Minimal composite equations and the stability of non-parallel flows

R. Narasimha and Rama Govindarajan

Fluid Dynamics Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Bangalore 560 064, India

The stability of a laminar boundary layer has classically been analysed in terms of the solutions of the Orr-Sommerfeld equation, which assumes that the flow is parallel. The purpose of this paper is to summarize the principles underlying the work done by the authors on non-parallel flows. This work adopts an asymptotic approach that involves the formulation of what we shall call 'minimal composite equations' in the limit of large Reynolds numbers. These equations include every term that is important somewhere, and none that is important nowhere, 'importance' being defined in terms of errors to some prescribed order in the local Reynolds number. This approach leads to a hierarchy of stability equations of successively increasing accuracy, including, in the lowest order, an ordinary differential equation for similarity flows, a low-order parabolic partial differential equation in the next order, and finally a 'full nonparallel' equation which is equivalent to the parabolized stability (partial differential) equations of Bertolotti *et al.*[1]. The o.d.e., written here in similarity variables, is similar to but not identical with the Orr-Sommerfeld. Typical results from the present approach are given to illustrate the nature of the stability 'surface' derived from the present theory, and the accuracy of the computed amplitude distributions.

Introduction

The theoretical demonstration by Tollmien[2] in 1929 that the laminar boundary layer on a flat plate can be unstable at Reynolds numbers higher than a critical value has been justly considered a major landmark in 20th century fluid mechanics. At the time that Tollmien's paper was published there was considerable scepticism about the instability of boundary layer flow. It took another fifteen years before Schubauer and Skramstad[3] showed, through careful experimental work in one of the world's first 'quiet' wind tunnels (as they are called nowadays), that the predictions of Tollmien's theory were correct in all essentials.

Tollmien achieved his demonstration of boundary-layer instability through approximate solutions of the Orr-Sommerfeld equation, which is strictly valid only for parallel flow such as that in a plane channel. His achievement consisted in the way he skillfully navigated through and across the singularities of the equation in the limit of large Reynolds numbers. These singularities lead to, or represent, the viscous sublayers that appear in the solutions in the limit: in general two in number, one at the critical layer where the phase velocity of the wave coincides with the flow velocity, and one at the wall, but the two are not always distinct.

The assumption that the parallel-flow Orr-Sommerfeld equation (O-S) is adequate for spatially evolving flows like the boundary layer has long been questioned. When approximate solutions of the equation were obtained[2,4], they were often based on the assumption that the wave number of the disturbance was low. For such an assumption to be plausible, one has to have waves long enough for the relevant series to converge, but not so long that one end of the wave sees a significantly different boundary layer from the other. It was thus not clear that solving the O-S for the boundary layer was a rational procedure.

With the advent of the digital computer, the need for making technically convenient assumptions like low wave numbers was no longer felt; and in the 1960s the O-S equation began to be solved numerically on the machine, without appeal to such assumptions. On the other hand the question of whether one is solving the *right* equation has become more pressing, as computational power ceases to be the limiting resource.

Beginning in the 1970s, a series of non-parallel flow theories emerged[1,5–7]. The results provided by Gaster[6] have stood the test of time, and have become a widely accepted standard of comparison. Today the most frequently used approach is probably that of Bertolotti *et al.*[1], who formulate a parabolized stability equation (PSE) to handle the problem. This is a partial differential equation, which can be solved numerically without much difficulty on a computer of modest power.

Implicit in the PSE approach is the assumption that one needs the full paraphernalia of a partial differential equation for handling non-parallel flow. Such an assumption leads to an important question, which stems from the various terms with a factor R^{-1} that appear therein, apart from those already present in O-S. (Here

For correspondence. (e-mail: roddam@caos.iisc.ernet.in; rama@jncasr.ac.in.)

It is a pleasure to dedicate this paper to Prof. S. Dhawan, who initiated research studies in India on the problem of transition in boundary layers more than forty years ago.

R is a local Reynolds number, based on say the thickness of the boundary layer and the velocity at its edge.) Now if all R^{-1} terms are included in the stability analysis, one must necessarily take account of higher order (i.e. R^{-1}) effects on the mean flow as well, as these will make comparable contributions. It cannot, however, be argued that non-parallel flow effects can be consistently included only if the mean flow is obtained from higher order boundary layer theory (of the kind that Van Dyke[8] has described). It then follows immediately that it must be possible to formulate equations for non-parallel flow stability to lower orders than PSE.

We review here the basic ideas governing recent work by the authors[9–12] (referred to respectively as GN95, 97, 99,00) in formulating such lower order theories. We believe a careful reexamination of linear stability theory is justified, even though it cannot directly predict the location of transition. However, it has been demonstrated[1] that linear stability can take one a large part of the way, since secondary instabilities occurring just upstream of transition onset can be predicted well. Since stability analyses are far more economical than direct numerical simulations, especially when the Reynolds numbers are not low, they are likely to remain a powerful tool in the foreseeable future. Furthermore, the only practical method for getting any idea of transition onset location (the so-called e^n method) depends heavily on linear stability theory. These are reasons enough for continued interest, apart from all the other insights that stability theories have continued to provide in fluid flow problems.

The present approach

The approach we have pursued recognizes the fact that the equation formulated will eventually be solved numerically on a computer. The approach relies on the idea of matched asymptotic expansions[13]. In usual applications of this method the objective is to find uniformly valid solutions to a given `primitive' equation in the limit as some small parameter (say e) in the problem goes to zero. This is accomplished by identifying the distinguished limits to the primitive equation[14], solving the simpler equations that arise in each such limit, and matching neighbouring solutions in an asymptotic sense. From such matched solutions a uniformly valid composite solution can be constructed by well-known methods[13,15].

This approach is of value in stability theory as well. However, our present objective is different, because we propose to solve the equations numerically: finding uniformly valid solutions is feasible but involves (in this view) unnecessary and unrewarding trouble. We do not therefore attempt to construct here uniformly valid solutions to the equation, but rather to derive an asymptotically consistent equation that, to an appropriate order, contains just those terms necessary to obtain uniformly valid solutions. Thus, once the necessary distinguished limits are identified, it is first ensured that solutions of neighbouring limit equations can match each other in the asymptotic sense. Then a minimal composite *equation* is constructed; this equation includes all – and only – those terms that are necessary to ensure the existence of matched uniformly valid solutions to some prescribed order. The minimal composite equation so constructed may be seen as a consistent, lower order, reduced primitive; it includes every term that is important *somewhere*, and none that is important *nowhere* – 'important' to some prescribed order in the distinguished limits.

We shall find it convenient to speak of an equation as 'nominally' valid to some order in the small parameter R^{-1}, meaning thereby that the equation contains all the terms required to construct the relevant distinguished limits necessary to obtain uniformly valid solutions to that or to lower orders. Our objective then is to find the *minimal* composite equation that includes just those terms that are necessary to obtain uniformly valid solutions to the problem by numerical methods.

This approach has led to new low-order equations that are simpler than those in current use seeking to include the effects of non-parallelism in the flow.

The basic non-parallel flow equation

In the classical linear stability analysis of the flow over a flat plate[4], the disturbance stream function is broken up into normal modes of the form

$$\hat{f}(x, y) = f(y)e^{i(ax - w t)},$$

where a and w are the wave number and frequency of the disturbance respectively, x is downstream distance and t is time. Only two-dimensional disturbances need be considered since, for a two-dimensional mean flow, they become unstable at a lower Reynolds number than three-dimensional disturbances (by Squire's theorem[4]). Since the disturbances are assumed small, their products may be neglected. If it is further assumed that the boundary layer is locally parallel, i.e. does not vary with x (so $\partial/\partial x = 0$ and the normal velocity is zero), f satisfies the Orr-Sommerfeld equation

$$\{\mathrm{OS}\}f \equiv \left[i(w - a\,\Phi')(D^2 - a^2) + ia\Phi''' + \frac{1}{R}(D^4 - 2a^2D^2 + a^4) \right] f = 0, \tag{1}$$

which defines the Orr-Sommerfeld operator {OS}. Equation (1) has been nondimensionalized using the freestream velocity U and a boundary-layer thickness (in the sequel the momentum thickness q) as scales; R is the Reynolds number based on the same scales,

$$D^k \equiv \frac{\mathrm{d}^k}{\mathrm{d}y^k}, \quad k = 1, 2, \ldots,$$

and primes on the mean streamfunction $\Phi(y)$ denote differentiation with respect to y. In spatial stability analysis, $a = a_\mathrm{r} + \mathrm{i}a_\mathrm{i}$ is taken to be complex and w to be real, the boundary layer being unstable to a given disturbance if a_i is negative.

To see how the assumption of parallelism in the Orr-Sommerfeld equation may be relaxed, we return to the incompressible Navier-Stokes equations in two-dimensional flow, which may be written in terms of the streamfunction y_d as

$$\begin{aligned} &\frac{\partial}{\partial t_d}\nabla_d^2 y_d + \frac{\partial y_d}{\partial y_d}\frac{\partial}{\partial x_d}\nabla_d^2 y_d \\ &\quad - \frac{\partial y_d}{\partial x_d}\frac{\partial}{\partial y_d}\nabla_d^2 y_d - n\nabla_d^4 y_d = 0, \end{aligned} \tag{2}$$

where the subscript d indicates a dimensional quantity. The stream-function may be expressed as the sum of a steady mean and a time-dependent perturbation,

$$y_d(x_d, y_t, t) = \Phi_d(x_d, y_d) + \hat{f}_d(x_d, y_d, t_d).$$

First the following nondimensionalization is used (GN95):

$$y_d = U(x_d)q(x_d)y, \quad \mathrm{d}x = \frac{\mathrm{d}x_d}{q(x_d)}, \quad y = \frac{y_d}{q(x_d)}, \quad t = \frac{Ut_d}{q_d},$$

$$a = a_d q \quad \text{and} \quad w = \frac{w_d q}{U},$$

$$y = \Phi(x, y) + f(x, y)\exp(\mathrm{i}[\textstyle\int a(x)\mathrm{d}x - wt]). \tag{3}$$

(Note incidentally that $wt \equiv w_d t_d$.)

It can be seen that this nondimensionalization represents a departure from all earlier work, in that the scaling is local and takes explicit account of variations with x; although not essential, this approach makes it easier (as we shall show) to establish connections with the Orr-Sommerfeld equation. It must be noted that as q is permitted to be a function of x, the variable y, here and in all subsequent equations, is proportional to what is usually written as h in similarity solutions of the boundary layer equations. For a Falkner-Skan profile, $U \propto x_d^m$, where m is a constant. Therefore

$$x = \frac{2x_d}{(1+m)q} = \frac{R}{p}, \quad \frac{\mathrm{d}q}{\mathrm{d}x_d} = \frac{q}{R}, \quad \frac{\mathrm{d}(Uq)}{\mathrm{d}x_d} = \frac{Up}{R}, \tag{4}$$

where p and q are constants given by

$$p = q^{*2}, \quad q = q^{*2}\frac{(1-m)}{(1+m)},$$

$$q^* \equiv \sqrt{\frac{(1+m)U}{2nx_d}}\int_0^\infty \Phi'(1-\Phi')\mathrm{d}y_d.$$

We note that $\mathrm{d}q/\mathrm{d}x_d = O(R^{-1})$, and assume that a and f cannot vary faster (in x) than does q, i.e. that their first derivatives with respect to x_d are at most of order R^{-1}, and that their second derivatives are $o(R^{-1})$ and can therefore be neglected in comparison.

Furthermore, while a is permitted to vary with x, the disturbance field at any station is assumed to vary harmonically in time with the same frequency w_d. This makes it feasible to handle experiments where a wave-maker imposes a disturbance of given frequency on the flow, as in the experiments of Schubauer and Skramstad and all those that have since been conducted by many workers. Any more general disturbance can always be handled through a suitable Fourier decomposition. However, as $w_d t_d = wt$, constant w_d, as in wave-maker experiments, does not correspond to constant w. When one is interested in following the downstream evolution of a disturbance of given frequency, it can be done in one of several ways. If a stability loop is presented in the (w, R) plane, constant w_d will correspond to a suitably curved trajectory in the (w, R) plane, and it is along such a trajectory that the disturbance would have to be tracked. An alternative procedure, followed in GN95, is to present the stability loop in terms of a transformed frequency variable F that is directly proportional to w_d at all R (i.e. x), so that a straight line in the (F, R) plane can represent the wave-maker experiment and a constant w_d trajectory.

One now substitutes (3) in (2) and expands Φ as

$$\Phi(x, y) = \Phi_0(y) + \frac{1}{R}\Phi_1(x, y) + \ldots,$$

where Φ_0 represents the classical 'Prandtl' solution and Φ_1 comes from higher-order boundary layer theory[8]. It will then be seen that the lowest order mean flow is given by

$$\Phi_0^{\mathrm{iv}} + p\Phi_0\Phi_0''' + (2q - p)\Phi_0'\Phi_0'' = 0, \tag{5}$$

which is the classical Falkner-Skan similarity equation differentiated once with respect to y. Unlike in the traditional Orr-Sommerfeld approach, the correct mean flow equation emerges naturally here. The disturbance streamfunction is given by

$$\{\mathrm{NP}\}f = 0,$$

where the operator, including all terms nominally of $O(R^{-1})$, is (from GN95)

$$
\begin{aligned}
\{\mathrm{NP}\} \equiv\ & i(w - a\Phi_0')[D^2 - a^2] + ia\Phi_0''' + \frac{1}{R}\Bigg(D^4 - 2a^2D^2 \\
& + a^4 + \Big\{ p\Phi_0 D^3 + [(2q - p)\Phi_0']D^2 + [2yqa(w - a\Phi_0') \\
& - pa^2\Phi_0 + (2q - p)\Phi_0'']D + [(q - 2p)aw + p\Phi_0''' \\
& + 3(p - q)a^2\Phi_0'] + (-w + 3a\Phi_0')Ra' \\
& + \left[\Phi_0''' + 3a^2\Phi_0' - 2aw - \Phi_0' D^2\right]R\frac{\partial}{\partial x}\Big\}\Bigg) \\
& + \frac{1}{R}[-ia\Phi_1'(D^2 - a^2) + \Phi_1'''].
\end{aligned} \tag{6}
$$

Here, terms of $O(y/R^2)$ have been neglected, hence the above equation is valid as long as $y \sim o(R)$. Since a'' and $\partial^2 f/\partial x^2$ are both negligible to the given order, at a given Reynolds number a' and $\partial f/\partial x$ are independent of x to the lowest order (a' is a number and $\partial f/\partial x$ is a function only of y). It therefore follows that the above partial differential equation can be treated like an ordinary differential equation in *y for any prescribed value of R*, which plays the role of a parameter just as in the Orr-Sommerfeld equation. Of course neither a' nor $\partial f/\partial x$ is known *a priori*, but they may be computed by an iterative procedure described in GN95.

The boundary conditions are

$$f = Df = 0 \text{ at } y = 0 \text{ and} \tag{7}$$

$$f \to 0,\ Df \to 0 \text{ as } y \to \infty. \tag{8}$$

The behaviour of f at large y has been discussed by GN95.

Structure of the equation

Equation (6), which may be called the 'full non-parallel equation', has the form

$$\{\mathrm{OS}\}f + \frac{1}{R}\{\mathrm{NP}_1 + \mathrm{NP}_h\}f = O\left[\frac{1}{R^2}\right], \tag{9}$$

with the Orr-Sommerfeld operator {OS} containing certain terms of $O(1)$ and others with a factor R^{-1}. The operator $\{\mathrm{NP}_1\}$, contained within curly brackets in (6), consists of non-parallel terms due to the change in the boundary layer thickness, streamwise variations in the freestream velocity as well as the x-dependence of the disturbance. The operator $\{\mathrm{NP}_h\}$, the last term in (6), accounts for higher order corrections to the mean flow, due to displacement thickness, surface curvature, etc. (the effect of displacment thickness on the mean flow for Falkner-Skan wedge flows was considered by GN95). Equation (9) includes all terms with the factor R^{-1} in the primitive variables, and will be termed the primitive 'nominally' correct to $O(R^{-1})$ in the following.

A major qualitative difference between the Orr-Sommerfeld eq. (1) and the 'full non-parallel' eqs (6) or (9), or PSE, is that while (1) is an ordinary differential equation in y, the other equations contain derivatives with respect to x as well. Bertolotti *et al.*[1] solve the PSE by space marching. GN95, on the other hand, note that $\partial f/\partial x$ is independent of the streamwise coordinate x to the order considered, treat it as a perturbation on an ordinary differential equation and solve (9) by a trial and error procedure. In the case of a boundary layer over a semi-infinite flat plate, the two methods when applied to the same equation lead to virtually identical solutions; details of the differences in the equations and approaches are discussed in GN95.

Now the mean flow in general contains contributions of $O(R^{-1})$. (The flow over an infinitesimally thin semi-infinite flat plate is a special case in which these higher order contributions happen to vanish.) A stability analysis conducted using a full non-parallel equation including all terms of $O(R^{-1})$ would be rational only if the mean flow were correct up to this order. Apart from it being not feasible always for the mean flow to be prescribed to this degree of accuracy, it would seem obvious that non-parallel effects must exist even when only the lowest order contributions to the mean flow are known or given. This question has been considered in GN97.

The lowest order theory

At first glance, it might appear from (6) that the Rayleigh equation

$$\{(w - a\Phi_0')(\mathrm{D}^2 - a^2) + a\Phi_0'''\}f = 0, \tag{10}$$

which is the result of omitting all terms containing the factor R^{-1} in (9), is a valid lowest order equation. It is, however, well known that the solution of eq. (10) has a singularity at the critical point $y = y_c$, and that in the associated `critical layer' it is necessary to invoke viscosity. Similarly, near the wall satisfaction of the no-slip boundary condition also demands that viscous effects be taken into account. At large R, the thicknesses of the critical and wall layers are respectively of $O(R^{-1/3})$ and $O(R^{-1/2})$ (e.g. ref. 4). On the lower branch of the Orr-Sommerfeld stability boundary the phase velocity c_r of the wave, and correspondingly also y_c, are so small that the two layers may even merge into each other. Without loss of generality, however, we can proceed by first considering the two separately in the present approach.

Thus, we can say that there are three distinguished limits to consider:

(i) the bulk of the flow (outside layers (ii) and (iii) below), governed by the outer inviscid ('Rayleigh') solutions, defined by y fixed, $R^{-1} \to 0$;

(ii) the critical layer, given by

$$h_c \equiv (y - y_c)/e_1 \text{ fixed, } e_1 \equiv (aR)^{-1/3} \to 0;$$

(iii) the wall layer, given by

$$h_w \equiv y/e_2 \text{ fixed, } e_2 = (aR)^{-1/2} \to 0.$$

In the critical layer, f may be expressed as the asymptotic expansion

$$f(y) \equiv c(h_c) = c_0(h_c) + e_1 c_1(h_c) + \ldots, \tag{11}$$

and Φ_0 expanded in a Taylor series around y_c,

$$\Phi_0 = \Phi_{0c} + \Phi'_{0c}(y - y_c) + \Phi''_{0c}(y - y_c)^2/2 + \ldots \tag{12}$$

On substituting (11) and (12) into the full non-parallel eq. (9), we get, to the leading two orders in e_1, the equations

$$c_0^{(iv)} - \mathrm{i} h_c \Phi''_{0c} c''_0 = 0, \tag{13}$$

$$c_1^{(iv)} - \mathrm{i} h_c \Phi''_{0c} c''_1 = \mathrm{i}\Phi'''_{0c}\left(\frac{1}{2}h_c^2 c''_0 - c_0\right) - p\Phi_{0c} c'''_0. \tag{14}$$

Compared to the well-known inner viscous layer equations in Orr-Sommerfeld theory[4], we see that the only difference is the presence of the additional term $p\Phi_{0c} c'''_0$ in eq. (14), which is in general comparable to the other terms in the equation. (It would become negligible only if y_c, and hence Φ_{0c} also, become small.) Now it is known from Orr-Sommerfeld theory that, to match the logarithmic behaviour of the Rayleigh solution near y_c, it is necessary to consider the two leading terms in the expansion (11). Of the three relevant independent solutions of eq. (13) (the fourth is exponentially large as $h \to \infty$ and so is to be ignored), only the Airy function solution c_{03} that is exponentially small as $h_c \to \infty$ will make an additional contribution to the solution c_1; the reason is that the third derivatives of the other two solutions c_{01}, c_{02} is zero.

We can now 'compose' the lowest order equations at the critical layer as follows. From the full non-parallel eq. (6), we select just those terms that yield, on the use of the expansions (11) and (12), the terms that appear in eqs (13) and (14). This gives us the 'minimal' subset of (6) that adequately represents the critical layer as

$$\{\mathrm{i}(w - a\Phi'_0)\mathrm{D}^2 + R^{-1}\mathrm{D}^4 + [\mathrm{i}a\Phi'''_0 + R^{-1}p\Phi_0\mathrm{D}^3]\}f = 0 \tag{15}$$

where the last two terms, within square brackets, are $O(R^{-1/3})$ relative to the first two. Recalling the definition of p from (4), we see that the term $p\Phi_{0c} c'''_0$ in eq. (14), and the corresponding term in eq. (15), are a direct result of flow non-parallelism, and we shall return to its significance presently.

An exactly similar analysis can be carried out for the wall layer. Expanding Φ_0 around the wall $y = 0$ and noting the wall boundary conditions $\Phi_0(0) = 0$, $\Phi'_0(0) = 0$, we have

$$\Phi_0(y) = \frac{1}{2}y^2\Phi''_0(0) + \cdots. \tag{16}$$

The appropriate viscous wall solution

$$f(y) \equiv \hat{c}(h_w) = \hat{c}_0(h_w) + e_2\hat{c}_1(h_w) + \cdots,$$

where $h_w \equiv y/e_2$, obeys the equations

$$\hat{c}_0^{(iv)} + \mathrm{i}c\hat{c}''_0 = 0$$

$$\hat{c}_1^{(iv)} + \mathrm{i}c\hat{c}''_1 = \mathrm{i}h_w\Phi''_0\hat{c}''_0, \tag{17}$$

which are the same as in classical Orr-Sommerfeld analysis, i.e. no additional term is introduced here by non-parallelism in the flow. A minimal composite equation is therefore

$$\{\mathrm{i}(w - a\Phi')\mathrm{D}^2 + R^{-1}\mathrm{D}^4\} f = 0, \tag{18}$$

which is already contained in eq. (15). It is easy to see why the two additional terms in eq. (15) are unnecessary near the wall. Compared to the terms retained in eq. (18), the Φ'''_0 term is $O(R^{-1})$ and the $p\Phi_0\mathrm{D}^3 f$ term is $O(R^{-1/2})$: the latter is a consequence of the fact that, from eq. (16), Φ_0 is $O(R^{-1})$ in the wall layer.

We also note that, if the critical layer is close to the wall, we have y_c 1 and $\Phi_{0c} = O(y^2_c)$ as in eq. (16), so the effect of the new term in eq. (14) may not be significant. On the other hand, there are flow situations in which y_c is not small (e.g. near the upper branch of the stability loop, especially in strong pressure gradients), hence it is not in general justifiable to ignore the new term.

In the 'bulk' of the flow we have the Rayleigh equation

$$\{\mathrm{i}(w - a\,\Phi_0')(\mathrm{D}^2 - a^2) + \mathrm{i}a\,\Phi_0'''\}f = O(R^{-1}). \qquad (19)$$

As the idea is to treat the problem numerically, GN97 do not handle these distinguished limits separately but instead construct the minimal 'composite' equation

$$\left\{\mathrm{i}(w - a\Phi_0')(\mathrm{D}^2 - a^2) + \mathrm{i}a\,\Phi_0''' + \frac{1}{R}\left(\mathrm{D}^4 + p\Phi_0\mathrm{D}^3\right)\right\}f = 0, \qquad (20)$$

which contains all terms that are of order $R^{-1/2}$ or lower anywhere in the boundary layer, and is therefore (in particular) the rational equation up to that order. A numerical solution of eq. (20), with the boundary conditions (eqs 7, 8), can therefore yield the lowest order stability boundaries for the (non-parallel) flow in a Falkner-Skan boundary layer.

The implication of this work is that the simplest approximation to the stability characteristics of a (non-parallel flow) Falkner-Skan boundary layer is given by the ordinary differential eq. (21); the Orr-Sommerfeld is in principle not appropriate, because it considers only parallel flow. Furthermore, the effects of non-parallelism appear in two different ways. The first is purely geometric, and is taken care of by the introduction of local coordinates through the transformation (3). The second is dynamic, and appears (in the lowest order) solely through the term involving p in eq. (20). As shown by GN97, this dynamic effect is the transport of disturbance vorticity at the critical layer by the mean wall-normal velocity of the (non-parallel) boundary layer.

A higher order treatment

The suggestion implicit in the above statements is that the spatial development of the flow affects stability at a lower order than $O(R^{-1})$. In eq. (20), however, there is no explicit effect of the downstream propagation of the disturbances on the stability. Indeed, a legitimate question about a theory of this type is the following: if an ordinary differential equation in y (like eq. (20)) has a solution $f(y)$, an arbitrary function of x times $f(y)$ is also a solution; so how does the x-dependence get determined? In practice this question has been answered, e.g. in e^n-type calculations, by noting that an o.d.e. like (eq. 20) or the Orr-Sommerfeld equation, through the dependence of R on x, carries x as a parameter. Thus the amplitude of the disturbance at any station is determined by the amplification or attenuation that it suffers through the stability characteristics computed at the immediately preceding station. A more satisfactory answer to this question must, however, proceed from a primitive equation in which the x-dependence is explicit; it is clear that the weak dependence on x, i.e. the fact that $\partial f/\partial x$ is a higher order term, making the parabolicity singular in some sense, holds the key to the answer. This question will be pursued elsewhere, but it can be shown (GN99) that parabolic effects first appear at order $R^{-2/3}$.

Following arguments similar to those above but including only the next round of higher order terms, GN99 show that, to $O(R^{-2/3})$, the equation governing stability is

$$\left[(w - a\,\Phi_0')(\mathrm{D}^2 - a^2) + a\,\Phi_0''' + \frac{1}{\mathrm{i}R}\right.$$
$$\left.\times\left\{\mathrm{D}^4 + p\Phi_0\mathrm{D}^3 + \left(-2a^2 + \Phi_0'\left(2q - p - \frac{\partial}{\partial x}\right)\right)\mathrm{D}^2\right\}\right]f = 0. \qquad (21)$$

It may be noticed that the last term here contains the streamwise derivative of the disturbance eigenfunction, which was absent in the lowest order eq. (20), i.e. the effects of the *parabolic* nature of the flow on its stability first appear in this equation. It is therefore appropriate to call it the 'Lowest-order Parabolic Stability Equation' (LOP). The boundary conditions in y remain the same as in eqs (7) and (8), but need to be supplemented by an initial condition at some x.

It is important to note that the higher order contributions to the mean flow, i.e. Φ_1 and so on, do not affect stability up to the order considered. If, in a quest for greater accuracy, we were to use an equation correct to $O(R^{-1})$, we need to know Φ_1 accurately in order to be consistent.

When the lowest order parabolic equation is compared to the O-S eq. (1), it is noticed that the term $a^4 f$, present in (1), is $O(R^{-1})$ or higher everywhere in the boundary layer and so has to be neglected in the LOP. Instead, the term containing $\mathrm{D}^3 f$ and two additional terms containing $\mathrm{D}^2 f$ are now included. We have already encountered the third derivative term, which has been traced by GN97 to be due to the advection of the disturbance vorticity z_d by the normal component of the mean velocity. The non-parallel component of the streamwise advection of z_d, on the other hand, gives rise to a new second derivative term as well as to the explicit parabolic term in eq. (21).

The origin of each of the terms in eq. (21) can be traced back to a corresponding term in eq. (6), and the primitive equation for eq. (21) may be derived to be (GN99)

$$\frac{\mathrm{D}_0 x_d}{\mathrm{D}t_d} = -\hat{v}_d\,\frac{\partial^3\Phi_{0d}}{\partial y_d^3} + n\,\frac{\partial^3\hat{u}_d}{\partial y_d^3} + 2n\,\frac{\partial^3\hat{u}_d}{\partial x_d^2\partial y_d}, \qquad (22)$$

where D_0/Dt_d stands for the total derivative following the mean flow. This eq. (22) contains all nonparallel effects up to $O(R^{-2/3})$, and so provides a sufficient basis to obtain stability characteristics up to this order. This observation is relevant, especially for non-similar flows where eq. (21) will not hold. In comparison with the primitive equation for the lowest order stability equation (equation (4.3) of GN97), it is seen that the only additional term is the last one, which represents streamwise diffusion of the dominant term in disturbance vorticity. In the LOP (as in the OS equation), this diffusion appears as an additional second derivative term ($-2a^2D^2f$). Note that the last term in eq. (22) is significant only at the critical layer, where the dominant contribution to z_d comes only from $\partial\hat{u}_d/\partial y_d$: the other term $\partial\hat{v}_d/\partial x_d$ in the definition of vorticity z_d will be of higher order.

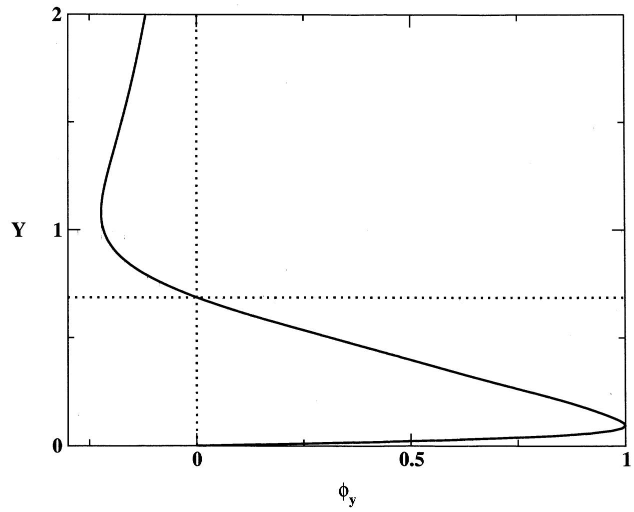

Figure 1. Typical eigenfunction for the Blasius boundary layer, showing the three zeroes respectively at the wall, at infinity and at an intermediate point.

Discussion

Various results that have come out of the present work have been published previously, but we would like to highlight two sets of results which are particularly revealing.

The first set concerns stability 'loops'. Since the work of Tollmein, such loops separating the stable and unstable regimes in the (w, R) or (a, R) space have become very familiar. In nonparallel flows, however, it is now well known that stability characteristics depend on distance y normal to the surface. Simple reflection shows that this is not really as surprising as it seems at first sight. To understand it we first note that the eigenfunction has in general three zeroes, one at the wall, one at infinity and one at some intermediate point in the boundary layer (Figure 1). Consider now a probe that is traversed downstream along a track in the (x, y) plane that passes through the intermediate zeroes of the eigenfunction. Clearly the disturbance level will remain zero all along this track, whether there is stability or instability in the flow along other tracks. Indeed, by considering eigenfunctions at successive stations, it is easy to imagine tracks along which the disturbances may increase or decrease as one moves downstream. Even if the track corresponds to y = constant, y_d will not remain constant. The stability loop thus becomes a function of distance from the *surface*. What is more, it can sometimes take surprising and unsuspected forms (GN97). It therefore becomes necessary to think not of a stability loop but rather of a stability *surface* in the space (y, w, R) or (y, a, R). The nature of such a stability surface for Blasius flow is illustrated in Figure 2, which shows several views of the surface. It is seen that the surface consists of two segments which are stuck to each other with almost a discontinuity located around the intermediate zero of the eigenfunction. At distances just above this location there is a little kink in the upper branch of the loop, of the kind shown in Figure 3. The back of the surface has a marked valley as well as ridge. All of these features basically stem from the way that the amplification rates are determined from the track y = constant cutting across successive eigenfunctions as they develop downstream. One can, of course, still take (by convention) some specific location to give us a *measure* of the stability characteristics of the boundary layer; a good candidate would be the location of the inner maximum in the disturbance distributions. This location has the advantage that it automatically tracks the highest wave amplitudes; incidentally results at this location also happen to be very close to those provided by the classical Orr-Sommerfeld theory (which of course are independent of y). An alternative procedure would be to examine the integral of the disturbance energy across the boundary layer[6].

These results suggest that the concept of stability loops and critical Reynolds numbers becomes somewhat fuzzy in non-parallel flow. Thus, to quote a critical Reynolds number one would have to specify the location in y, and in fact even the disturbance quantity; values will be different for u', v', u'^2, etc. (as Bouthier[5] and Gaster[6] pointed out). In other words, the concept of critical Reynolds number in non-parallel flow is rather like that of the thickness of the boundary layer: there are no unique values, and a good 'measure' is all one can provide.

This of course suggests that looking at the stability loops is perhaps not the best way of characterizing the stability characteristics of the flow. A more direct parameter would be the disturbance intensity as it evolves in the streamwise direction. Figure 4 shows a comparison between the results of GN95 with the calculations

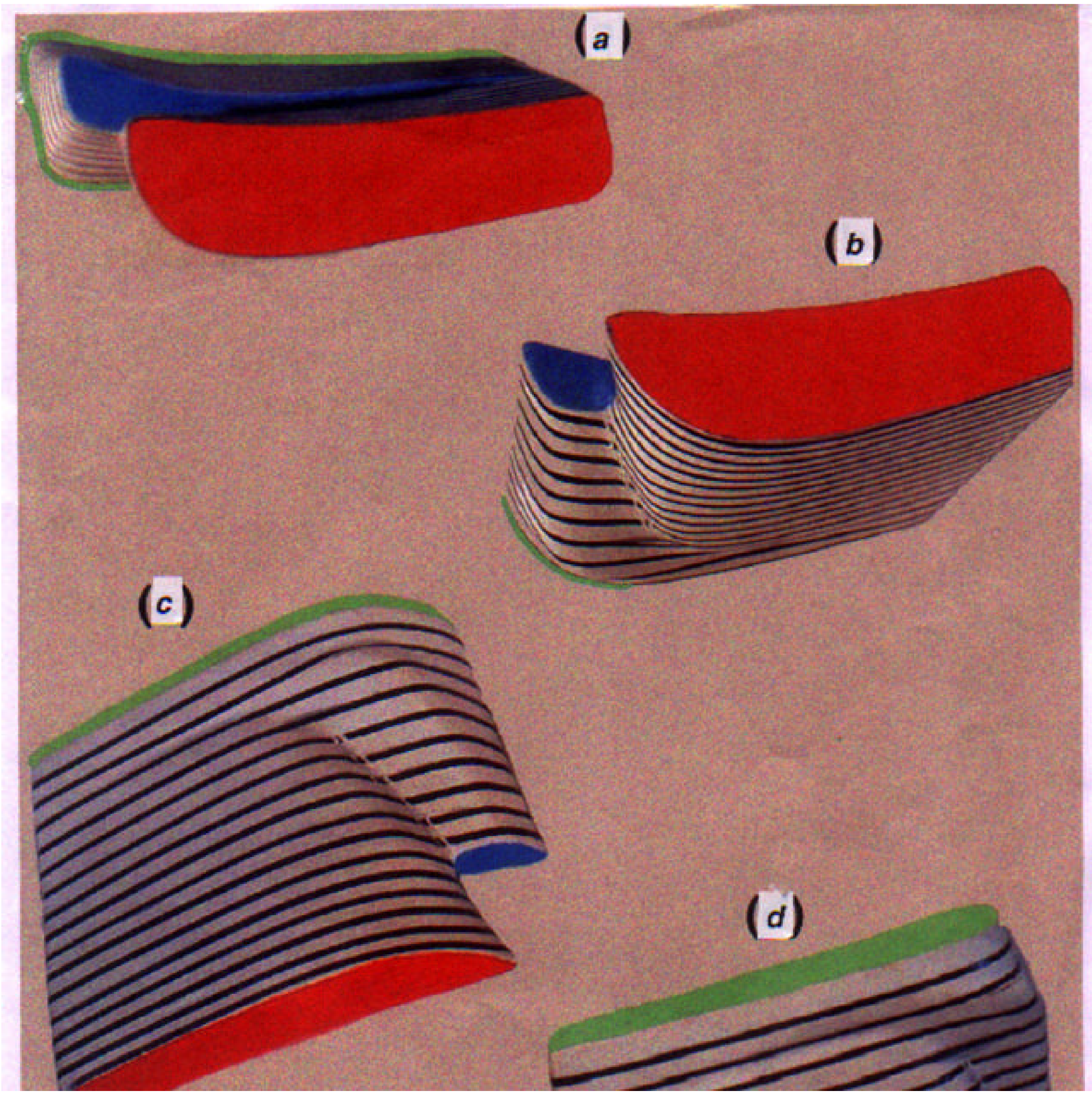

Figure 2. Four views of the stability surface for the Blasius boundary layer, in (y, w, R) space. The surface is generated by stacking up, along the y-axis, stability loops generated at various values of y. The red surface is close to the wall, the blue surface is near the intermediate zero, and the pink surface is near the top of the eigenfunction shown in Figure 1. ***a***, View with R to the right, w towards the top and y into paper. The red region is close to the unstable regime shown in the classical Orr-Sommerfeld stability loop. Note the barely discernible cut-back near the blue loop (shown in greater detail in Figure 3). ***b***, View from below, showing the lower branches of the stability loop stacked along y. ***c***, ***d***, Other views, chiefly of the lower branches, showing the valley and ridge nature of the topography of the stability surface.

of Gaster and the direct numerical simulation of Fasel and Konzelmann[16]. It will be seen that all three curves are close to each other. However, the rather closer agreement between the present results and the numerical simulations does not appear to be accidental. A detailed study (GN00) has shown that the neglect of the change in the shape of the eigenfunction in Gaster's calculations is responsible for the differences noted in Figure 2. Gaster's calculation shows that a very good approximation can in fact be obtained without taking into account the small correction of the eigenfunction that results from accounting for non-parallelism in the theory. Such a calculation is correct to an order less than $R^{-2/3}$. The fact that GN95 includes terms of order R^{-1} appears to be responsible for the closer agreement with DNS.

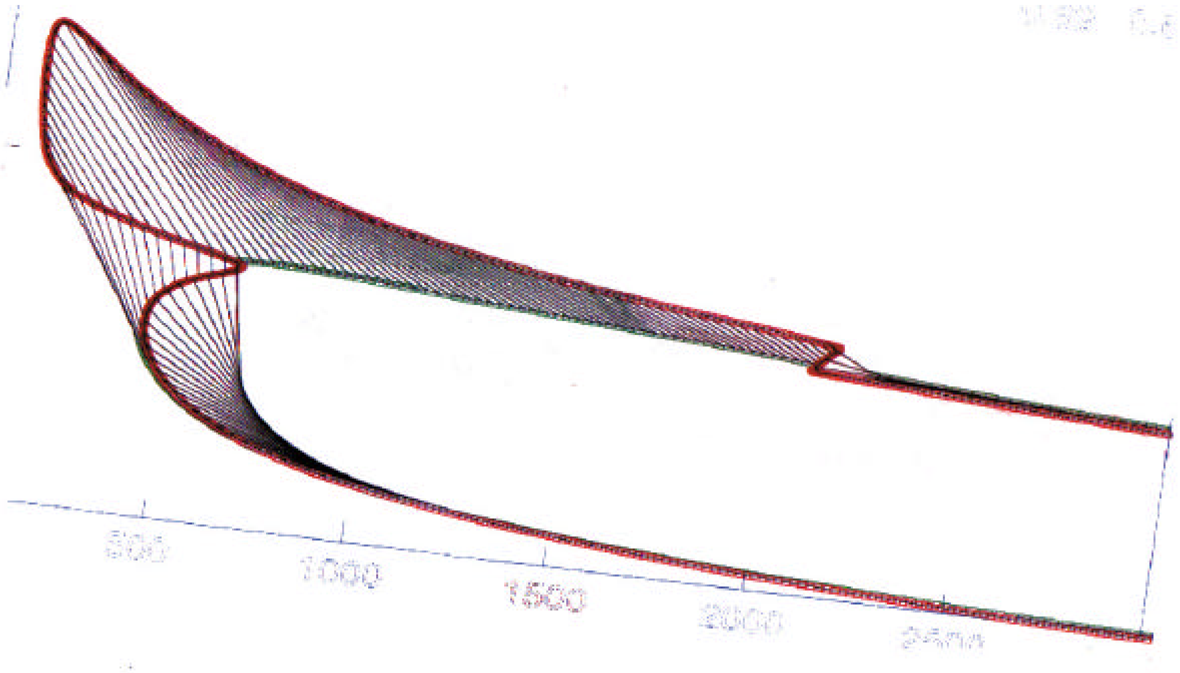

Figure 3. A slice of the stability surface of Figure 2, taken around the blue loop, bounded by $y = 0.69, 0.70$. The axis shown in R. Note the fold-back on the upper branch.

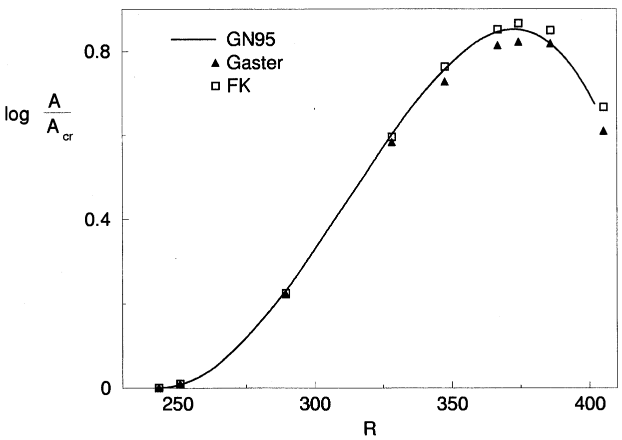

Figure 4. Comparison of the streamwise variation of the amplitude ratios as obtained from three calculations.

In general the differences between Orr-Sommerfeld, the present lowest order theory and the higher order non-parallel flow theories become appreciable as the pressure gradient becomes adverse. The chief reason is that in adverse pressure gradients the critical Reynolds numbers drop, and the higher order terms make significant numerical contributions to the final result. In favourable pressure gradients, on the other hand, the opposite effect (namely that the critical Reynolds numbers increase substantially) makes the higher order contributions less significant.

Conclusions

We could summarize the position as follows. The Orr-Sommerfeld equation, which has been used for a very long time to study stability of boundary layer flows, is valid only for parallel flow; its great advantage is that it is universal, i.e. it is valid for *all* parallel flows and all Reynolds numbers. However the universality is much less powerful than it seems, because the mean flow is supposed to be given independently, and does not always follow from the parallel flow assumption. The number of possible velocity distributions in strictly parallel flow is very limited, but in practice the Orr-Sommerfeld is often used (inconsistently and indiscriminately, in the present view) on *any* 2D velocity profile. Most modern stability analyses use not the OS, but the PSE. Since the lowest order equation for boundary layer stability is (20), it would be more appropriate to think of the PSE as being a higher order correction on eq. (20) rather than on the OS.

When we are looking at stability of boundary layer flows, the limit as Reynolds number tends to infinity is already implicit in the equation itself – and of course goes back to Prandtl's formulation of the theory. Thus, taking the limit $R \rightarrow \infty$ to discuss the stability of the flow is not only legitimate, but indeed both automatic and consistent. The use of minimal composite equations, in this high Reynolds limit, leads in the lowest order to an ordinary differential equation. This equation is rather like the Orr-Sommerfeld, but is not the same. It already takes into account the non-parallelism in the flow; indeed there is one term which explicitly represents the effect. However the ordinary differential equation is valid only for similarity solutions of the boundary layer equation. For zero or favourable pressure gradients the solutions of the lowest order composite equation are very close to those of the Orr-Sommerfeld, because the terms that represent the differences happen to be small. (The reasons for this are that the critical Reynolds numbers – several hundreds or more – are high, and y_c is relatively low.) This of course is why no major deficiencies in the Orr-Sommerfeld have come to light, and it has held sway over boundary layer stability studies for such a long time. On the other hand, in strong adverse pressure gradients the critical Reynolds numbers drop, and the critical layer moves up; and so the present equations give results appreciably different from the Orr-Sommerfeld (GN99).

It is relevant to mention here that the present theory is different from the triple-deck approach of Smith[17], who also proposes a rational theory for the nonparallel stability of boundary layers. The chief difference is that in the present work the frequency and wave number do not participate in the limiting process. Smith's equations are therefore simpler, but they are valid only for $R \quad R_{cr}$, and can predict neither the critical Reynolds number nor the upper branch of the stability loop (although a separate five-deck theory can be formulated for the asymptotic part of the upper branch).

The streamwise evolution of disturbance amplitudes cannot, in principle, be obtained directly by solving the present ordinary differential equation; this, of course, is true for the Orr-Sommerfeld as well. However the equation solved, in either case, does contain x-dependent parameters, for example through the local Reynolds number R which is in general a function of x. It is therefore possible to calculate an amplitude evolution in space by going from one station to the next, as engineers have been doing for more than forty years in the e^n method. A rigorous treatment of streamwise evolution demands a higher order theory which appears here in the form of the lowest order parabolic equation. Being a partial differential equation the streamwise evolution is explicitly present in this equation. However, the streamwise derivative of the eigen-function is a higher order quantity, therefore the parabolicity of this equation is also singular.

If the boundary layer is not similar, the present lowest order equation is strictly speaking not valid. In this case one can make approximate calculations either assuming local similarity or adopting a weakly non-similar approach (GN 95). For more accurate results, one can appeal to the reduced primitive equation that underlies the lowest order theory.

More importantly, the lowest order minimal composite equation is not universal. Thus for each type of flow the governing equation has to be specially derived. As an example, we may consider here the plane far wake, with (constant) free-stream velocity U, and a centre-line velocity defect w_0. (For a preliminary account, see ref. 18.) With appropriate non-dimensionalization, the mean streamwise velocity obeys the similarity solution

$$\Phi_0'(y) = \Lambda - g(y),$$

where $\Lambda(x) = U/w_0(x)$ is the reciprocal of the velocity defect ratio and $g(y)$ is the appropriate similarity function for the defect-velocity profile. Similarity obtains in the far field, where the local Reynolds number is independent of x but Λ varies like $x^{1/2}$. It can then be shown[19] that the lowest order linear stability equation for this flow is

$$\left\{ \mathrm{i}[w - a(\Lambda - g)](\mathrm{D}^2 - a^2) - \mathrm{i}ag'' + \frac{1}{R}\left(\mathrm{D}^4 + \frac{y}{2}\mathrm{D}^3\right)\right\} f = 0,$$

with the boundary conditions

$$\mathrm{D}f = \mathrm{D}^2 f = 0 \text{ at } y = 0,$$
$$f \to 0,\ \mathrm{D}f \to 0 \text{ as } y \to \infty.$$

The lowest order parabolic equation will use only the lowest order mean flow. We believe that the use of more elaborate equations in which the higher order effects on the mean flow (such as surface curvature, displacement, free-stream vorticity, etc.) are ignored is not consistent. Computing higher-order mean flow in the boundary layer presents many problems, and is generally not worthwhile except possibly in high-altitude hypersonic flow applications.

The present theory has been extended to include effects of compressibility[20] and three-dimensionality[21].

1. Bertolotti, F. P., Herbert, Th. and Spalart, P.R., *J. Fluid Mech.*, 1992, **242**, 441–474.
2. Tollmien, W., *Nachr. Ges. Wiss. Göettingen, Math.-phys. Klasse*, 1929, 21–44; also NACA T.M. 609, 1931.
3. Schubauer, G. B. and Skramstad, H. K., *J. Aero. Sci.*, 1947, **14**, 69–78.
4. Drazin, P. G. and Reid, W. H., *Hydrodynamic Stability*, Cambridge Univ. Press, Cambridge, UK, 1981.
5. Bouthier, M., *J. Mech.*, 1973, **12**, 75–95.
6. Gaster, M., *J. Fluid Mech.*, 1974, **66**, 465–480.
7. Saric, W. S. and Nayfeh, A. H., *Phys. Fluids*, 1975, **18**, 945–950.
8. Van Dyke, M., *Ann. Rev. Fluid Mech.*, 1969, **1**, 265–292.
9. Govindarajan, R. and Narasimha, R., *J. Fluid Mech.*, 1995, **300**, 117–147.
10. Govindarajan, R. and Narasimha, R., *Proc. R. Soc. London*, 1997, **453**, 2537–2549.
11. Govindarajan, R. and Narasimha, R., *Phys. Fluids*, 1999, **11**, 1449–1458.
12. Govindarajan, R. and Narasimha, R., *Report FD003*, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore.

13. Van Dyke, M., *Pertubation Methods in Fluid Mechanics*, Academic Press, New York, 1964.
14. Lagerstrom, P. A. and Casten, R. G., *SIAM Review*, 1972 **14**, 63–120.
15. Kevorkian, J. and Cole, J. D., *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York, 1981.
16. Fasel, H. F. and Konzelmann, U., *J. Fluid Mech*., 1990, **221**, 311–347.
17. Smith, F. T., *Proc. R. Soc. London*, 1979, **A366**, 91.
18. Govindarajan, R., Narasimha, R. and Gajjar, J.S.B., Proceedings of the 7th Asian Congress of Fluid Mechanics, Chennai, 1997, pp. 163–166.
19. Govindarajan, R. and Narasimha, R., Stability of the plane far wake. (to be published.)
20. Mitra, D., Seshadri, R., Rao, K. S. and Govindarajan, R., *IUTAM Symposium on Laminar–Turbulent Transition*, Sedona, Arizona, 1999.
21. Govindarajan, R. and Narasimha, R., IUTAM Symposium on Laminar-Turbulent Transition, Sedona, Arizona, 1999.

ACKNOWLEDGEMENTS. Figure 2, which shows different views of the stability surface of the Blasius boundary layer, has been possible because of the considerable effort put in by Mr Ramesh Babu and Mr S. S. Padbidri in creating a physical model of the surface from computed stability loops, and of Dr Sangeetha Menon at NIAS in creating digital imageries of the surface. We thank all three for their help.