A RECONSIDERATION OF THE EFFECTIVE DEPTH OF LINE FORMATION IN PLANETARY ATMOSPHERES

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Abstract. A new definition of the effective depth of line formation is given which incorporates its dependence on the angle of emergence as well as on the position within the line. Solution obtained for isotropic scattering in the third approximation of discrete ordinates explains: (i) Variation of equivalent widths over the disc; (ii) Inverse dependence of this variation on line strength; and (iii) The phase effect for relative equivalent widths.

1. Introduction

Theoretical and observational studies for planetary atmospheres show that the change of equivalent widths and line depths with phase angle, or with colatitude and longitude for a particular phase angle, is an inverse function of line strength (see e.g., Bhatia and Abhyankar 1982; Young *et al.*, 1980). An explanation for these phenomena requires the consideration of the effective depth of line formation. This useful concept was introduced by Chamberlain (1965), who defined it by the equation

$$\langle\tau\rangle = \int_0^\infty \tau S(\tau)\,\mathrm{d}\tau \bigg/ \int_0^\infty S(\tau)\,\mathrm{d}\tau, \qquad (1)$$

where

$$S(\tau) = (1-\tilde{\omega})J(\tau,\mu_0) \qquad (2)$$

is the sink function; here $\tilde{\omega}$ is the single scattering albedo, J the mean intensity of diffuse radiation and $\mu_0 = \cos\theta_0$, θ_0 being the angle of incidence. This may be called the effective absorption depth. In the first approximation in the method of discrete ordinates for an isotropically scattering semi-infinite atmosphere, it can be written (cf., Lestrade and Chamberlain, 1980) as

$$\langle\tau\rangle = \frac{\mu_1}{(1-\tilde{\omega})^{1/2}} + \mu_0 + \mu_1. \qquad (3)$$

We can also define an effective scattering optical depth by Equation (1) with a source function $S = S_{sc}$ given by

$$S_{sc}(\tau) = \tilde{\omega}J(\tau,\mu_0). \qquad (4)$$

One can easily verify that it is identical with the absorption optical depth given by equation (3).

An alternate definition has been given by Kattawar (1979) as

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$$\tau_{\text{eff}} = \int_0^\infty \tau N_{sc}(\tau)\,\mathrm{d}\tau \Big/ \int_0^\infty N_{sc}(\tau)\,\mathrm{d}\tau, \tag{5}$$

where $N_{sc}(\tau)$ represents the number of photon scatterings occurring at depth τ for light which emerges. It is also an effective scattering optical depth. Lestrade and Chamberlain (1980) have shown that the behaviour of τ_{eff} is similar to that of $\langle\tau\rangle$; in particular both of them go to infinity as $\tilde{\omega}$ approaches unity.

All three above definitions do not tell us anything about the dependence of the effective depth of line formation on the angle of emergence, which is very much essential for the interpretation of the variation of line intensities over the disk of the planet and also with phase angle. Therefore, a new formulation is required which establishes both these dependences. We consider two approaches to this problem in this paper.

2. Generalisation of the Eddington–Barbier Relation

(a) Definition of the effective optical depth $\langle\tau\rangle$: – It is well known that in a semi-infinite atmosphere with a linear source function: $S(\tau) = a + b(\tau)$, the emergent intensity is given by

$$I(0,\mu) = a + b\mu = S(\tau = \mu). \tag{6}$$

This is Eddington–Barbier relation which can be interpreted to mean that the effective optical depth $\langle\tau\rangle$ for radiation emerging at an angle $\theta = \cos^{-1}\mu$ is μ. Equation (6) is a consequence of the mean value theorem according to which, since

$$I(0,\mu) = \int_0^\infty S(\tau)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu, \tag{7}$$

and $(1/\mu)\exp(-\tau/\mu)$ does not change sign in the interval $0 \leqslant \tau \leqslant \infty$ we have

$$I(0,\mu) = S(\langle\tau\rangle) \int_0^\infty e^{-\tau/\mu}\, \mathrm{d}\tau/\mu = S(\langle\tau\rangle) \tag{8}$$

for at least one value of $\langle\tau\rangle$ between 0 and ∞.

Now, since Equation (7) is valid for all semi-infinite plane parallel stratified atmospheres – stellar as well as planetary – we can make Equation (8) the basis for defining the effective optical depth. It is evident that $\langle\tau\rangle$ defined in this way is also a function of μ, or θ, the angle of emergence.

(b) Application to an isotropically scattering atmosphere: – For a semi-infinite isotropically scattering atmosphere illuminated by a flux πF at $-\mu_0$, the source function for diffuse radiation (cf. Chandrasekhar 1960) is of the form

$$S(\tau) = \tfrac{1}{4}\tilde{\omega}F\left[\sum_{r=1}^{n} L_r\, e^{-k_r\tau} + \gamma\, e^{-\tau/\mu_0}\right] \tag{9}$$

in the nth approximation of discrete ordinates. The corresponding emergent intensity is

$$I(0,\mu) = \tfrac{1}{4}\tilde{\omega}F\left[\sum_{r=1}^{n}\frac{L_r}{1+k_r\mu} + \frac{\gamma}{1+\mu/\mu_0}\right]. \tag{10}$$

In this latter equation γ, k_r, and L_r are given, for an albedo $\tilde{\omega}$, by the equations

$$\frac{1}{\gamma} = 1-\tilde{\omega}\sum_{j=1}^{n} a_j/(1-\mu_j^2/\mu_0^2) \tag{11}$$

$$\frac{1}{\tilde{\omega}} = \sum_{j=1}^{n} a_j/(1-\mu_j^2k^2) \tag{12}$$

and

$$\sum_{j=1}^{n} L_r/(1-\mu_i k_r) + \gamma/(1-\mu_i/\mu_0) = 0, \qquad (i=1,n), \tag{13}$$

respectively. Hence, we can define an effective optical depth by the equation

$$\sum_{r=1}^{n} L_r\, e^{-k_r\langle\tau\rangle} + \gamma\, e^{-\langle\tau\rangle/\mu_0} = \sum_{r=1}^{n}\frac{L_r}{1+k_r\mu} + \frac{\gamma}{1+\mu/\mu_0}. \tag{14}$$

We have solved Equation (14) in the 3rd approximation for various points on the disc of a planet viewed at different phase-angles α, which stands for the usual planetocentric elongation of the Earth from the Sun denoted by ES in Figure 1. Points on the disc like × are specified by the planetocentric longitude η (EY) and colatitude β(PX) measured with respect to the sub-earth point, η being measured positive towards the limb. The values of μ and μ_0 for them are given (cf. Bhatia and Abhyankar, 1982) by

$$\mu = \sin\beta\cos\eta, \qquad \mu_0 = \sin\beta\cos(\eta-\alpha). \tag{15}$$

In this paper we will deal with points on the intensity equator (AB, $\beta = 90$) and on the mirror meridian (PMP′) on which $\eta = \alpha/2$ and $\mu = \mu_0$.

(c) Ambiguity of $\langle\tau\rangle$ for non-monotonic source functions: – Figure 2 shows the source function $S(\tau)$ of Equation (9) for $\tilde{\omega} = 0.997$ for various points on the intensity equator for $\alpha = 80°$. The points at which it becomes equal to the emergent intensity of Equation (10) are marked by a cross on each curve. We see that for small values of η – i.e. large valeus of μ – the effective optical depth is uniquely determined because the source function monotonically decreases with increasing optical depth. On the other hand, for larger values of η, i.e. small vlaues of μ, the source function first increases and then decreases with increasing optical depth. Consequently, there are two possible values of $\langle\tau\rangle$, say τ_1 and τ_2 ($\tau_2 > \tau_1$). Then the actual effective depth will be some weighted mean of the two:

$$\langle\tau\rangle = \frac{a_1\tau_1 + a_2\tau_2}{a_1 + a_2}. \tag{16}$$

For example, on physical grounds we can say that $\langle\tau\rangle$ should approach zero as μ goes to 0

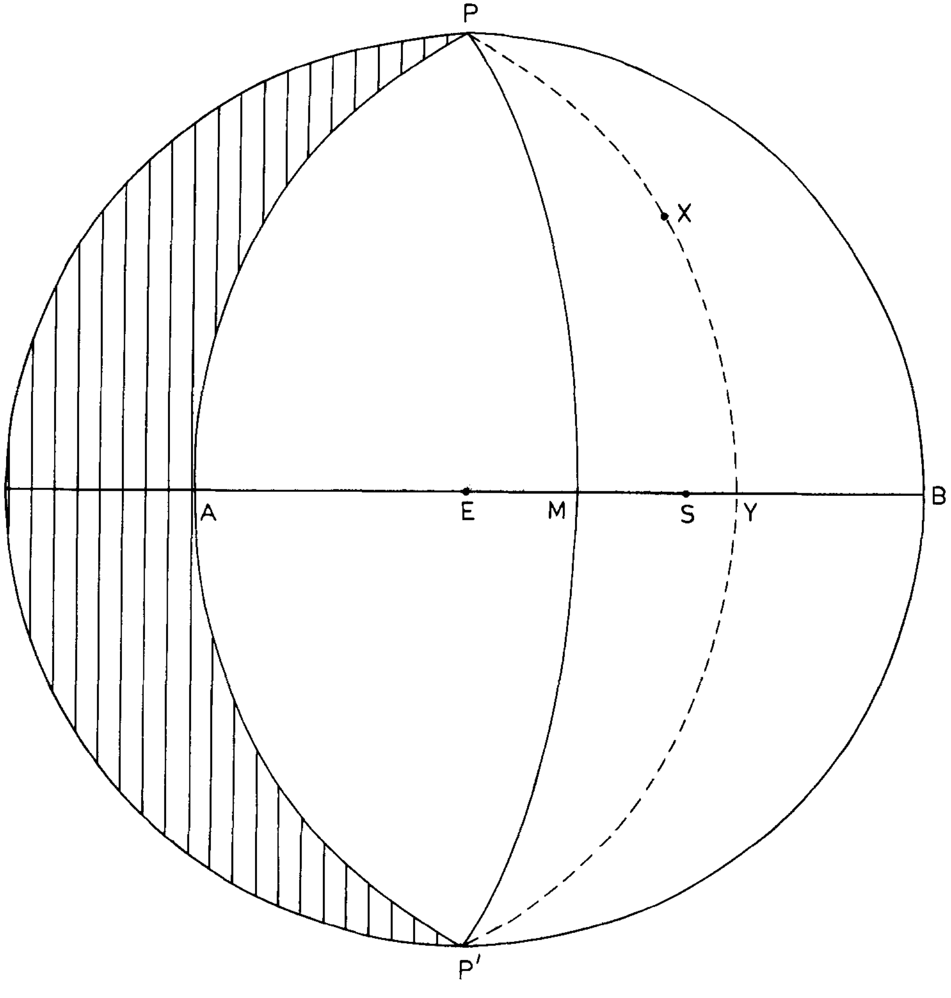

Fig. 1. Visible disc of a planet at phase angle $\alpha = \mathrm{SE} < 90°$. E is the subearth point, S the subsolar point, AB the intensity equator, M the mirror point and PMP′ the mirror meridian. X is any point with $\eta = \mathrm{EY}$ and $\beta = \mathrm{PX}$.

due to the grazing emergence of radiation; hence $a_1 \to 1$ and $a_2 \to 0$ as $\mu \to 0$. However, there is no satisfactory criterion for assigning the weights a_1 and a_2 which would be applicable in general. We will always encounter this problem for large values of $\tilde{\omega}$. The same will be true for a stellar atmosphere in which there is a temperature reversal at some depth giving rise to a source function of type $S(\tau) = a - b\tau + c\tau^2$, where a, b, c are all positive.

Hence we have to abandon this approach of generalising the Eddington–Barbier relation and consider an alternate approach.

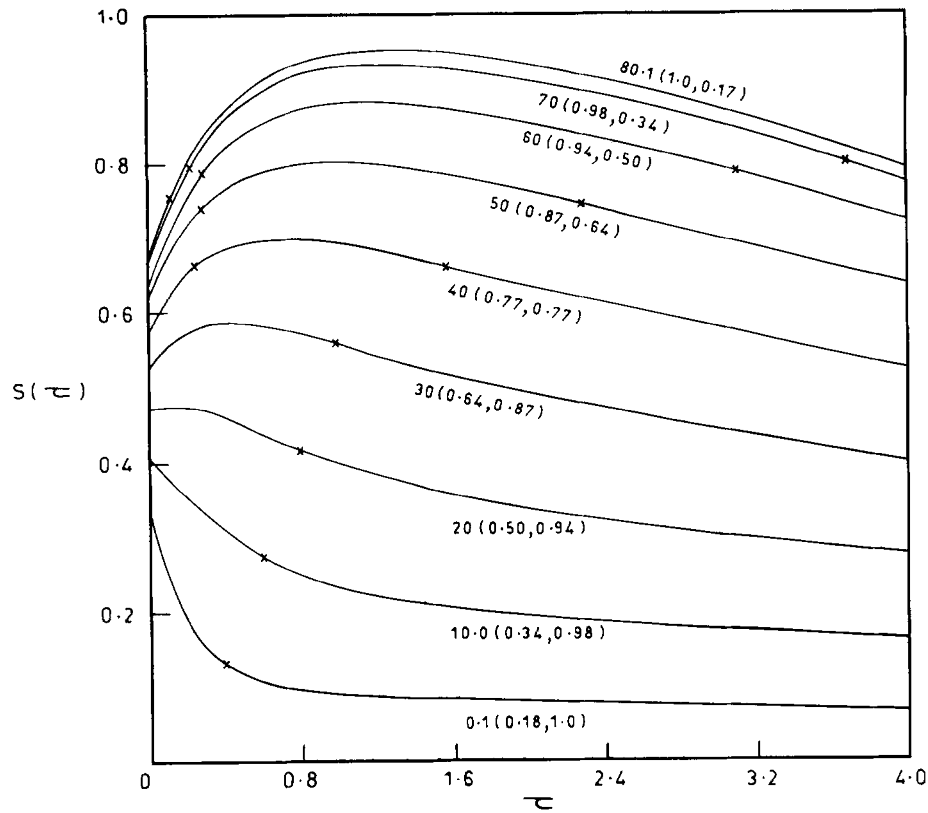

Fig. 2. Source function for diffuse radiation for $\tilde{\omega} = 0.997$ when $\alpha = 80°$ for various points on the intensity equator. The numbers on the curves denote the longitude η with μ_0 and μ in the bracket.

3. Modification of Chamerberlain's Definition

(a) *General case*: – We have already noted that Lestrade and Chamberlain (1980) had found that the definitions (1) and (5) both lead to an infinite value of $\langle\tau\rangle$ as $\tilde{\omega}$ approaches unity. This difficulty arises because there $S(\tau)$ itself is used as a weighting function for averaging τ. In fact it is not $S(\tau)$ but its reduced value $S(\tau)\, e^{-\tau/\mu}$ which contributes to the emergent intensity. Hence, it is proper to modify Chamberlain's definition to the effective optical depth and put

$$\langle\tau\rangle = \frac{\int_0^\infty \tau S(\tau)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}{\int_0^\infty S(\tau)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}, \tag{17}$$

where $S(\tau)$ is the actual source function of thermal or diffuse radiation, (or a combination of both) in the atmospheres under consideration. It may be mentioned that $S(\tau)$ takes into account the full effect of multiple scattering – i.e., diffusion of photons.

Now integrating Equation (17) once by parts we obtain

$$\langle\tau\rangle = \frac{\int_0^\infty \mu[S(\tau) + \tau S'(\tau)]\, e^{-\tau/\mu}\, d\tau/\mu}{\int_0^\infty S(\tau)\, e^{-\tau/\mu}\, d\tau/\mu}. \tag{18}$$

Successive repetition of this process gives

$$\langle\tau\rangle = \mu\left[1 + \frac{\sum_{n=1}^{\infty} \mu^n \int_0^\infty S^{(n)}(\tau)\, e^{-\tau/\mu}\, d\tau/\mu}{\int_0^\infty S(\tau)\, e^{-\tau/\mu}\, d\tau/\mu}\right], \tag{19}$$

where $S^{(n)}(\tau)$ is the n-th derivative of $S(\tau)$. The series on the right hand side of Equation (19) will converge for all values of μ $(0 \leqslant \mu \leqslant 1)$ if $S(\tau)$ is either a polynomial in τ or if it contains terms of type $e^{-k\tau}$, $(k > 0)$. These conditions are necessarily satisfied in all physical atmospheres.

Equation (19) shows that the first term in the series for $\langle\tau\rangle$ is μ which corresponds to the Eddington–Barbier relation. We also see that $\langle\tau\rangle = 0$ for $\mu = 0$, and it normally increases with increasing value of μ reaching a maximum at $\mu = 1$. For example in the case of the linear source function $S(\tau) = a + b\tau$, we have $\langle\tau\rangle = \mu[1 + b\mu/(a + b\mu)]$.

(b) *Application to an isotropically-scattering atmosphere*: – Substituting $S(\tau)$ of Equation (9) in Equation (17) we obtain

$$\langle\tau\rangle = \mu\left[\frac{\sum_{r=1}^{n} \frac{L_r}{(1 + k_r\mu)^2} + \frac{\gamma}{(1 + \mu/\mu_0)^2}}{\sum_{r=1}^{n} \frac{L_r}{(1 + k_r\mu)} + \frac{\gamma}{(1 + \mu/\mu_0)}}\right]. \tag{20}$$

We have evaluated $\langle\tau\rangle$ by Equation (20) with $n = 3$ for various points on the disc of a planet taking several values of $\tilde{\omega}$ and α. The results are shown in Figures 3, 4, and 5; they are discussed below.

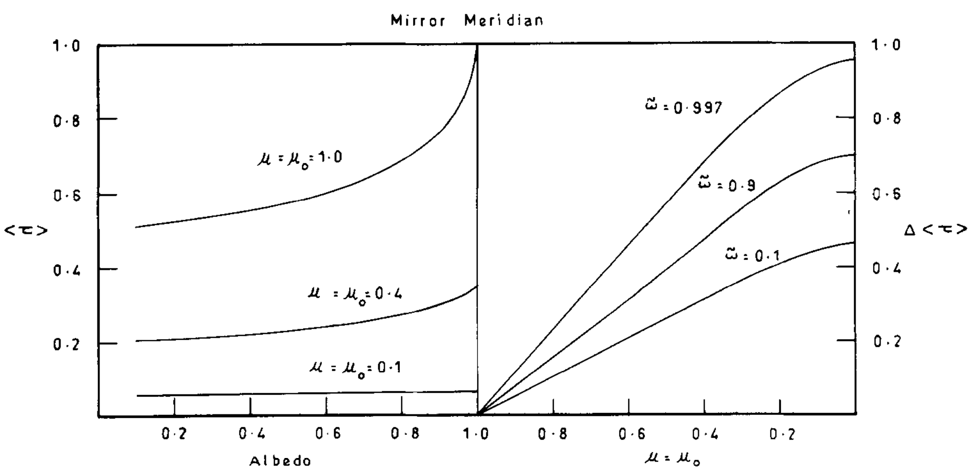

Fig. 3. Left: Variation of the effective optical depth with albedo for three points on the mirror meridian. Right: Change in the effective optical depth from equator to pole for three values of albedo.

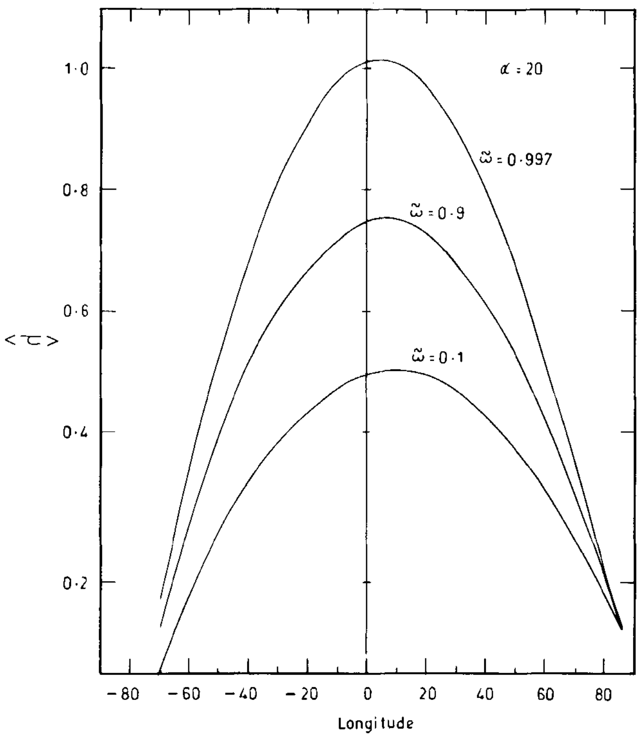

Fig. 4(a). Variation of the effective depth along the intensity equator for $\alpha = 20°$.

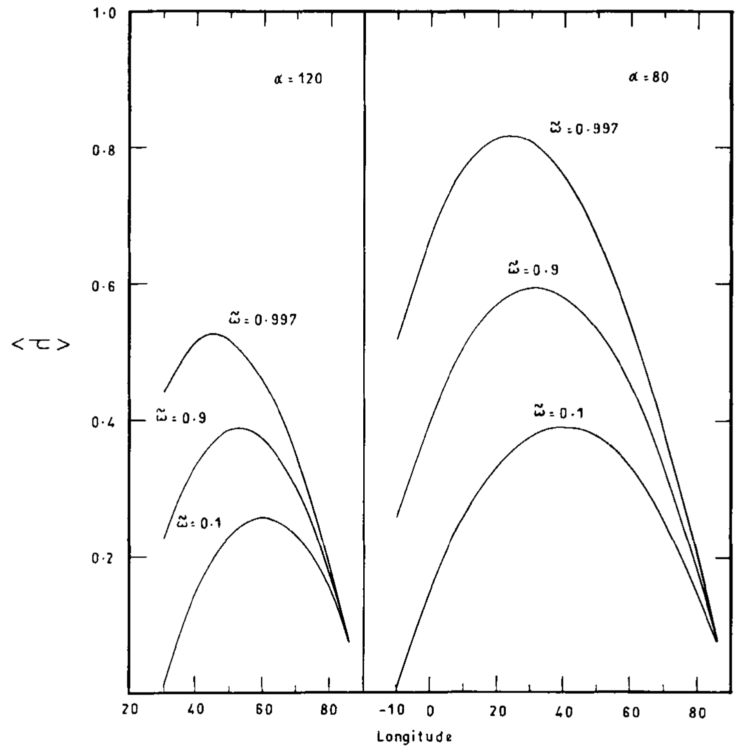

Fig. 4(b). Same as Figure 4(a) for $\alpha = 80°$ and $120°$.

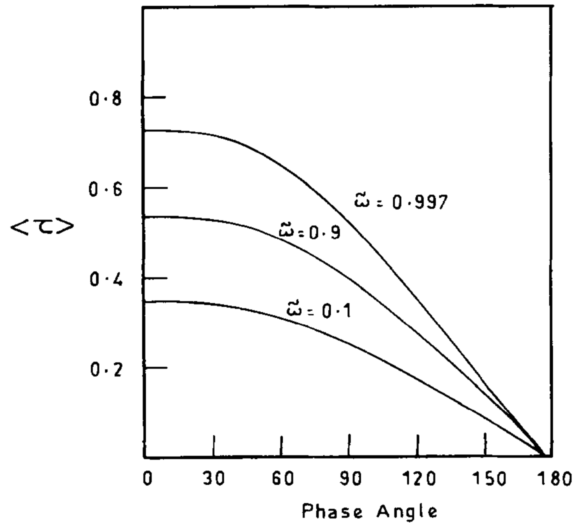

Fig. 5. Variation of the flux averaged effective optical depth with phase angle.

The left part of Figure 3 shows the effective optical depth as a function of $\tilde{\omega}$ for three values of $\mu = \mu_0$ on the mirror meridian for $\alpha = 80^\circ$. It is seen that $\langle\tau\rangle$ increases with increasing μ for all values of $\tilde{\omega}$ from which it can be inferred that the absorption lines should increase in strength from the poles to the equator. Further, the change in optical depth with μ is larger for larger values of $\tilde{\omega}$ as shown in the right part of the figure. Since increasing $\tilde{\omega}$ reduces the strength of the line we expect weaker lines to show larger fractional increase in their strength as one goes from the poles to the equator.

Figures 4(a) and (b) show the effective optical depth for points along the intensity equator for three phase angles. Again the variation is larger for larger values of $\tilde{\omega}$ – i.e., for weaker lines. Further, the variation towards the limb ($\eta = 90^\circ$) is larger than that towards the terminator, for large values of $\tilde{\omega}$, i.e. for weaker lines.

The effective optical depth for lines observed in the total flux can be obtained in two ways. In one we can first calculate the flux averaged values of μ and μ_0 by

$$\langle \mu \text{ or } \mu_0 \rangle = \frac{\int (\mu \text{ or } \mu_0) I\mu \, d\mu}{\int I\mu \, d\mu}, \tag{21}$$

and then calculate $\langle\tau\rangle$ by substituting these values of μ and μ_0 in Equation (20). In the second method we first calculate $\langle\tau(\mu, \mu_0)\rangle$ for each pair of μ and μ_0 by Equation (20) and then obtain $\langle\tau\rangle$ by

$$\langle\tau\rangle = \frac{\int \tau(\mu, \mu_0) I\mu \, d\mu}{\int I\mu \, d\mu}. \tag{22}$$

Integrals on the right hand sides of Equations (21) and (22) can be evaluated by the Gauss–Chebyshev quadrature as descirbed by Horák (1950); we have used a 36-point quadrature in our calculations. Both methods give similar results which are shown in Figure 5; it again shows that the variation is larger for higher values of $\tilde{\omega}$ i.e. for weaker lines. Further there is a very small increase in $\langle\tau\rangle$ with α at small α before the setting in of the decrease at larger phase angles. This would give rise to a small inverse phase effect for line strength even for isotropic scattering (cf., Section 4 below).

4. Comparison with Computed Line Strengths

The strength of a spectral line is characterised by its equivalent width defined by

$$W = \int_{\text{line}} (1 - I_\nu/I_c)\,\mathrm{d}\nu \tag{23}$$

where I_ν and I_c are the intensities or fluxes in the line and the continuum, respectively. In an isotropically scattering atmosphere illuminated by a flux πF at $-\mu_0$, the emergent intensity is given by

$$I(0, \mu) = \tfrac{1}{4}\tilde{\omega}_\nu F \frac{\mu_0}{\mu + \mu_0} H(\mu)H(\mu_0) \tag{24}$$

where the required H-functions are tabulated by Abhyankar and Fymat (1971). Assuming a Lorentz profile for the line, one can write

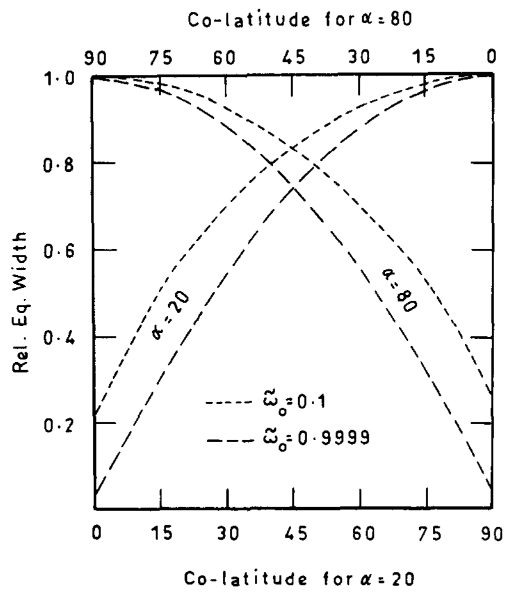

Fig. 6. Variation of the relative equivalent width $W(X)/W(M)$ for points on the mirror meridian in an isotropically scattering semi-infinite atmosphere.

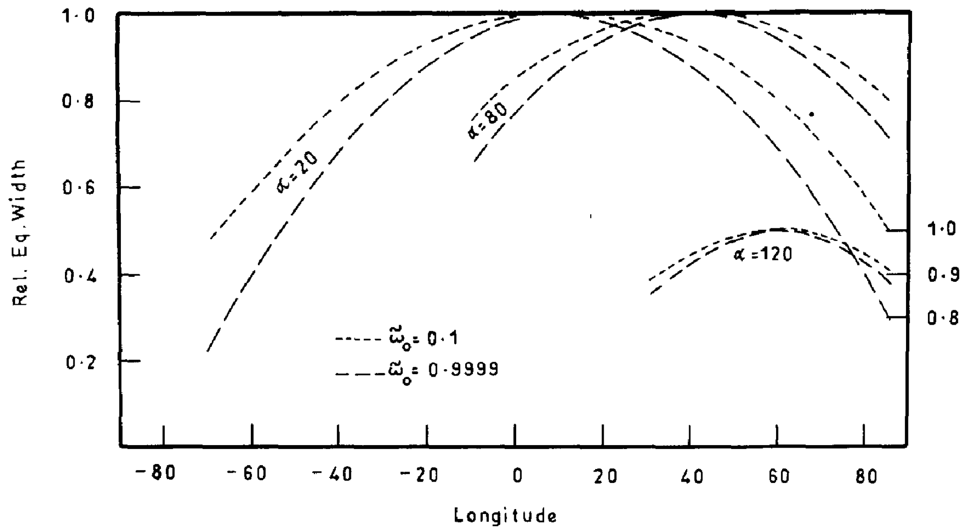

Fig. 7. Variation of the relative equivalent width along the intensity equator for an isotropically scattering semi-infinite atmosphere. For $\alpha = 120°$ see the ordinate scale at right.

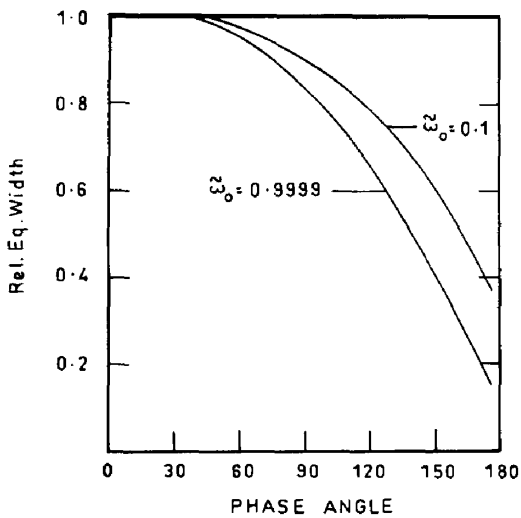

Fig. 8. The phase effect for equivalent widths in an isotropically scattering semi-infinite atmosphere.

$$\tilde{\omega}_\nu = \left[\frac{1-\tilde{\omega}_0}{\tilde{\omega}_0(1+x)^2} + \frac{1}{\tilde{\omega}_c}\right]^{-1} \tag{25}$$

where $\tilde{\omega}_c$ is the continuum albedo, $\tilde{\omega}_0$ is the albedo at line centre in the absence of continuum and $x = (\nu - \nu_0)/\alpha_L$, α_L being the Lorentz half width. Substitution of Equations (25) and (24) in Equation (23) will give the equivalent width of a line of a strength characterised by a particular value of $\tilde{\omega}_0$, at any point on the disc of a planet. A Gauss–Chebychev cubature over the visible part of the disc gives the equivalent widths in the integrated light; for details of such calculations see Bhatia and Abhyankar (1982). The relevant results for isotropic scattering are shown in Figures 6, 7, and 8 for three values of $\tilde{\omega}_0$ taking $\tilde{\omega}_c = 0.997$. It may be noted that smaller the value of $\tilde{\omega}_0$ stronger is the line.

Figure 6 shows the variation of relative equivalent width for two lines along the mirror meridian for $\alpha = 20$ and 80. We see that both lines become weaker as we go from equator to pole and the variation is larger for the weaker line with $\tilde{\omega}_0 = 0.9999$. These variations nicely correlate with the variations of the effective depth of line formation illustrated in Figure 3.

Figure 7 shows the variation of the relative equivalent widths for points on the intensity equator for three phase angles $\alpha = 20$, 80, and 120. The relative equivalent widths are expressed in terms of their values at the mirror point M which is the intersection of the intensity equator and the mirror meridian. We see a close similarity between Figures 7 and 4 which indicates that variation of equivalent width is a direct consequence of the variation of the effective optical depth. Both of them show more change for weaker lines with larger $\tilde{\omega}_0$.

Finally Figure 8 shows the variation of equivalent widths in the integrated light as a function of phase angle. A comparison with Figure 5 immediately makes it apparent that the normal phase effect in which the equivalent widths decrease with increasing phase angle is a consequence of the decrease in the effective optical depth with increasing phase angle. The small increase in effective depth at small phase angles is also reflected in the small inverse phase effect, even for isotropic scattering, as displayed in Table I.

TABLE I

Relative equivalent widths for isotropic scattering

Phase angle	Relative equivalent widths $W(\alpha)/{}^{+}W(0)$		
	$\tilde{\omega}_0 = 0.1$	$\tilde{\omega}_0 = 0.9$	$\tilde{\omega}_0 = 0.9999$
0	1.0000	1.0000	1.0000
20	1.0024	1.0024	1.0029
40	0.9953	0.9922	0.9897
60	0.9723	0.9597	0.9501
80	0.9311	0.9030	0.8814
100	0.8697	0.8203	0.7823
120	0.7849	0.7075	0.6527
135	0.7031	0.6017	0.5355
150	0.6021	0.4766	0.4027
175	0.3677	0.2131	0.1452
$\dfrac{{}^{+}W(0)}{2\alpha_L}$	14.2268	11.4844	0.0017

5. Concluding Remarks

(a) Effective optical depth for a general phase function: – For non-isotropic scattering the source function will be a function of direction: $S(\tau, +\mu, \varphi)$. Then

$$I(0, +\mu, \varphi) = \int_0^\infty S(\tau, +\mu, \varphi)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu;$$

form which we define

$$\langle\tau\rangle = \frac{\int_0^\infty \tau S(\tau, +\mu, \varphi)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}{\int_0^\infty S(\tau, +\mu, \varphi)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}, \tag{26}$$

where $\langle\tau\rangle$ is now a function of μ and φ. If polarisation is taken into account both I and S will be vectors and we can define an effective optical depth for each component I_1, I_r, V and U separately. For example,

$$\langle\tau_1\rangle = \frac{\int_0^\infty \tau S_1(\tau, +\mu, \varphi)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}{\int_0^\infty S_1(\tau, +\mu, \varphi)\, e^{-\tau/\mu}\, \mathrm{d}\tau/\mu}.$$

Although we have not made such calculations the results obtained for isotropic scattering can be qualitatively indicative of what would be expected. For example the variation of equivalent widths for Stokes parameters I_1^e and I_r^e parallel and perpendicular to the intensity equator as well as total intensity over the disc for a Rayleigh scattering atmosphere found by Bhatia and Anhyankar (1983) are qualitatively in agreement with the variation of effective depth of line formation for isotropic scattering. Even the small increase in equivalent widths in the integrated light is present between phase angles 0 and 40 for the I_r^e component which is essentially scattered isotropically.

(b) Effective optical depth for the whole line: – Since $\tilde{\omega}_\nu$ is a function of frequency we will have a different $\langle\tau\rangle_\nu$ for each point in the line. Then we can define a mean optical depth for the whole line through the equation

$$\langle\tau\rangle_{\text{line}} = \frac{\int_{\text{line}} \langle\tau\rangle_\nu (1 - I_\nu/I_c)\, \mathrm{d}\nu}{\int_{\text{line}} (1 - I_\nu/I_c)\, \mathrm{d}\nu} \tag{27}$$

or

$$\langle\tau\rangle_{\text{line}} = \frac{1}{W} \int_{\text{line}} \langle\tau\rangle_\nu (1 - I_\nu/I_c)\, \mathrm{d}\nu.$$

Calculation of $\langle\tau\rangle_{\text{line}}$ for various line strengths will assist in determining the change of temperature with altitude and similar stratification effects as inferred by Kattawar (1979).

(c) Consideration of a finite atmosphere: – Our typical calculations show that $\langle\tau\rangle \leqslant 1$ for most cases although the atmosphere is taken to be semi-infinite. Hence any atmosphere, whose optical depth exceeds a few times unity, can be considered semi-infinite for all practical purposes. Thus the atmospheres of Venus and Jovian planets can be taken as semi-infinite – albeit inhomogeneous – for most calculations of line profiles and equivalent widths.

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