a $\mathscr{D}_X[\log S]$-module F supported on S will mean a filtration of the vector bundle $F|S$ by subbundles F_p which are $\mathscr{D}_X[\log S]$-submodules. A filtration of a pre-$\mathscr{D}$-module (E_0, E_1, s, t) is an increasing filtration $(E_i)_p$ of the logarithmic connection E_i $(i = 0, 1)$ such that s and t are filtered* *morphisms with respect to the specialized filtration of* E_0 *and the filtration of* E_1. *A filtration of a reduced module* (E_0, E_1, u), *with* $u = s \otimes t$ *where we take* $s = 0$ *and* $t = 0$ *if* $u = 0$, *is a filtration of the pre-*$\mathscr{D}$*-module* (E_0, E_1, s, t). *We shall always assume that this filtration is exhaustive, that is,* $(E_i)_p = 0$ *for* $p \ll 0$ *and* $(E_i)_p = E_i$ *for* $p \gg 0$. *A filtration is* **nontrivial** *if some* $(E_i)_p$ *is a proper subbundle of* E_i *for* $i = 0$ *or* 1.

For a filtered pre-$\mathscr{D}$-module (or reduced module), each step of the filtration as well as the graded object are pre-$\mathscr{D}$-modules (or reduced modules).

Remark 4.10 There is a natural family $(\mathbf{E}_\tau)_{\tau \in A^1}$ of pre-$\mathscr{D}$-modules or reduced modules parametrized by the affine line $A^1 = \operatorname{Spec}\mathbb{C}[\tau]$, which fibre at $\tau = 0$ is the graded object $\mathbf{E}'$ and the fibre at $\tau_0 \neq 0$ is isomorphic to the original pre-$\mathscr{D}$-module or reduced module $\mathbf{E}$: put (for $i = 0, 1$) $\mathscr{E}_i = \oplus_{p \in \mathbf{Z}} (E_i)_p \tau^p \subset E_i \otimes \mathbb{C}[\tau, \tau^{-1}]$ and the relative $\mathscr{D}$ log-structure is the natural one.

Definition 4.11 *A* **special filtration of a coherent** $\mathscr{O}_X$**-module** E *is a filtration for which each* E_p *has the same normalized Hilbert polynomial as* E. *A* **special filtration of a reduced module** (E_0, E_1, u) *is a filtration of this reduced module which is special on* E_0 *and on* E_1.

The graded reduced module $\mathbf{E}'$ associated with a special filtration of a semistable reduced module $\mathbf{E}$ is again semistable.

Definition 4.12 *The equivalence relation on the set of isomorphism classes of all semistable reduced modules generated by this relation (by which* $\mathbf{E}'$ *is related to* $\mathbf{E}$*) will be called S-equivalence.*

Definition 4.13 *We say that a semistable reduced module is* **stable** *if it does not admit any nontrivial special filtration.*

Remarks 4.14 (1) Note in particular that if each E_0, E_a is stable as a Λ-module, then the reduced module $\mathbf{E}$ is stable. Consequently we have the following. Though the definition of stability depends on the ample line bundle L on X, irrespective of the choice of the ample bundle, for any pre-$\mathscr{D}$-module such that the monodromy representation of $E_0|(X - S)$ is irreducible, and the monodromy representation of $\pi_a^* E_a|(N_{S_a,X} - S_a)$ is irreducible for each component S_a, the corresponding reduced module is stable. The converse is not true – a pre-$\mathscr{D}$-module, whose reduced module is stable, need not have irreducible monodromies. The example 2.4.1 in [N] gives a logarithmic connection, whose associated pre-$\mathscr{D}$-module in which s is identity and t is the residue, gives a stable reduced module, but the monodromies are not irreducible.

(2) If $u = 0$, the reduced module is stable if and only E_0 and each E_a is stable.

(3) When X is a curve, a reduced module with $u \neq 0$ is stable if and only if the logarithmic connection E_0 is stable. If $u = 0$, each E_a must moreover have length at most one as an $\mathscr{O}_X$-module. Hence over curves, there is a plentiful supply of stable reduced modules.

Lemma 4.15 *Let (E_0, E_1, u) be a reduced module and let $(E_i)_p$ be filtrations of E_i $(i = 0, 1)$. Then s and t are filtered morphisms with respect to the specialized filtrations if and only if there exists some point $P \in S$ such that the restrictions of s and t to the fibre $E_{i,P}$ at P are filtered with respect to the restricted filtrations.*

Proof This follows from the fact that if a section σ of a vector bundle with a flat connection has a value $\sigma(P)$ in the fibre at P of a sub flat connection, then it is a section of this subbundle: we apply this to s (resp. t) as a section of $Hom((E_0)_{p|S}, (E_1)_{|S})$ (resp. $Hom((E_1)_{p|S}, (E_0)_{|S})$).

A criterion for stability

Let $\mathbf{E} = (E_0, E_1, u = s \otimes t)$ be a reduced module. Assume that we are given filtrations $0 = F_0(E_i) \subset F_1(E_i) \subset \cdots \subset F_{\ell_i}(E_i) = E_i$ of E_i $(i = 0, 1)$ by vector subbundles which are $\mathscr{D}_X[\log S]$-submodules.

For $j = 0, \ldots, \ell_i$ let $k(j)$ be the smallest k such that $s(\mathrm{sp}_S F_j(E_0)) \subset F_k(E_1)$ and let $J(s)$ be the graph of the map $j \to k(j)$. A *jump point* is a point $(j, k(j))$ on this graph such that $k(j-1) < k(j)$. Consider also the set G_s made by points under the graph: $G_s = \{(j,k) \mid k \leq k(j)\}$. For t there is an equivalent construction: we have a map $k \to j(k)$ and a set G_t on the left of the graph $I(t)$: $G_t = \{(j,k) \mid j \leq j(k)\}$.

Definition 4.16 *$u = s \otimes t$ is* **compatible** *with the filtrations if the two sets G_s and G_t intersect at most at (common) jump points (where if $u = 0$, take $s = 0$ and $t = 0$).*

Proposition 4.17 *Let $\mathbf{E} = (E_0, E_1, u)$ be a semistable reduced module. The following conditions are equivalent:*

(1) $\mathbf{E}$ is not stable,

(2) there exists a nontrivial special filtration $F_j(E_i)$ $(j = 0, \ldots \ell_i)$ of each E_i where all inclusions are proper and u is compatible with these filtrations.

Proof (1) $\Rightarrow$ (2): If $\mathbf{E}$ is not stable, we can find two nontrivial special filtrations $(E_0)_p$ and $(E_1)_q$ such that s and t are filtered morphisms. Let p_j $(j = 1, \ldots, \ell_0)$ be the set of jumping indices for $(E_0)_p$ and q_k $(k = 1, \ldots, \ell_1)$ for $(E_1)_q$. For each j_0 and k_0 we have $j(k(j_0)) \leq j_0$ and $k(j(k_0)) \leq k_0$. We define $F_j(E_0) = (E_0)_{p_j}$ and $F_k(E_1) = (E_1)_{q_k}$. We get nontrivial filtrations of E_0 and E_1 where all inclusions are proper. Moreover there cannot exist two distinct points $(j_0, k(j_0))$ and $(j(k_0), k_0)$ with $j_0 \leq j(k_0)$ and $k_0 \leq k(j_0)$ otherwise we would have $j_0 \leq j(k_0) \leq j(k(j_0)) \leq j_0$ and the same for k_0 so the two points would be the equal. Consequently u is compatible with these filtrations.

(2) $\Rightarrow$ (1): We shall construct a special filtration $((E_0)_p, (E_1)_q)$ of the reduced module from the filtrations $F_j(E_i)$ of each E_i. Choose a polygonal line with only positive slopes, going through each jump point of G_s and for which each jump point of G_t is on or above this line (see Fig. 1). Choose increasing functions $p(j)$ and $q(k)$ such that $p(j) - q(k)$ is identically 0 on this polygonal line, is

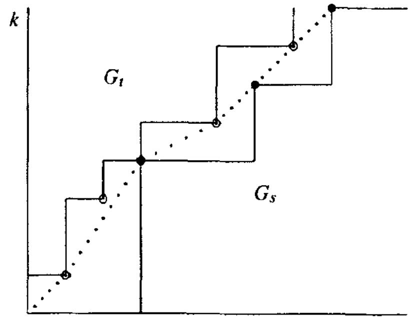

Fig. 1. $\bullet$ = jump points of s, $\circ$ = jump points of t

< 0 above it and > 0 below it (for instance, on each segment $[(j_0, k_0), (j_1, k_1)]$ of this polygonal line, parametrised by $j = j_0 + m\varepsilon_1$, $k = k_0 + m\varepsilon_2$, put $p(j) = p(j_0) + \varepsilon_2(j - j_0)$ and $q(k) = q(k_0) + \varepsilon_1(k - k_0)$, and $p(0) = q(0) = 0$). For $p(j) \le p < p(j+1)$ put $(E_0)_p = F_j(E_0)$ and for $q(k) \le q < q(k+1)$ put $(E_1)_q = F_k(E_1)$. The filtration $((E_0)_p, (E_1)_q, u)$ is then a nontrivial special filtration of the reduced module $\mathbf{E}$.

Proposition 4.18 *Semistability and stability are Zariski open conditions on the parameter scheme of any family of reduced modules.*

Proof As semistability is an open condition on $\mathscr{D}_X[\log S]$-modules, it follows it is an open condition on reduced modules. Now, for any family of semistable reduced modules parametrised by a scheme T, all possible special filtrations of the form given by 4.17 on the specializations of the family are parametrised by a scheme U which is projective over T. The image of U in T is the set of non stable points in T, hence its complement is open.

Points of the moduli

We are now ready to prove the following theorem.

Theorem 4.19 *Let X be a projective variety together with an ample line bundle, and let $S \subset X$ be a smooth divisor.*

(1) There exists a coarse moduli scheme $\mathscr{P}$ for semistable pre-$\mathscr{D}$-modules on (X, S) with given Hilbert polynomials p_i. The scheme $\mathscr{P}$ is quasiprojective, in particular, separated and of finite type over $\mathbb{C}$.

(2) The points of $\mathscr{P}$ correspond to S-equivalence classes of semistable pre-$\mathscr{D}$-modules.

(3) The S-equivalence class of a semistable reduced module $\mathbf{E}$ *equals its isomorphism class if and only if* $\mathbf{E}$ *is stable.*

(4) $\mathscr{P}$ has an open subscheme $\mathscr{P}^s$ whose points are the isomorphism classes of all stable reduced modules. This is a coarse moduli for (isomorphism classes of) stable reduced modules.

Proof Let $\mathscr{P} = H//\mathscr{G}$. Then (1) follows by the construction of $\mathscr{P}$. To prove (2), first note that by the existence of the deformation $\mathbf{E}_t$ (see 4.10) of any reduced module $\mathbf{E}$ corresponding to a weighted special filtration, and by the separatedness of $\mathscr{P}$, the reduced module $\mathbf{E}$ and its limit $\mathbf{E}'$ go to the same point of $\mathscr{P}$. Hence an S-equivalence class goes to a common point of $\mathscr{P}$. For the converse, first recall that $\mathscr{G} = \mathscr{H} \times GL(1)$, and the quotient $\mathscr{P}$ can be constructed in two steps: $\mathscr{P} = R//\mathscr{H}$ where $R = H/\mathscr{G}$. The scheme R parametrizes a canonical family of reduced modules. Let the $\mathscr{H}$ orbit of a point x of R corresponding the reduced module $\mathbf{E}$ not be closed in R. Let x_0 be any of its limit points. Then there exists a 1-parameter subgroup λ of $\mathscr{H}$ such that $x_0 = \lim_{t\to 0}\lambda(t)x$. This defines a map from the affine line A^1 to R, which sends $t \mapsto \lambda(t)x$. Let $\mathbf{E}_t$ be the pull back of the tautological family of reduced modules parametrized by R. Then from the description of the limits of the actions of 1-parameter subgroups on a quot scheme given in Sect. 1 of Simpson [S], it follows that $\mathbf{E}$ has a special filtration such that the family $\mathbf{E}_t$ is isomorphic to a deformation of the type constructed in 4.10 above. Hence the reduced modules parametrized by x and x_0 are S-equivalent. This proves (2).

If the orbit of x is not closed, then it has a limit x_0 outside it under a 1-parameter subgroup, which by above represents a reduced module $\mathbf{E}'$ which is the limit of $\mathbf{E}$ under a special filtration. As by assumption $\mathbf{E}'$ is not isomorphic to $\mathbf{E}$, the special filtration must be nontrivial. Hence $\mathbf{E}$ is not stable. Hence stable points have closed orbits in R. If x represents a stable reduced module, then x cannot be the limit point of any other orbit. For, if x is a limit point of the orbit of y, then by openness of stability (see 4.18), y should again represent a stable reduced module. But then by above, the orbit of y is closed. This proves (3).

Let $H^s \subset H$ be the open subscheme where the corresponding pre-$\mathscr{D}$-module is stable. By (2) and (3) above, H^s is saturated under the quotient map $H \to \mathscr{P}$, hence by properties of a good quotient, its image $\mathscr{P}^s$ is open in $\mathscr{P}$. Moreover by (2) and (3) above, H^s is the inverse image of $\mathscr{P}^s$. Hence $H^s \to \mathscr{P}^s$ is a good quotient, which again by (2) and (3) is an orbit space. Hence points of $\mathscr{P}^s$ are exactly the isomorphism classes of stable reduced modules, which proves (4).

5 Perverse sheaves, Verdier objects and finite descriptions

Let X be a nonsingular projective variety and let S be a smooth divisor. The abelian category of perverse sheaves constructible with respect to the stratification $(X-S, S)$ of X is equivalent to the category of 'Verdier objects' on (X, S). Before defining this category, let us recall the notion of specialization along S.

Let $\mathscr{E}$ be a local system (of finite dimensional vector spaces) on $X - S$. The *specialization* $\mathrm{sp}_S\,\mathscr{E}$ is a local system (of the same rank) on $N_{S,X} - S$ equipped with an endomorphism $T_{\mathscr{E}}$. It is constructed using the nearby cycle functor ψ defined by Deligne applied to the morphism which describes the canonical deformation from X to the normal bundle $N_{S,X}$.

A local system $\mathcal{F}$ on $N_{S,X} - S$ equipped with an endomorphism $\tau_{\mathcal{F}}$ is said to be **monodromic** if $\tau_{\mathcal{F}}$ is equal to the monodromy of $\mathcal{F}$ around S. Then $\mathrm{sp}_S\,\mathcal{E}$ is monodromic.

Definition 5.1 *A* **Verdier object** *on* (X,S) *is a tuple* $\mathbf{V} = (\mathcal{E}, \mathcal{F}, C, V)$ *where*

(1) $\mathcal{E}$ *is a local system on* $X - S$,

(2) $\mathcal{F}$ *is a monodromic local system on* $N_{S,X} - S$,

(3) $C : \mathrm{sp}_S\,\mathcal{E} \to \mathcal{F}$ *and* $V : \mathcal{F} \to \mathrm{sp}_S\,\mathcal{E}$ *are morphisms of (monodromic) local systems on* $N_{S,X} - S$ *satisfying*

(4) $CV = \tau_{\mathcal{F}} - \mathrm{id}$ *and* $VC = \tau_{\mathcal{E}} - \mathrm{id}$.

Remark 5.2 The morphisms between Verdier objects on (X,S) are defined in an obvious way, and the category of Verdier objects is an abelian category in which each object has finite length. Hence the following definition makes sense.

Definition 5.3 *We say that two Verdier objects are* **S-equivalent** *if they admit Jordan-Hölder filtrations such that the corresponding graded objects are isomorphic.*

Remark 5.4 Let B be a tubular neighbourhood of S in X, diffeomorphic to a tubular neighbourhood of S in $N_{S,X}$. Put $B^* = B - S$. The specialized local system $\mathrm{sp}_S\,\mathcal{E}$ can be realized as the restriction of $\mathcal{E}$ to B^*, its monodromy $\tau_{\mathcal{E}}$ at some point $x \in B^*$ being the monodromy along the circle normal to S going through x. Hence a Verdier object can also be described as a tuple $\mathbf{V}$ where $\mathcal{F}$ is a local system on B^* and C, V are morphisms between $\mathcal{E}|B^*$ and $\mathcal{F}$ subject to the same condition (4).

The notion of a family of perverse sheaves is not straightforward. We can however define the notion of a family of Verdier objects. Let us define first a family of local systems on $X - S$ (or on $N_{S,X} - S$) parametrized by a scheme T. This is a locally free $p^{-1}\mathcal{O}_T$-module of finite rank, where p denotes the projection $X - S \times T \to T$. Morphisms between such objects are $p^{-1}\mathcal{O}_T$-linear. The notion of a family of Verdier objects is then straightforward.

In order make a moduli space for Verdier objects, we shall introduce the category of 'finite descriptions' on (X,S). Let us fix the following data (D):

(D1) finitely generated groups G and G_a for each component S_a of S,

(D2) for each a an element τ_a which lies in the center of G_a and a group homomorphism $\phi_a : G_a \to G$.

Definition 5.5 *A finite description* $\mathbf{D}$ *(with respect to the data (D)) is a tuple* $(E, \rho, F_a, \rho_a, C_a, V_a)$ *where*

(1) $\rho : G \to GL(E)$ *is a finite dimensional complex representation of the group* G*; for each* a *we will regard* E *as a representation of* G_a *via the homomorphism* $\phi_a : G_a \to G$*;*

(2) for each a, $\rho_a : G_a \to GL(F_a)$ *is a finite dimensional complex representation of the group* G*;*

(3) for each a, $C_a : E \to F_a$ *and* $V_a : F_a \to E$ *are* G_a*-equivariant linear maps such that* $V_aC_a = \rho(\tau_a) - \mathrm{id}$ *in* $GL(E)$ *and* $C_aV_a = \rho_a(\tau_a) - \mathrm{id}$ *in* $GL(F_a)$.

A morphism between two finite descriptions has an obvious definition.

Remark 5.6 Let $P_0 \in X - S$ and let P_a be a point in the component $B*_a$ of B^*. Choose paths $\sigma_a : [0,1] \to X - S$ with $\sigma_a(0) = P_0$ and $\sigma_a(1) = P_a$. Let G be the fundamental group $\pi_1(X - S, P_0)$, and let $G_a = \pi_1(B_a^*, P_a)$. Let $\tau_a \in G_a$ be the positive loop based at P_a in the fiber of $B_a^* \to S_a$. Finally, let $\phi_a : G_a \to G$ be induced by the inclusion $B_a^* \hookrightarrow X - S$ by using the path σ_a to change base points. Then, under the equivalence between representations of fundamental group and local system, the category of finite description with respect to the previous data is equivalent to the category of Verdier objects on (X, S).

Remark 5.7 The category of finite descriptions is an abelian category in which each object has finite length. Therefore the notion of S-equivalence as in definition 5.3 above makes sense for finite descriptions.

Definition 5.8 *A family of finite descriptions parametrized by a scheme T is a tuple $(E_T, \rho_T, F_{T,a}, \rho_{T,a}, C_{T,a}, V_{T,a})$ where E_T and the $F_{T,a}$ are locally free sheaves on T, ρ and $\rho_{T,a}$ are families of representations into these, and the $C_{T,a}$ and $V_{T,a}$ are $\mathcal{O}_T$-homomorphisms of sheaves satisfying the analogues of condition 5.5.3 over T. The pullback of a family under a morphism $T' \to T$ is defined in an obvious way, giving a fibered category. Let PS denote the corresponding groupoid.*

Remark 5.9 It can be checked (we omit the details) that the groupoid *PS* is an Artin algebraic stack.

6 Moduli for perverse sheaves

Let us fix data (D) as above.

Theorem 6.1 *There exists an affine scheme of finite type over $\mathbb{C}$, which is a coarse moduli scheme for finite descriptions* $\mathbf{D} = (E, \rho, F_a, \rho_a, C_a, V_a)$ *relative to* (D) *with fixed numerical data $n = \dim E$ and $n_a = \dim F_a$. The closed points of this moduli scheme are the S-equivalence classes of finite descriptions with given numerical data (n, n_a).*

Using remark 5.6 we get

Corollary 6.2 *There exists an affine scheme of finite type over $\mathbb{C}$, which is a coarse moduli scheme for Verdier objects* $\mathbf{V} = (\mathscr{E}, \mathscr{F}, C, V)$ *(or perverse sheaves on (X, S)) with fixed numerical data $n = \operatorname{rank}\mathscr{E}$ and $n_a = \operatorname{rank}\mathscr{F}|B_a^*$. The closed points of this moduli scheme are the S-equivalence classes of Verdier objects with given numerical data (n, n_a).*

The above corollary and its proof does not need X to be a complex projective variety, and the algebraic structure of X does not matter. All that is needed is that the fundamental group of $X - S$ and that of each S_a is finitely generated.

The rest of this section contains the proof of the above theorem.

Proposition 6.3 *(1) Let* $\mathbf{D}$ *be a finite description, and let* $\mathrm{gr}(\mathbf{D})$ *be its semisimplification. Then there exists a family* $\mathbf{D}_T$ *of finite descriptions parametrized by the affine line* $T = A^1$ *such that the specialization* $\mathbf{D}_0$ *at the origin* $0 \in T$ *is isomorphic to* $\mathrm{gr}(\mathbf{D})$*, while* $\mathbf{D}_t$ *is isomorphic to* $\mathbf{D}$ *at any* $t \neq 0$.

(2) In any family of finite descriptions parametrized by a scheme T*, each S-equivalence class (Jordan-Hölder class) is Zariski closed in* T.

Proof The statement (1) has a proof by standard arguments which we omit. To prove (2), first note that if $\mathbf{D}_T$ is any family and $\mathbf{D}'$ a simple finite description, then the condition that $\mathbf{D}' \times \{t\}$ is a quotient of $\mathbf{D}_t$ defines a closed subscheme of T. From this, (2) follows easily.

Construction of Moduli Let E and F_a be vector spaces with $\dim(E) = n$ and $\dim(F_a) = n_a$. Let $\mathscr{R}$ be the affine scheme of all representations ρ of G in E, made as follows. Let $h_1, \ldots, h_r$ be generators of G. Then $\mathscr{R}$ is the closed subscheme of the product $GL(E)^r$ defined by the relations between the generators. Similarly, choose generators for each G_a, and let $\mathscr{R}_a$ be the corresponding affine scheme of all representations ρ_a of G_a in F_a.

Let

$$A \subset \mathscr{R} \times \prod_a (\mathscr{R}_a \times Hom(E, F_a) \times Hom(F_a, E))$$

be the closed subscheme defined by condition 5.5.3 above. Its closed points are tuples (ρ, ρ_a, C_a, V_a) where the linear maps $C_a : E \to F_a$ and $V_a : F_a \to E$ are G_a-equivariant under the representations $\rho\phi_a : G_a \to GL(E)$ and $\rho_a : G_a \to GL(F_a)$, and satisfy $V_a C_a = \rho(\tau_a) - 1$ in $GL(E)$, and $C_a V_a = \rho_a(\tau_a) - 1$ in $GL(F_a)$ for each a.

The product group $\mathscr{G} = GL(E) \times (\prod_a GL(F_a))$ acts on the affine scheme A by the formula

$$(\rho, \rho_a, C_a, V_a) \cdot (g, g_a) = (g^{-1}\rho g, g_a^{-1}\rho_a g_a, g_a^{-1} C_a g, g^{-1} V_a g_a).$$

The orbits under this action are exactly the isomorphism classes of finite descriptions. The moduli of finite descriptions is the good quotient $\mathscr{F} = A//\mathscr{G}$, which exists as A is affine and $\mathscr{G}$ is reductive. It is an affine scheme of finite type over $\mathbb{C}$. It follows from 6.3.1 and 6.3.2 and properties of a good quotient that the Zariski closures of two orbits intersect if and only if the two finite descriptions are S-equivalent. Hence closed points of $\mathscr{F}$ are S-equivalence classes (Jordan-Hölder classes) of finite descriptions.

7 Riemann-Hilbert morphism

To any Malgrange object $\mathbf{M}$, there is an obvious associated Verdier object $\mathbf{V}(\mathbf{M})$ obtained by applying the de Rham functor to each component of $\mathbf{M}$. This defines a functor, which is in fact an equivalence of categories from Malgrange objects to Verdier objects. We have already defined a functor η from pre-$\mathscr{D}$-modules with good residual eigenvalues to Malgrange objects. Composing, we get an exact functor from pre-$\mathcal{D}$-modules with good residual eigenvalues to Verdier objects. Choosing base points in X and paths as in remark 5.6 we get an exact functor $\mathcal{R}\mathcal{H}$ from pre-$\mathcal{D}$-modules to finite descriptions. This construction works equally well for families of pre-$\mathcal{D}$-modules, giving us a holomorphic family $\mathcal{R}\mathcal{H}(\mathbf{E}_T)$ of Verdier objects (or finite descriptions) starting from a holomorphic family $\mathbf{E}_T$ of pre-$\mathcal{D}$-modules with good residual eigenvalues.

Remark 7.1 Even if $\mathbf{E}_T$ is an algebraic family of pre-$\mathcal{D}$-modules with good residual eigenvalues, the associated family $\mathcal{R}\mathcal{H}(\mathbf{E}_T)$ of Verdier objects may not be algebraic.

Remark 7.2 If a semistable pre-$\mathcal{D}$-module has good residual eigenvalues, then any other semistable pre-$\mathcal{D}$-module in its S-equivalence class has (the same) good residual eigenvalues. Hence the analytic open subset T_g of the parameter space T of any analytic family of semistable pre-$\mathcal{D}$-modules defined by the condition that residual eigenvalues are good is saturated under S-equivalence.

Lemma 7.3 *If two semistable pre-$\mathcal{D}$-modules with good residual eigenvalues are S-equivalent (in the sense of definition 4.13 above), then the associated finite descriptions are S-equivalent (that is, Jordan-Hölder equivalent).*

Proof Let $\mathbf{E} = (E_0, E_1, s, t)$ be a pre-$\mathcal{D}$-module with good residual eigenvalues (that is, the logarithmic connection E_0 has good residual eigenvalues on each component of S) such that $s \otimes t = 0$. Then one can easily construct a family of pre-$\mathcal{D}$-modules parametrized by the affine line A^1 which is the constant family $\mathbf{E}$ outside some point $P \in A^1$, and specializes at P to $\mathbf{E}' = (E_0, E_1, 0, 0)$. Let $\phi : A^1 \to F$ be the resulting morphism to the moduli $\mathcal{F}$ of finite descriptions. By construction, ϕ is constant on $A^1 - P$, and so as $\mathcal{F}$ is separated, ϕ is constant. As the points of $\mathcal{F}$ are the S-equivalence classes of finite descriptions, it follows that the finite descriptions corresponding to $\mathbf{E}$ and $\mathbf{E}'$ are S-equivalent. Hence the S-equivalence class of the finite description associated to a pre-$\mathcal{D}$-module depends only on the reduced module made from the pre-$\mathcal{D}$-module. Now we must show that any two S-equivalent (in the sense of 4.13) reduced semistable modules have associated finite descriptions which are again S-equivalent (Jordan-Hölder equivalent). This follows from the deformation given in 4.10 by using the separatedness of $\mathcal{F}$ as above.

Now consider the moduli $\mathcal{P} = H//\mathcal{G}$ of semistable pre-$\mathcal{D}$-modules. Let H_g be the analytic open subspace of H where the family parametrized by H has good residual eigenvalues. By the above remark, H_g is saturated under $H \to \mathcal{P}$. Hence its image $\mathcal{P}_g \subset \mathcal{P}$ is analytic open. Let $\phi : H_g \to \mathcal{F}$ be the classifying map to the moduli $\mathcal{F}$ of finite descriptions for the tautological family of pre-$\mathcal{D}$-modules parametrized by H, which is defined because of the the above lemma. By the analytic universal property of GIT quotients (see Proposition 5.5 of Simpson [S] and the remark below), ϕ factors through an analytic map $\mathcal{R}\mathcal{H} : P_g \to \mathcal{F}$, which we call as the *Riemann-Hilbert morphism.*

Remark 7.4 In order to apply Proposition 5.5 of [S], note that a $\mathscr{G}$-linear ample line bundle can be given on H such that all points of H are semistable. Moreover, though the proposition 5.5 in [S] is stated for semisimple groups, its proof works for reductive groups.

Remark 7.5 The Riemann-Hilbert morphism can also be thought of as a morphism from the analytic stack of pre-$\mathscr{D}$-modules with good residual eigenvalues to the analytic stack of perverse sheaves.

8 Some properties of the Riemann-Hilbert morphism

In this section we prove some basic properties of the morphism $\mathscr{RH}$, which can be interpreted either at stack or at moduli level.

Lemma 8.1 (Relative Deligne construction) *(1) Let T be the spectrum of an Artin local algebra of finite type over $\mathbb{C}$, and let ρ_T be a family of representations of G (the fundamental group of $X - S$ at base point P_0) parametrized by T. Let E be a logarithmic connection with eigenvalue not differing by nonzero integers, such that the monodromy of E equals ρ, the specialization of ρ_T. Then there exists a family E_T of logarithmic connections parametrized by T such that $E_0 = E$ and E_T has monodromy ρ_T.*

(2) A similar statement is true for analytic germs of G-representations.

Proof For each a, choose a fundamental domain Ω_a for the exponential map $(z \mapsto \exp(2\pi\sqrt{-1}z))$ such that the eigenvalues of the residue $R_a(E)$ of E along S_a are in the interior of the set Ω_a. As the differential of the exponential map $M(n,\mathbb{C}) \to GL(n,\mathbb{C})$ is an isomorphism at all those points of $M(n,\mathbb{C})$ where the eigenvalues do not differ by nonzero integers, using the fundamental domains Ω_a we can carry out the Deligne construction locally to define a family E_T of logarithmic connections on (X,S) with $E_0 = E$, which has the given family of monodromies.

Note that for the above to work, we needed the inverse function theorem, which is valid for Artin local algebras.

Remark 8.2 If in the above, the family ρ_T of monodromies is a constant family (that is, pulled back under $T \to \mathrm{Spec}(\mathbb{C})$), then E_T is also a constant family as follows from Proposition 5.3 of [N].

Proposition 8.3 ('Injectivity' of $\mathscr{RH}$) *Let $\mathbf{E} = (E,F,t,s)$ and $\mathbf{E}' = (E',F',t',s')$ be pre-$\mathscr{D}$-modules having good residual eigenvalues, such that for each a, the eigenvalues of the residues of E and E' over S_a belong a common fundamental domain Ω_a for the exponential map $\exp : \mathbb{C} \to \mathbb{C}^* : z \mapsto \exp(2\pi\sqrt{-1}z)$. Then $\mathbf{E}$ and $\mathbf{E}'$ are isomorphic if and only if the finite descriptions $\mathscr{RH}(\mathbf{E})$ and $\mathscr{RH}(\mathbf{E}')$ are isomorphic.*

Proof It is enough to prove that if the Malgrange objects $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic, then so are the pre-$\mathscr{D}$-modules $\mathbf{E}$ and $\mathbf{E}'$. First use the fact that, in a given meromorphic connection M on $X - S$ (or on $N_{S,X} - S$), there exists one and only one logarithmic connection having its residue along S_a in Ω_a for each a, to conclude that E and E' (resp. F and F') are isomorphic logarithmic modules. To obtain the identification between s and s' (resp. t and t'), use the fact that these maps are determined by their value at a point in each connected component $N_{S_a,X} - S_a$ of $N_{S,X} - S$ and this value is determined by the corresponding C_a or C'_a (resp. V_a or V'_a).

Proposition 8.4 (Surjectivity of $\mathcal{R}.\mathcal{H}$) *Let* $\mathbf{D}$ *be a finite description, and let* $\sigma_a : \mathbb{C}^* \to \mathbb{C}$ *be set theoretic sections of* $z \mapsto \exp(2\pi\sqrt{-1}z)$. *Then there exists a pre-*$\mathcal{D}$*-module* $\mathbf{E}$ *whose eigenvalues of residue over* S_a *are in image*(σ_a), *for which* $\mathcal{R}.\mathcal{H}(\mathbf{E})$ *is isomorphic to* $\mathbf{D}$.

Proof This follows from Proposition 3.7.

Remark 8.5 The Propositions 8.3 and 8.4 together say that the set theoretic fiber of $\mathcal{R}.\mathcal{H}$ over a given finite description is in bijection with the choices of 'good' logarithms for the local monodromies of the finite description (here 'good' means eigenvalues do not differ by nonzero integers).

Proposition 8.6 (Tangent level injectivity for $\mathcal{R}.\mathcal{H}$) *Let* $(E, F, t, s)_T$ *be a family of pre-*$\mathcal{D}$*-modules having good residual eigenvalues parametrized by the spectrum* T *of an Artinian local algebra. Let the family* $\mathcal{R}.\mathcal{H}(E, F, t, s)_T$ *of finite descriptions parametrized by* T *be constant (pulled back under* $T \to \operatorname{Spec}\mathbb{C}$*). Then the family* $(E, F, t, s)_T$ *is also constant.*

Proof This is just the rigidity result of Proposition 3.9.

Proposition 8.7 (Infinitesimal surjectivity for $\mathcal{R}.\mathcal{H}$) *Let* T *be the spectrum of an Artin local algebra of finite type over* $\mathbb{C}$, *and let* $\mathbf{D}$ *be a family of finite descriptions parametrized by* T. *Let* $\mathbf{E}$ *be a pre-*$\mathcal{D}$*-module having good residual eigenvalues such that* $\mathcal{R}.\mathcal{H}(\mathbf{E}) = \mathbf{D}_\xi$, *the restriction of* $\mathbf{D}$ *over the closed point* ξ *of* T. *Then there exists a family* $\mathbf{E}'_T$ *of pre-*$\mathcal{D}$*-modules having good residual eigenvalues with* $\mathbf{E}'_\xi = \mathbf{E}$ *and* $\mathcal{R}.\mathcal{H}(\mathbf{E}_T) = \mathbf{D}_T$.

Proof This follows from Lemma 8.1 and the proof of Proposition 3.7 which works for families over Artin local algebras.

Theorem 8.8 *The analytic open substack of the stack (or analytic open subset of the moduli) of pre-*$\mathcal{D}$*-modules on* (X, S), *where* $\mathbf{E}$ *has good residual eigenvalues, is an analytic spread over the stack (or moduli) of perverse sheaves on* (X, S) *under the Riemann-Hilbert morphism.*

Proof This follows from Propositions 8.4, 8.6 and 8.7 above.

Note that we have not defined $\mathcal{R}.\mathcal{H}$ on the closed analytic subset T_o of the parameter space of a family where $\mathbf{E}$ does not have good residual eigenvalues. Note that T_o is defined by a 'codimension one' analytic condition, that is, if T is nonsingular, and if T_o is a nonempty and proper subset of T, then T_o

has codimension 1 in T. However, it follows from Proposition 8.9 below that the morphism $\mathcal{RH}$ on $T - T_0$ can be extended to an open subset of T of complementary codimension at least two. However, on the extra points to which it gets extended, it may not represent the de Rham functor.

Proposition 8.9 (Removable singularities for $\mathcal{RH}$) *Let T be an open disk in $\mathbb{C}$ centered at 0. Let $\mathbf{E}_T = (E, F, t, s)_T$ be a holomorphic family of pre-$\mathcal{D}$-modules parametrized by T. Let the restriction E_z have good residual eigenvalues for all $z \in T - \{0\}$. Then there exists a holomorphic family $\mathbf{D}_U$ of finite descriptions parametrized by a neighbourhood U of $0 \in T$ such that on $U - \{0\}$, the families $\mathcal{RH}(\mathbf{E}_U | U - \{0\})$ and $\mathbf{D}_{U - \{0\}}$ are isomorphic.*

If at $z = 0$ the logarihmic connection E does not have good residual eigenvalues, it is possible to change it to obtain a new logarithmic connection having good residual eigenvalues. This is done by the classical 'shearing transformation' that we adapt below (**inferior and superior modifications** for pre-$\mathcal{D}$-modules). This can be done in family and has no effect on the Malgrange object at least locally.

Definition 8.10 *If E is a vector bundle on X, and V a subbundle of the restriction $E|S$, then the* **inferior modification** *${}_VE$ is the sheaf of all sections of E which lie in V at points of S. This is a locally free subsheaf of E (but not generally a subbundle). The* **superior modification** *${}^VE$ is the vector bundle $\mathcal{O}_X(S) \otimes {}_VE$.*

Remark 8.11 If $E|S = V \oplus V'$, then we have a canonical isomorphism

$$ {}_VE|S \to V \oplus (\mathcal{I}_{S,X}^* \otimes V') $$

and hence also a canonical isomorphism

$$ {}^VE|S \to (\mathcal{I}_{S,X}^* \otimes V) \oplus V' $$

Remark 8.12 If (E, ∇) is a logarithmic connection on (X, S) and V is invariant under the residue, then it can be seen that ${}_VE$ is invariant under ∇, so is again a logarithmic connection. We call it the inferior modification of the logarithmic connection E along the residue invariant subbundle $V \subset E|S$. It has the effect that the residual eigenvalues along V get increased by 1 when going from E to ${}_VE$. As $\mathcal{O}_X(S)$ is canonically a logarithmic connection, the superior modification ${}^VE$ is also a logarithmic connection, with the residual eigenvalues along V getting decreased by 1.

Let (E, F, t, s) be pre-$\mathcal{D}$-module on (X, S) such that E has good residual eigenvalues. Let us for simplicity of writing assume that S is connected. Let $E|S = \oplus_\alpha E^\alpha$ and $F = \oplus_\alpha F^\alpha$ be the respective direct sum decompositions into generalized eigen subbundles for the action of θ. Then (see also Remark 3.5) as θ commutes with s and t, it follows that $t(E^\alpha) \subset F^\alpha$ and $s(F_\alpha) \subset E^\alpha$. Moreover, when $\alpha \neq 0$, the maps s and t are isomorphisms between E^α and F^α.

Now let $\alpha \neq 0$. Let $V = E^{\alpha}$ and $V' = \oplus_{\beta\neq\alpha} E^{\beta}$. Let $F'' = \oplus_{\beta\neq\alpha} F^{\beta}$. Let $F' = F^{\alpha} \oplus . I_{S,X}^{*} \otimes F''$. Let $E' = {}_V E$. Then using 8.11 and the above, we get maps $t' : E'|S \to F'$ and $s' : F' \to E'|S$ such that (E', F', s', t') is a pre-$\mathcal{S}$-module.

Definition 8.13 *We call the pre-$\mathcal{S}$-module (E', F', s', t') constructed above as the inferior modification of (E, F, s, t) along the generalized eigenvalue $\alpha \neq 0$.*

Similarly, we can define the superior modification along a generalized eigenvalue $\alpha \neq 0$ by tensoring with $\mathcal{O}_X(S)$.

Remark 8.14 The construction of inferior or superior modification of pre-$\mathcal{S}$-modules can be carried out over a parameter space T (that is, for families) provided the subbundles V and V' form vector subbundles over the parameter space T (their ranks are constant).

Proof of 8.9 If the restriction $E = E_{T|z=0}$ has good residual eigenvalues, then $\mathcal{R}\mathcal{H}\mathbf{E}_T$ has the desired property. So suppose E does not have good residual eigenvalues.

We first assume for simplicity of writing that E fails to have good residual eigenvalues because its residue R_a on S_a has exactly one pair $(\alpha, \alpha - 1)$ of distinct eigenvalues which differ by a positive integer, with $\alpha - 1 \neq 0$. Let f_T be the characteristic polynomial of $R_{a,T}$. Then f_0 has a factorization $f_0 = gh$ such that the polynomials g and h are coprime, $g(\alpha) = 0$ and $h(\alpha - 1) = 0$. On a neighbourhood U of 0 in T we get a unique factorization $f_T|U = g_U h_U$ where g_U specializes to g and h_U specializes to h at 0. By taking U small enough, we may assume that g_U and h_U have coprime specializations at all points of U. Let V_U be the kernel of the endomorphism $g_U(R_{a,U})$ of the bundle $E_{a,U}$. If U is small enough then F_U is a subbundle. Now take the inferior modification $\mathbf{E}' = ({}_V E_U, F'_U, t'_U, s'_U)$ of the family $(E, F, t, s)_U$ as given by construction 8.13. Then ${}_V E_U$ is a family of logarithmic connections having good residual eigenvalues, so by definition $\mathbf{E}'$ has good residual eigenvalues.

If $(0, 1)$ are the eigenvalues, then use superior modification along the eigenvalue 1.

If R_a has eigenvalues $(\alpha, \alpha - k)$ for some integer $k \geq 1$, then repeat the above inferior (or superior) modification k times (whether to choose an inferior or superior modification is governed by the following restriction : the multiplicity of the generalized eigenvalue 0 should not decrease at any step). By construction, we arrive at the desired family (E', F', s', t').

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