PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Cascade Theory with Collision Loss

H. J. Bhabha and S. K. Chakrabarty

Proc. R. Soc. Lond. A 1943 **181**, 267-303
doi: 10.1098/rspa.1943.0007

The cascade theory with collision loss

By H. J. Bhabha, F.R.S. and S. K. Chakrabarty

(*Received* 2 *March* 1942)

Electrons are assumed to suffer a constant energy loss β by collision, and the radiation loss and pair creation are taken to be described by the formulae of Bethe and Heitler valid for complete screening. With these assumptions a solution of the cascade equations is given in the form of a series, and it is shown that the series is so rapidly convergent that in general it is necessary only to calculate the first term. Collision loss enters into each of the terms in an essential way, and as a result the first term alone gives to a very considerable degree of accuracy *the whole energy spectrum of electrons from the highest energy to energies far below the critical energy*. For thicknesses greater than 1·5 in the characteristic unit of length the number of particles of energy E increases monotonically with decreasing E, but the spectrum gets flattened for energies below the critical energy. For thicknesses t below 1·5, the spectrum has a very different shape, decreasing first as E decreases from the primary energy and then increasing again to the smallest E, the flattening taking place now only for $E < \beta t$. It is shown that neglect of collision loss sometimes causes the number of electrons of even the critical energy to be as much as seven times too large. Tables of the spectra of cascade electrons due to primaries of different energies are given for five typical thicknesses.

The solution is also valid when the energy of the primary electron starting the cascade is comparable with or lower than the critical energy, and gives in a compact form the complete solution of the problem of the absorption of a low-energy electron by collision loss and cascade production.

It is generally accepted that the cascade theory put forward by Carlson & Oppenheimer (1937) and Bhabha & Heitler (1937) correctly describes all the general features of the behaviour of the soft component of cosmic radiation. Moreover, it appears to be in rough quantitative agreement with the observed absorption curve of cosmic radiation in the atmosphere and the production of electron showers in heavy materials by fast electrons. Nevertheless, a very exact comparison between theory and experiment has not yet been made, due, on the one hand, to inaccuracies in the theory introduced by the inexactness of the physical assumptions and approximations in the mathematical treatment, and on the other hand to uncertainties in the experimental data itself. A stage is, however, being reached when the gradually increasing data on the atmospheric absorption curves at different latitudes makes a more accurate knowledge of the theoretical behaviour of cascade processes necessary in order that a more exact comparison between theory and experiment may be made with a view to finding out whether experiment reveals other processes besides those explained by the cascade theory, and to what extent. The purpose of this paper is to give a more accurate theoretical treatment of cascade processes than has been done hitherto.

All the previous theoretical treatments of the subject have been inexact, first, because approximations have been made in describing the physical processes which play a role in the cascade theory, and secondly, because the mathematical treatment has sometimes not been exact even on the basis of these physical assumptions. The weakest feature in all the previous treatments has been their very inadequate

consideration of the effect of the ionization or collision loss suffered by the cascade electrons. In the paper of Bhabha & Heitler the concept of critical energy was introduced, defined as the energy of an electron for which its radiation loss is equal to its collision loss in the substance, and it was assumed that the collision loss would not appreciably affect the cascade process above the critical energy, while it would materially reduce the multiplication of electrons below the critical energy and lead to an absorption of the cascade. But the collision loss was not treated quantitatively. An attempt to calculate the number of electrons below the critical energy more accurately was made by Arley (1938) on the basis of very rough physical assumptions. The more accurate calculations of this paper show, however, that the figures he got for the distribution of electrons below the critical energy are not even qualitatively right for small thicknesses,* and give a justification for the very rough assumption of Bhabha & Heitler that for all except extremely small thicknesses the number of particles per unit energy range below the critical energy always increases slowly with decreasing energy.

Carlson & Oppenheimer, and later Snyder (1938) and Serber (1938) have given approximate formal solutions of the cascade equations with collision loss in the form of double integrals in the complex plane. These expressions cannot be calculated directly, and for evaluating the number of cascade electrons of energy E, they all get in essence an infinite series in powers of β/E, where β is approximately the critical energy. Closer analysis shows that these series are *divergent for all values of* E, and the one which is based on the most accurate physical assumptions, namely, Serber's series, is essentially the formal solution given by equation (42) of this paper, which we have rejected for the same reason. It will appear that this series cannot even be considered as an asymptotic solution of the problem when $E \lesssim \beta$. It can be shown that the divergence of Snyder and Serber's series is intimately connected with the fact that their solutions do not satisfy the correct boundary conditions at the surface of the layer. In our opinion the divergence of their series makes it impossible to say that they satisfy the actual boundary conditions even approximately. Snyder and Serber have used their solutions for calculating the total number of particles of all energies at any depth, but the serious defects mentioned above seem to us to throw doubt on the reliability of their numerical results. It is also obvious that even without the above defects all these methods would completely fail to give the energy spectrum of electrons with energies in the neighbourhood of the critical energy, which is a region of very great interest both theoretically and from the experimental point of view.

It is clear that a solution of the cascade problem valid over the whole range of energies cannot be obtained in the form of a series in powers of β. For if the collision loss be neglected, then the spectrum of electrons is of the form $1/E$ for small E and

* Arley has completely neglected the production of electrons below the critical energy by pair creation. This process is just the one that gives the most important contribution at small thicknesses, as will be seen in the last section of this paper. But his treatment of radiation processes is also much too crude to claim any quantitative accuracy.

hence tends to infinity as $E \to 0$. The effect of collision loss at depths which are not too small is to cut the spectrum off for energies below the critical energy, and to make the number of electrons finite as $E \to 0$. For low energies therefore the collision loss does not introduce a small correction, but plays an essential part in the process, and hence must enter in some essential way into the solution. In incorporating this feature the solution of this paper differs essentially from all the previous treatments of the problem, and may be regarded as an advance on all the previous work.

In the present paper the collision loss is treated as a constant loss independent of the energy of the electron. With this assumption a mathematical solution of the cascade equations is given which allows the energy distribution of electrons to be calculated for *all* energies, *including those near and below the critical energy*. The solution is formally in the form of an infinite series, but this series is not strictly a series in powers of β, since β enters essentially into the expression for each term. The first few terms of the solution can be evaluated without undue difficulty, and it can be shown that the main contribution comes from the first term, the second being always smaller than the first, and for all except the very end of a shower less than 30 % of the first, *even for electrons of energy near or less than the critical energy*. Moreover, the solution is valid irrespectively of how small the energy of the electron starting the shower may be, in contrast with the work of the other authors mentioned above. Indeed, for an electron of initial energy less than the critical energy it can be shown that all but the first term of the solution are negligible, so that our method incidentally also gives the complete solution to the problem of the absorption of a low-energy electron by collision loss with a very small accompanying cascade. It also allows one to calculate the energy spectrum of electrons in a shower at small thicknesses, that is, precisely in the region where the usual methods fail. It is found that the energy spectrum at small thicknesses is quite different to what has been usually supposed. We therefore get, for the first time, a quantitative insight into the effect of collision loss on the energy spectrum of cascade electrons at all depths.

For the processes of radiation loss and pair creation, we take the exact cross-sections calculated by Bethe & Heitler valid for extremely high energies where screening is complete. The decrease of these cross-sections at energies so low that screening is incomplete will not be considered. These, or cruder approximations thereto, also underlie all the previous work. For this general problem a rigorous mathematical solution has been given by K. S. K. Iyengar (1942), but his method is not suitable for obtaining numerical results without laborious calculation.*

When the effect of collision loss is negligible, i.e. for energies large compared with the critical energy, it appears that the figures of Bhabha & Heitler are not in general in error by more than 30 %, and give the position of the maximum of the shower correctly. The approximations of Carlson & Oppenheimer make the maximum

* An empirical method of allowing for incomplete screening has recently been given by Corben (1941). But his treatment of collision loss follows that of Serber and therefore has the same defects. Corben's approximate method of allowing for incomplete screening could be incorporated into the solution of this paper.

number of particles appear at a depth which is too large by 30 % to 60 %, and give an energy spectrum which is incorrect at small thicknesses. Their figures are in error by much more than 30 % even in the region where collision loss is negligible. On the other hand, the figures given by Landau & Rumer (1938) are often in error by a factor 30, due to the fact that these authors have failed to carry their elegant method through to the end with mathematical rigour. For energies large compared with the critical energy the figures of this paper are correct to within 5 %, and hence are a considerable advance on all the previous work. But even for electrons of energy near or below the critical energy the figures for the energy spectrum of electrons calculated in this paper by taking account of collision loss are accurate for all but the end of a shower to within 30 %, and are hence as accurate as those which have been given hitherto for energies far above the critical energy where collision loss is negligible. Moreover, the present calculations show that even for electrons of the critical energy, the effect of collision loss is so large that the number of electrons may be as low as one-seventh of what was previously supposed.

Physical assumptions and the general equations

The effective differential cross-section for the emission by an electron of energy E of a quantum of energy lying between E' and $E'+dE'$ while passing through the field of an atom of nuclear charge Ze have been calculated by Bethe & Heitler (1934) and can be written in the form

$$4\frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \log 183Z^{-\frac{1}{3}}\, R(E, E')\frac{E'dE'}{E^2}, \tag{1}$$

where

$$R(E, E') = \left(1-\frac{4}{3}\frac{E}{E'}+\frac{4}{3}\frac{E^2}{E'^2}\right)(\chi_1+\chi_2)-\chi_2 \tag{2}$$

and

$$\chi_1 = \frac{\phi_1(\rho)+4\log Z^{-\frac{1}{3}}}{4\log 183Z^{-\frac{1}{3}}}, \quad \chi_2 = \frac{\phi_1(\rho)-\phi_2(\rho)}{8\log 183Z^{-\frac{1}{3}}}, \tag{3}$$

with

$$\rho = \frac{100}{Z^{\frac{1}{3}}}\left|\frac{mc^2E'}{E(E-E')}\right|, \tag{4}$$

m being the electron mass. Here ϕ_1 and ϕ_2 are functions of ρ only, and are given in the paper of Bethe & Heitler. χ_1 and χ_2 decrease monotonically as ρ increases, and take their largest value when $\rho = 0$, in which case $\chi_1 = 1$ and $\chi_2 = (12\log 183Z^{-\frac{1}{3}})^{-1}$. Thus, even for the heaviest atoms χ_2/χ_1 is less than 5 % when $\rho = 0$, and it becomes rapidly smaller as ρ increases, being already less than 1 % even for $\rho = 0{\cdot}5$. It is completely negligible for $\rho \geqslant 0{\cdot}8$. Thus, if an accuracy of not more than 5 % is aimed at χ_2 can be neglected altogether. Now $\rho \ll 1$, that is, by (4)

$$\left|\frac{E(E-E')}{E'mc^2}\right| \gg 100Z^{-\frac{1}{3}} \tag{5}$$

corresponds to the case in which the screening of the nucleus by the surrounding atomic electrons is complete, and then, introducing the values of χ_1 and χ_2 given above for $\rho = 0$, (2) simplifies to

$$R_0(E, E') = 1 - (\tfrac{4}{3} + \alpha)\left(\frac{E}{E'} - \frac{E^2}{E'^2}\right), \tag{6}$$

with

$$\alpha = \frac{1}{9 \log 183 Z^{-\frac{1}{3}}}.$$

This is the expression which will be used in the cascade calculations of this paper. It depends only on the ratio E/E', whereas the general expression (2) depends not only on this ratio but on the absolute magnitudes of E and E'. Moreover, the properties of the atom only affect α which is about 0·06 for lead, and still less for lighter atoms, so that the expression (6) is practically independent of Z. To the accuracy of the calculations of this paper α could equally well be neglected, but we shall not do so since its inclusion presents no difficulty.

The effective differential cross-section for the creation by a quantum of energy E of an electron pair, one particle of which has an energy between E' and $E' + dE'$, while passing through the field of an atom of charge Ze has also been calculated by Bethe & Heitler, and can be written in the form

$$4 \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \log 183 Z^{-\frac{1}{3}} R(E', E) \frac{dE'}{E}, \tag{7}$$

with R given by (2). Here again, for very high energies, more accurately, for the case of complete screening, R takes on the simpler form given by (6), and we shall use this specialized form in the calculations of this paper.

Since in the calculations of this paper we use R_0 instead of R, it is important to know the error which this introduces. R_0 given by (6) is accurate only when the inequality (5) is satisfied. It therefore differs from the more general expression (2) provided $E - E'$ is small enough, however large E may be. But in practice, more detailed considerations show that, except for very small values of $E - E'$ which are not important in our calculations, the difference between R_0 and R is less than 20 % provided E is greater than 30 MeV in lead and 70 MeV in air. Now, as will be seen below, the critical energy at which collision loss becomes important is 7MeV in lead and 103 MeV in air. Thus the fact that the cross-section (6) is not the exact one for low energies* restricts the accuracy of our calculations to 20 % in lead even for electrons above the critical energy, while in air the error is within about 10 % above the critical energy, and considerably less than 20 % even for particles below the critical energy, since the majority of these are produced by radiation loss and pair creation by particles and quanta above the critical energy for which the formula (6) is accurate to a much greater degree. It should be noticed that whereas the use of (6) instead of (2) even for low energies introduces relatively small errors, the neglect of collision loss completely alters the whole picture near and below the critical energy.

* An approximate allowance for this could be made here by following Corben.

Moreover, as has already been mentioned, (6) is always greater than (2), so that the use of R_0 instead of R, even when the inequality (5) is not satisfied, results in slightly more multiplication than is actually the case.

The probability that in travelling a thickness dx of a substance containing N atoms of nuclear charge Ze per unit volume an electron of energy E emits a quantum of energy E', or a quantum of energy E creates a pair of which one particle has an energy E', is got by multiplying (1) or (7) respectively by $N\,dx$. It is therefore convenient to introduce a quantity l of the dimensions of length, defined by

$$l^{-1} = 4\frac{Z^2N}{137}\left(\frac{e^2}{mc^2}\right)^2 \log 183Z^{-\frac{1}{3}}. \tag{8}$$

For substances containing several types of atoms one must take on the right-hand side of (8) a sum of similar expressions, one for each type of atom. This quantity l will be known as the characteristic unit of length for that substance. As in all the previous papers, it is convenient to measure all lengths in a given substance in terms of the characteristic length l and denote length so measured by t, so that $t \equiv x/l$ and is a dimensionless quantity. The values of l for different substances are given in the first row of table 1.

TABLE 1

	air	H_2O	Al	Fe	Pb
l in cm.	34,200	43·4	9·80	1·84	0·525
mean collision loss in MeV per cm. $= (-\partial E/\partial x)_{\text{coll.}}$	$3{\cdot}03 \times 10^{-3}$	2·64	5·67	14·1	13·2
β in MeV $= l(-\partial E/\partial x)_{\text{coll.}}$	103·0	114·6	55·56	25·88	6·927

The only other physical process which plays an essential part in the cascade theory is the 'ionization' or collision loss of electrons. This loss is a minimum at energies near twice the rest energy of an electron, and increases rapidly for lower energies so that any electron whose energy has fallen as low as twice the rest energy, i.e. about one million electron volts, can be regarded as completely stopped. For higher energies the collision loss increases logarithmically with the energy, and in the whole range from 5 to 150 MeV in which alone the collision loss plays a dominating part, it rises by less than 50 %. It will therefore be treated as a constant loss independent of the energy. Hence, measuring length in the characteristic unit (8), the collision loss is

$$-\left(\frac{\partial E}{\partial t}\right)_{\text{coll.}} = -l\left(\frac{\partial E}{\partial x}\right)_{\text{coll}} \equiv \beta, \tag{9}$$

where β is a constant independent of the energy. Since the collision loss is very nearly proportional to ZN, while l is proportional to $Z^{-2}N^{-1}$, it follows that β is very nearly proportional to Z^{-1} and is independent of N. It has the dimensions of energy. The second row in table 1 gives the mean values that are taken in this paper for the collision loss $-(\partial E/\partial x)_{\text{coll.}}$ in different substances, and the third row then gives the corresponding values of β as defined by (9).

From (1) the total energy loss of an electron due to radiation is

$$-\left(\frac{\partial E}{\partial t}\right)_{\text{rad.}} = E\int_0^E R(E, E')\frac{E'^2 dE'}{E^3}. \tag{10}$$

Using (2) for R, it appears that the integral only increases logarithmically with E, and for very large E, where in fact (2) tends to (6), it takes the constant value $(1+\frac{1}{2}\alpha)\approx 1$. Thus, for high energies,

$$-\left(\frac{\partial E}{\partial t}\right)_{\text{rad.}} \approx E. \tag{11}$$

The critical energy in a substance is usually defined as the energy at which the radiation loss of an electron is equal to its collision loss. By equating (9) and (11) the critical energy is seen to be very nearly β. For the purposes of this paper, in which the collision loss is treated accurately, it is logical to define the critical energy as exactly equal to β, for this is the physically significant quantity which appears in the equations. The critical energy β has therefore another physically significant interpretation. It is the collision energy loss suffered by an electron in travelling the characteristic unit of length in the substance. Hence an electron whose energy has fallen to the critical energy will at the most travel a distance of one unit before its energy is reduced to zero.

For quanta of energy below 5 MeV in lead and 25 MeV in air, the Compton effect becomes larger than the pair creation. But even for energies down to 5 MeV the angular scattering of the quanta is small, and in consequence in each Compton process a small fraction of the energy of the quantum is given to the electron. The effect of successive Compton processes is therefore to reduce the energy of low-energy quanta in a large number of small steps just as collision loss reduces the energy of electrons, and indeed this energy loss can be taken into account in our theory in the same way. Calculation shows that the resulting energy loss of a quantum per unit length of path traversed is between one-eighth and one-tenth of the collision energy loss of electrons. It can therefore be omitted altogether, since its effect would be to modify the spectrum of quanta of energies below about one-tenth of the critical energy.

Consider a layer of substances, upon the surface of which electrons and quanta with a known energy distribution impinge normally. We wish to know the energy distribution of electrons and quanta at some distance t from the surface of the layer. The problem may be treated as a one-dimensional one as far as calculating the number of particles at a depth t is concerned, since the angular deflexion of a particle or quantum of energy E resulting from a radiation or pair creation process is of the order mc^2/E, and is hence small. Denote by $P(E,t)\,dE$ the number of particles (electrons and positrons) whose energies lie between E and $E+dE$ at a distance t from the surface of the substance, all distances t being now measured normally from the surface of the layer in the characteristic unit of length l. Let the corresponding expression for the number of quanta be $Q(E,t)\,dE$. Then, owing to the radiation, pair

creation and collision processes given by (1), (7) and (9), this number varies with the thickness, and the differential equations determining their variation can be written down exactly as in the paper of Landau & Rumer. They are

$$\frac{\partial P(E,t)}{\partial t} = \int_E^\infty P(E',t)\,R(E',E'-E)\frac{(E'-E)\,dE'}{E'^2} - P(E,t)\int_0^E R(E,E')\frac{E'dE'}{E^2} + 2\int_E^\infty Q(E',t)\,R(E,E')\frac{dE'}{E'} + \beta\frac{\partial P(E,t)}{\partial E}, \tag{12a}$$

$$\frac{\partial Q(E,t)}{\partial t} = \int_E^\infty P(E',t)\,R(E',E)\frac{E\,dE'}{E'^2} - Q(E,t)\int_0^E R(E',E)\frac{dE'}{E}. \tag{12b}$$

The first two terms on the right-hand side of (12a) give the change in the number of particles of energy E due to radiation loss, the third term that due to pair creation by quanta, and the last term that due to collision loss. Similarly, the first term on the right-hand side of (12b) gives the increase in the number of quanta of energy E due to radiation by electrons and the second term their decrease due to disappearance by pair creation. In these equations R is strictly given by the general expression (2), which depends not only on the ratio E/E' but also on the value of E/mc^2. For this general problem no rigorous solution* can be obtained by the method of this paper and the general method developed by K. S. K. Iyengar must be used. While his method has the advantage of great generality and mathematical rigour, it is not so easy to derive numerical results from it. Moreover, it depends on an ingenious use of the solution neglecting collision loss derived in this paper. But quite apart from this, as the previous discussion shows, it is possible with considerable accuracy to take for R over the whole energy range the form (6) which it assumes for very high energies, and which is only a function of the ratio E/E'. It is then possible to solve the equations (12) by using the Mellin transformation. This method has the advantage that it is suitable for obtaining numerical results.

Introduce two new functions $p(s,t)$ and $q(s,t)$ of a variable s defined by

$$p(s,t) = \int_0^\infty E^{s-1}\,P(E,t)\,dE, \tag{13a}$$

$$q(s,t) = \int_0^\infty E^{s-1}\,Q(E,t)\,dE. \tag{13b}$$

It then follows from the theory of the Mellin transform that

$$P(E,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E^{-s}\,p(s,t)\,ds, \tag{14a}$$

$$Q(E,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E^{-s}\,q(s,t)\,ds, \tag{14b}$$

where σ is such that when $R(s)>\sigma$, $p(s,t)$ and $q(s,t)$ are analytic. The path of the s integration is a line parallel to the imaginary axis and to the right of it.

* An approximate treatment can be given by following Corben.

Following Landau & Rumer, multiply the equations (12) by E^{s-1} and integrate with respect to E from 0 to ∞. The last term on the right-hand side of (12a) then gives

$$\beta\int_0^\infty E^{s-1}\frac{\partial P(E,t)}{\partial E}\,dE = \beta\Big[E^{s-1}P(E,t)\Big]_0^\infty - \beta(s-1)\int_0^\infty E^{s-2}P(E,t)\,dE,$$

on integrating by parts. It is obvious from the physics of the problem that $E^{s-1}P(E,t)$ must vanish at infinity, since initially there are only particles of finite energy. Moreover, when collision loss is taken into account this expression also vanishes at $E=0$ provided $R(s)>1$. Thus the term on the left of the above equation just reduces to the second term on the right, which by (13a) can be written as $-\beta(s-1)p(s-1,t)$. The first term on the right of (12a) gives, if we interchange the orders of the E and E' integrations,

$$\int_0^\infty dE'\,P(E',t)\,E'^{s-1}\int_0^{E'} dE\,R(E',E'-E)\,\frac{(E'-E)\,E^{s-1}}{E'^{s+1}}.$$

Now if R is a function only of the ratio E/E', as is the case if we take for it the expression (6), then the E integral becomes a function of s only and is independent of the value of E', so that remembering (13) the double integral becomes $p(s,t)$ multiplied by a function of s.* All the other integrals can be treated similarly and the equations reduce to

$$\frac{\partial p(s,t)}{\partial t} = -A_s p(s,t) + B_s q(s,t) - \beta(s-1)p(s-1,t), \tag{15a}$$

$$\frac{\partial q(s,t)}{\partial t} = C_s p(s,t) - Dq(s,t), \tag{15b}$$

where A_s, B_s and C_s are functions of s denfied by

$$\begin{aligned} A_s &\equiv \int_0^1 \left\{\epsilon - (\tfrac{4}{3}+\alpha)\left(1-\frac{1}{\epsilon}\right)\right\}\{1-(1-\epsilon)^{s-1}\}\,d\epsilon \\ &= (\tfrac{4}{3}+\alpha)\left\{\frac{d}{ds}\log\Gamma(s)+\gamma-1+\frac{1}{s}\right\}+\frac{1}{2}-\frac{1}{s(s+1)}, \end{aligned} \tag{16a}$$

$$\begin{aligned} B_s &\equiv 2\int_0^1 \{1-(\tfrac{4}{3}+\alpha)(\epsilon-\epsilon^2)\}\,\epsilon^{s-1}\,d\epsilon \\ &= 2\left\{\frac{1}{s}-(\tfrac{4}{3}+\alpha)\frac{1}{(s+1)(s+2)}\right\}, \end{aligned} \tag{16b}$$

$$\begin{aligned} C_s &\equiv \int_0^1 \left\{1-(\tfrac{4}{3}+\alpha)\left(\frac{1}{\epsilon}-\frac{1}{\epsilon^2}\right)\right\}\epsilon^s\,d\epsilon \\ &= \frac{1}{s+1}+(\tfrac{4}{3}+\alpha)\frac{1}{s(s-1)} \end{aligned} \tag{16c}$$

and

$$\begin{aligned} D &\equiv \int_0^1 \{1-(\tfrac{4}{3}+\alpha)(\epsilon-\epsilon^2)\}\,d\epsilon \\ &= \tfrac{7}{9}-\tfrac{1}{6}\alpha. \end{aligned} \tag{16d}$$

* Owing to the form of R_0 the E integral diverges when $E=E'$, and the first and second integrals in (12a) have to be taken together in the Cauchy sense to give a convergent result. These together then give $-A_s p(s,t)$.

γ is the Euler-Mascheroni constant. The equations (15) were given by Landau & Rumer, but with the important omission of the last term in (15a) representing the effect of collision loss.*

Differentiating (15a) with respect to t and then eliminating $q(s,t)$ by the use of (15b) and then (15a) we obtain the second-order equation

$$\frac{\partial^2}{\partial t^2}p(s,t)+(A_s+D)\frac{\partial}{\partial t}p(s,t)+(A_sD-B_sC_s)\,p(s,t)$$
$$=-\beta(s-1)\left\{\frac{\partial}{\partial t}+D\right\}p(s-1,t). \qquad (17)$$

From the definitions (13) it is obvious that $p(2,t)$ gives the total energy carried by particles and $q(2,t)$ that carried by quanta at the depth t, while $p(1,t)$ gives the total number of particles at this depth. Adding (15a) and (15b) and putting $s=2$, we get

$$\frac{\partial}{\partial t}\{p(2,t)+q(2,t)\}=(C_2-A_2)\,p(2,t)+(B_2-D)\,q(2,t)-\beta p(1,t).$$

Since the left-hand side of the above equation gives the rate of increase of the total energy of the shower with depth, and the radiation and pair creation processes do not change the total energy of the shower, the terms independent of β on the right-hand side of the above equation must vanish irrespective of the values of $p(2,t)$ and $q(2,t)$. It follows that the coefficients of these terms must vanish, and hence that $A_2=C_2$ and $B_2=D$, relations which can be verified directly from (16). The above equation therefore reduces to

$$\frac{\partial}{\partial t}\{p(2,t)+q(2,t)\}=-\beta p(1,t),$$

which expresses the physically evident result that the rate of diminution of the total energy of a shower is equal to the rate of energy loss by collision. This equation also shows that, since the left-hand side is always finite, *the total number of* particles $p(1,t)$ at a thickness t *must always be finite when collision loss is taken into account.* This is not so when collision loss is neglected ($\beta=0$).

Cascades without collision loss

It is first necessary to give a complete treatment of the cascade problem neglecting collision loss, that is, the solution of the equations (12), with $\beta=0$, since the work of the next section is based upon it.

As has been shown by Landau & Rumer, the general solution of the equations

* With the approximations made by Carlson & Oppenheimer, we should have $A_s=2(1-1/s)$, $B_s=4/3s$, $C_s=1/(s-1)$, $D=2/3$. Hence A_s goes seriously wrong for $s\sim1$. Although their values of λ_s are not in general in error by more than 50 %, since λ appears in (32) in the form $\exp(-\lambda_s t)$, the resulting error in the number of particles may be very large. See footnote on p. 283.

(15) when $\beta = 0$, and consequently of equation (17) neglecting the right-hand side, is of the form

$$p(s,t) = a_s e^{-\lambda_s t} + b_s e^{-\mu_s t}, \tag{18}$$

with a similar expression for $q(s,t)$, where a_s, b_s are arbitrary functions of s independent of t which have to be chosen to satisfy the boundary conditions, and λ_s and μ_s are the two roots of the quadratic

$$X^2 - (A_s + D)X + (A_s D - B_s C_s) = 0,$$

whence $\lambda_s + \mu_s = A_s + D$, a relation which will often be used later, and

$$\lambda_s = \tfrac{1}{2}(A_s + D) - \tfrac{1}{2}\sqrt{\{(A_s - D)^2 + 4B_s C_s\}}, \tag{19a}$$

$$\mu_s = \tfrac{1}{2}(A_s + D) + \tfrac{1}{2}\sqrt{\{(A_s - D)^2 + 4B_s C_s\}}. \tag{19b}$$

It follows from the definitions (19) that $\mu_s > \lambda_s$ when s is real and greater than 1.

Certain general results can be deduced at once from the form of the solutions (18). Since, as was shown at the end of the last section, $A_2 = C_2$ and $B_2 = D$, hence $A_2 D - B_2 C_2 = 0$, and the smaller of the roots λ and μ must vanish at $s = 2$. Hence $\lambda_s = 0$ when $s = 2$ and this is a consequence of the conservation of energy. It further follows from (19b) and (16) that

$$\mu_2 = C_2 + D = 1 + \tfrac{1}{2}\alpha + \tfrac{7}{9} - \tfrac{1}{6}\alpha = \tfrac{16}{9} + \tfrac{1}{3}\alpha.$$

Putting $s = 2$ in (18) we see that the total energy of the particles or quanta in a shower tends to a constant value since $\lambda_2 = 0$ and the terms proportional to $\exp(-\mu_2 t)$ become negligible for $t \gg 1/\mu_2 = 9/(16 + 3\alpha)$. Thus, at large thicknesses, $\partial q(2,t)/\partial t$ vanishes, and it follows from (15b) that

$$\frac{p(2,t)}{q(2,t)} = \frac{D}{C_2} = \frac{14 - 3\alpha}{18 + 9\alpha} \approx \frac{7}{9}.$$

Hence, after a thickness of about one unit, the ratio of the total energy carried by the particles to that carried by the quanta reaches a constant value, the quanta carrying somewhat more than half the total energy of the shower. The energy is of course carried by progressively more and more particles and quanta of lower energy. This conclusion is altered if collision loss is taken into account, the energy carried by the particles then being relatively less, especially at large thicknesses. The above discussion also shows that the effect of conditions at the boundary does not make itself felt for more than a distance of about one unit. This is the average range in which 'transition effects' must take place when the shower passes from one substance to another.

Now consider a shower started by one electron or positron of energy E_0 entering the surface of the substance unaccompanied by any quanta. The boundary conditions at $t = 0$ are then

$$P(E,0) = \delta(E - E_0), \quad Q(E,0) = 0. \tag{20}$$

From this, using (13), it follows that at $t = 0$

$$p(s, 0) = E_0^{s-1}, \tag{21a}$$

$$q(s, 0) = 0, \tag{21b}$$

and from (15a)
$$\left\{\frac{\partial}{\partial t} p(s,t)\right\}_{t=0} = -A_s E_0^{s-1}. \tag{22}$$

We have therefore to find a solution of equations (15) satisfying the boundary conditions (21) or of equation (17) with the boundary conditions (21a) and (22).

The conditions (21a) and (22) determine a_s and b_s, and give finally

$$p(s,t) = E_0^{s-1}\left\{\frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\mu-D}{\mu-\lambda} e^{-\mu t}\right\}, \tag{23}$$

where for brevity we have omitted the suffix s. Owing to singularities in A_s, B_s or C_s, λ_s and μ_s are singular when $s = 1$, 0 or a negative integer. It should be noticed at once that, although λ_s and μ_s are not one-valued functions of s due to the appearance of a square root in their definition, the functions $p(s,t)$ and $q(s,t)$ are symmetrical in the two roots λ and μ and are therefore one-valued everywhere in the s plane, although a part of them, for example the part containing $e^{-\lambda t}$, is by itself not one-valued. This point is of importance in the general theory of the next section.

The number of particles of energy E at the depth t namely $P(E,t)$ can be calculated at once by using (14a) and is

$$P(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^s\left\{\frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\mu-D}{\mu-\lambda} e^{-\mu t}\right\} ds. \tag{24}$$

(24) is the *exact* solution of the cascade equations for a shower started by an incident particle of energy E_0 if collision loss is neglected.

To proceed further it is necessary to know the dependence of A, λ and μ on s. This can be calculated at once by using (16) and (19). The values of λ and μ are given in table 2 for real values of s from 1 to 6 at intervals of 0·1. The asymptotic form of these functions for large s will also be required. From (16) and (19) it follows that asymptotically for large s

$$A_s \approx \alpha'\left\{\log s + \gamma - 1 + \frac{1}{2s}\right\} + \tfrac{1}{2}, \tag{25a}$$

$$\lambda_s \approx D - \frac{B_s C_s}{A_s - D} \approx D - \frac{2}{\alpha' s(s+1)\log s}, \tag{25b}$$

$$\mu_s \approx A_s + \frac{B_s C_s}{A_s - D} \approx A_s + \frac{2}{\alpha' s(s+1)\log s}, \tag{25c}$$

where we have written $\alpha' = \frac{4}{3} + \alpha$ for brevity.

TABLE 2

s	λ_s	μ_s	$\log\left(\frac{D-\lambda_s}{\mu_s-\lambda_s}\right)$	λ_s'	$\frac{d}{ds}\log\left(\frac{D-\lambda}{\mu-\lambda}\right)$	λ_s''	$\frac{d^2}{ds^2}\log\left(\frac{D-\lambda}{\mu-\lambda}\right)$
1·0	$-\infty$	$+\infty$					
1·1	−3·787	+4·712					
1·2	−2·279	3·339	−0·6102	9·450	+0·043	−65·42	−1·34
1·3	−1·569	2·749	−0·6120	5·410	−0·070	−24·67	−0·941
1·4	−1·125	2·413	−0·6225	3·653	−0·152	−12·47	−0·778
1·5	−0·8121	2·200	−0·6414	2·691	−0·225	− 7·400	−0·699
1·6	−0·5751	2·054	−0·6672	2·091	−0·294	− 4·855	−0·665
1·7	−0·3876	1·951	−0·7005	1·683	−0·359	− 3·422	−0·646
1·8	−0·2348	1·877	−0·7400	1·388	−0·423	− 2·539	−0·639
1·9	−0·1079	1·824	−0·7844	1·165	−0·487	− 1·960	−0·631
2·0	−0·0000	1·786	−0·8365	0·9904	−0·550	− 1·561	−0·621
2·1	+0·0915	1·760	−0·8944	0·8493	−0·610	− 1·275	−0·600
2·2	+0·1706	1·743	−0·9577	0·7330	−0·669	− 1·060	−0·575
2·3	+0·2389	1·734	−1·029	0·6357	−0·725	− 0·8940	−0·537
2·4	+0·2982	1·731	−1·103	0·5531	−0·775	− 0·7615	−0·490
2·5	+0·3499	1·733	−1·182	0·4825	−0·821	− 0·6540	−0·433
2·6	0·3950	1·740	−1·268	0·4218	−0·861	− 0·5639	−0·372
2·7	0·4345	1·750	−1·355	0·3693	−0·896	− 0·4882	−0·306
2·8	0·4689	1·763	−1·446	0·3237	−0·921	− 0·4231	−0·226
2·9	0·4995	1·778	−1·539	0·2841	−0·944	− 0·3672	−0·174
3·0	0·5261	1·795	−1·635	0·2500	−0·959	− 0·3220	−0·115
3·1	0·5496	1·815	−1·683	0·2204	−0·967	− 0·2769	−0·0585
3·2	0·5703	1·835	−1·734	0·1945	−0·970	− 0·2405	−0·0095
3·3	0·5886	1·857	−1·786	0·1721	−0·968	− 0·2094	+0·0354
3·4	0·6048	1·879	−1·840	0·1527	−0·964	− 0·1819	+0·0676
3·5	0·6192	1·903	−1·895	0·1355	−0·955	− 0·1585	+0·0987
3·6	0·6321	1·926					
3·7	0·6436	1·950					
3·8	0·6536	1·975					
3·9	0·6628	1·999					
4·0	0·6710	2·023					
4·1	0·6785	2·048					
4·2	0·6851	2·072					
4·3	0·6910	2·097					
4·4	0·6969	2·121					
4·5	0·7019	2·145					
4·6	0·7061	2·169					
4·7	0·7102	2·193					
4·8	0·7143	2·216					
4·9	0·7176	2·239					
5·0	0·7207	2·262					
5·1	0·7239	2·284					
5·2	0·7266	2·307					
5·3	0·7289	2·329					
5·4	0·7318	2·351					
5·5	0·7335	2·372					
5·6	0·7356	2·394					
5·7	0·7372	2·415					
5·8	0·7389	2·436					
5·9	0·7409	2·456					
6·0	0·7424	2·480					

It is convenient, as in the paper by Bhabha & Heitler, to introduce a variable y defined by

$$y = \log \frac{E_0}{E}. \tag{26}$$

It should be noted at once that when $E > E_0$ (24) vanishes exactly. For when $E > E_0$, y is negative, and the path of the s integration in (24) can be deformed into an infinite semicircle to the right with the point σ as centre. The expression in curly brackets tends to zero on this semicircle, as also the term outside it, namely, $\exp(-|y|s)$, so that (24) vanishes. This is, of course, necessary for the physical interpretation.

Now consider (24) when $E < E_0$. Since $\lambda_s < \mu_s$ for real $s > 1$, as is also shown clearly by table 2, it follows that, except for very small t, the main contributions to (24) come from the first term in curly brackets having λ_s in the exponent. Write $P(E,t)$ given by (24) as the sum of two terms P_λ and P_μ defined by

$$P_\lambda \equiv \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s \frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} ds, \tag{27}$$

$$P_\mu \equiv \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s \frac{\mu-D}{\mu-\lambda} e^{-\mu t} ds. \tag{28}$$

It will now be shown that (27) can be evaluated by the saddle-point method with an error well within 5 %. Introduce a function ψ such that e^ψ is equal to the integrand of (27). Then, remembering (26),

$$\psi(s) \equiv ys - \lambda_s t + \log \frac{D-\lambda_s}{\mu_s - \lambda_s}. \tag{29}$$

Since $C_s \to \infty$ as $s \to 1$ it follows that $\lambda_s \to -\infty$ and hence $\psi \to \infty$ as $s \to 1$. Moreover, it follows from (25) that $\psi \to \infty$ as $s \to \infty$. Hence ψ must have a minimum as s increases along the real axis from 1 to ∞. Denote the particular value of s, where

$$\frac{\partial \psi}{\partial s} = 0$$

by s_0. We shall call this the saddle-point. Hence, differentiating (29), s_0 is determined by

$$y - \lambda'_{s_0} t + \left(\frac{d}{ds} \log \frac{D-\lambda_s}{\mu_s - \lambda_s}\right)_{s=s_0} = 0. \tag{30}$$

A dash affixed to a symbol will be used to denote differentiation with respect to s. The value of s_0 depends on the values of y and t. Now shift the contour of the integration in (27) to a parallel contour to the right or left so as to make it pass through the point s_0, in other words, take $\sigma = s_0$. This can always be done since the only restriction on σ is $\sigma > 1$. Writing $i\tau \equiv s - s_0$, by Taylor's theorem

$$\psi(s) = \psi(s_0) - \frac{\tau^2}{2} \psi''(s_0) + \ldots. \tag{31}$$

We now make the usual approximation of the saddle-point method and replace ψ by the first three terms of its Taylor expansion, namely, the two terms on the right of (31). Then

$$P_\lambda(E,t) = \frac{1}{2\pi E_0}\int_{-\infty}^{\infty} e^{\psi(s_0)-\frac{1}{2}\tau^2\psi''(s_0)}\,d\tau$$

$$= \frac{e^{\psi(s_0)}}{E_0\sqrt{\{2\pi\psi''(s_0)\}}} = \frac{1}{E_0}\frac{e^{ys_0-\lambda_{s_0}t}}{\sqrt{\{2\pi\psi''(s_0)\}}}\frac{D-\lambda_{s_0}}{\mu_{s_0}-\lambda_{s_0}}, \tag{32a}$$

where, from (30)
$$\psi''(s_0) = -\lambda''_{s_0}t + \left(\frac{d^2}{ds^2}\log\frac{D-\lambda_s}{\mu_s-\lambda_s}\right)_{s=s_0} \tag{32b}$$

It should be noticed that $\psi''(s_0)$ contains t explicitly but not y.

The approximation in deriving (32a) consists in replacing $\exp\{\psi(s_0+i\tau)\}$ by $\exp\{\psi(s_0)-\tau^2\psi''(s_0)/2\}$. Now, owing to the symmetry of the contour of integration in $\pm i\tau$, it is only the real part of $\exp\psi(s_0+i\tau)$ which makes a non-vanishing contribution to the integral (27). Hence to check the accuracy of the approximation, it is necessary to compare the values of

$$R\{\exp\psi(s_0+i\tau)\}\exp\{-\psi(s_0)\} \quad \text{with} \quad \exp\{-\tau^2\psi''(s_0)/2\}.$$

This has been done for two examples, namely, $y = 4$, $t = 10$ and $y = 10$, $t = 10$, the corresponding values of s_0 being 2·84 and 2·03 respectively. The figures are given in table 3 for different values of τ, which shows that the error introduced by evaluating (27) by the saddle-point method is about 2 %. It appears, therefore, that the accuracy of (32) depends on an accurate determination of the saddle-point s_0 from equation (30). λ'_s and $\frac{d}{ds}\left(\log\frac{D-\lambda}{\mu-\lambda}\right)$ are given in the fifth and sixth columns of table 2. Since s is given at intervals of 0·1 in the table, s_0 has been determined by interpolation by using Newton's formula. The method is described in another paper by Chakrabarty (1942). In the seventh and eighth columns of table 2 are also given the values of λ''_s and $\frac{d^2}{ds^2}\left(\log\frac{D-\lambda}{\mu-\lambda}\right)$, which are required for calculating (32b).

TABLE 3. ACCURACY OF THE SADDLE-POINT METHOD

	$y=4,\ t=10$; $s_0=2{\cdot}84$, $\psi(s_0)=1{\cdot}06$, $\psi''(s_0)=3{\cdot}789$		$y=10,\ t=10$; $s_0=2{\cdot}03$, $\psi(s_0)=9{\cdot}16$, $\psi''(s_0)=14{\cdot}06$	
τ	$R\{\exp\psi(s_0+i\tau)-\exp\psi(s_0)\}$	$\exp\{-\frac{1}{2}\tau^2\psi''(s_0)\}$	$R\{\exp\psi(s_0+i\tau)-\exp\psi(s_0)\}$	$\exp\{-\frac{1}{2}\tau^2\psi''(s_0)\}$
0·0	1·00	1·00	1·00	1·00
0·1	0·961	0·981	0·934	0·932
0·2	0·925	0·927	0·753	0·755
0·4	0·744	0·739	0·337	0·325
0·6	0·529	0·506	0·078	0·078
0·8	0·305	0·298	0·009	0·011

Since inaccuracies in the determination of the saddle-point s_0 cause large alterations in the result (32a), the logarithmic term in (29) can in no circumstances be neglected, as has been done by Landau & Rumer. Nor have these authors used the reverse of the Mellin transform equation (14a) for calculating $P(E,t)$, so that their numerical figures are in error by large factors, being sometimes thirty times larger than those of this paper.

The second part P_μ given by (28) cannot be evaluated in the same manner. Fortunately its order of magnitude can be accurately estimated, from which it appears that it is small compared with P_λ except for very small t. To estimate the order of magnitude of the integral (28) consider the path of integration moved to the right so that $\sigma \gg 1$. Then, everywhere on the contour we can approximately use the asymptotic forms of the functions A, λ and μ for large s given by (25). For large s, $(\mu - D)/(\mu - \lambda) \to 1$, and neglecting terms of order $1/s^2$ in the exponent, we get

$$P_\mu(E,t) \approx \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{ys-\mu_s t}\, ds$$

$$\approx \frac{e^{-\{\alpha'(\gamma-1)+\frac{1}{2}\}t}}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \exp\left\{ys - \frac{\alpha' t}{2s}\right\} s^{-\alpha' t}\, ds.$$

Writing s for ys the integral can at once be transformed into the well-known expression for a Bessel function (Whittaker & Watson 1927, p. 359) and we get

$$P_\mu(E,t) \approx \frac{1}{E_0} e^{(\alpha'-\gamma\alpha'-\frac{1}{2})t} \left(\frac{2y}{\alpha' t}\right)^{\frac{1}{2}(\alpha' t-1)} J_{\alpha' t-1}\{\sqrt{(2\alpha' t y)}\}. \tag{33}$$

Calculation shows that the contribution of P_μ is comparable with P_λ only for small t, as we should expect. For $t > 4$ it is less than 2 % of P_λ and may therefore be neglected altogether. A method of calculating (24) when t is small is given in the last section of this paper.

To get the total number of particles $N(E,t)$ whose energy is greater than E, one must integrate (24) with respect to E from E to infinity. By interchanging the order of the E and s integrations the E integration can be carried out first, giving

$$N(E,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{s-1} \frac{1}{s-1} \left\{\frac{D-\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\mu-D}{\mu-\lambda} e^{-\mu t}\right\} ds. \tag{34}$$

The integral (34) can be evaluated in the same way as (24). The only difference is that on the right of (29) ys is replaced by $y(s-1)$ and $-\log(s-1)$ is added to it. Consequently, for determining the saddle-point, (30) must be replaced by

$$y - \lambda'_{s_0} t + \left(\frac{d}{ds} \log \frac{D-\lambda_s}{\mu_s - \lambda_s}\right)_{s=s_0} - \frac{1}{s_0 - 1} = 0. \tag{35}$$

To calculate the maximum value of $N(E,t)$ for a given E, and the value of t at which this maximum occurs, it is only necessary to consider the part in the curly

brackets in (34) which is proportional to $e^{-\lambda t}$, since the other part is negligible at these thicknesses. The maximum of N therefore occurs when

$$\frac{\partial}{\partial t}N(E,t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{y(s-1)}\frac{1}{s-1}\frac{D-\lambda_s}{\mu_s-\lambda_s}\lambda_s e^{-\lambda_s t}\,ds = 0. \tag{36}$$

This integral can also be calculated by the saddle-point method and it will vanish when $\lambda_s = 0$. Now $\lambda_s = 0$ when $s = 2$. Hence, for a given y, the maximum will occur at such a value of t that the corresponding value of s_0 is 2. Calling this value t_m, it follows from (35) that*

$$t_m = \frac{1}{\lambda_2'}\left\{y + \left(\frac{d}{ds}\log\frac{D-\lambda}{\mu-\lambda}\right)_{s=2} - 1\right\} = 1{\cdot}01y - 1{\cdot}57. \tag{37}$$

Evaluating (34) by the saddle-point method just like (32a), and then putting $s_0 = 2$ and substituting for t_m by (37), the value of N at this point is

$$\begin{aligned} N_{\max.}(E) &= \frac{e^y}{\sqrt{\{2\pi(\psi''(2)+1)\}}}\frac{D}{\mu_2} \\ &= 0{\cdot}137\frac{E_0}{E}\left\{\log\frac{E_0}{E} - 1{\cdot}31\right\}^{-\frac{1}{2}}. \end{aligned} \tag{38}$$

(38) justifies the formula obtained empirically by Bhabha & Heitler, namely, $N_{\max.} = 0{\cdot}12(E_0/E)^{0{\cdot}93}$. The lower power of the exponent in their formula has the effect of roughly accounting for the variation of the logarithmic term in the denominator of (38). It appears that (38) is less than Bhabha & Heitler's approximate formula by about 30 %, confirming the estimate of the error in their calculations made by these authors. The numerical factors in (38) are of importance, for they show that even at the maximum, the number of particles is about a tenth of E_0/E, that is, the number that would be obtained if the whole primary energy E_0 were divided equally among particles of energy E. This is easy to understand. It is because there are a number of particles and quanta with energy greater than E, while at the same time a lot of the energy has already been degraded into particles and quanta of energy lower than E.

It should be noticed that according to (30) s_0 increases for a fixed y, as t increases, and the preceding discussion shows that $s_0 = 2$ at the thickness at which the maximum number of particles occur for this y. Hence the saddle-point lies at $s_0 < 2$ before the shower has reached its maximum, and at $s_0 > 2$ after the maximum.

It is interesting to note that $p(s,t)$ given by (23) and $q(s,t)$ are finite for all real values of $s > 1$ but tend to ∞ as $s \to 1$. Hence, writing $s = 1+\delta$ it follows from (13) that

$$p(1+\delta,t) = \int_0^\infty E^\delta\, P(E,t)\,dE$$

* With the approximations of Carlson & Oppenheimer (see footnote on p. 276) equation (37) is replaced by

$$t_m = 1{\cdot}25y - 1{\cdot}92.$$

This equation makes the maximum lie at a depth which is too great by 25–40 %.

is convergent for $\delta > 0$, but becomes infinite for $\delta = 0$. Hence $P(E,t)$ given by (14) must tend to infinity as $1/E$ for small E. Thus the spectrum of particles and quanta is of the form dE/E for very low energies. This is the result mentioned in the introduction, that if collision loss be neglected then the spectrum of electrons is of the form dE/E for small E. Collision loss completely alters the low-energy spectrum of electrons, as will be seen in the next section. The singularities in $p(s,t)$ and $q(s,t)$ at $s = 1$ are due entirely to the singularity in C at this point, as shown by (16). It can be seen quite easily that this singularity in C comes from the fact that the spectrum of quanta emitted by an electron as given by (1) is of the form dE/E for small E.

The creation of showers started by a quantum instead of by an electron can be treated by the same method. In the boundary condition (20) $P(E,0)$ and $Q(E,0)$ are now interchanged, from which it follows that $p(s,0)$ and $q(s,0)$ are interchanged in (21), so that the coefficients of the terms proportional to $e^{-\lambda t}$ and $e^{-\mu t}$ are different functions of s. It can be easily shown that a result of this is to replace $(D-\lambda)/(\mu-\lambda)$ by $B/(\mu-\lambda)$ in (27). Since the maximum number of particles occurs when $s_0 = 2$, and at this point $\lambda = 0$ and $B_2 = D$, it follows from (32a) that the number of particles at the maximum is very nearly the same, the difference being due only to the difference in $\psi''(2)$ in the denominator of (32a). It can also be shown that the maximum of $N(E,t)$ always occurs at a thickness which is 0·82 unit greater than the corresponding thickness for electron excited showers as given by (37). Detailed calculations have been carried out in a paper by Chakrabarty, and numerical results are given there. They show, as we should expect, that the curves giving the number of particles as a function of the thickness t are practically the same for a shower produced by an electron or a quantum of the same initial energy, with the difference that the latter curve is shifted to greater thickness by approximately one unit of length over the whole of its range.

Cascades with collision loss

We now proceed to solve the equations (15) when $\beta \neq 0$. Consider a shower started by an electron of energy E_0. The boundary conditions at $t = 0$ are then (20), from which (21) can be deduced again. Substituting (21) into (15a) the boundary condition (22) is now replaced by

$$\left\{\frac{\partial}{\partial t}p(s,t)\right\}_{t=0} = -E_0^{s-1}\left\{A_s + \frac{\beta}{E_0}(s-1)\right\}. \tag{39}$$

The problem can therefore be reduced to the solution of the second order equation (17) with the boundary conditions (21a) and (39) at $t = 0$.

One point should be noted immediately. Since $p(s,t)$ is proportional to E_0^{s-1} and on the right of (17) we have $p(s-1,t)$ it follows that in effect β only appears in equation (17) and the boundary condition (39) in the ratio β/E_0. Thus, if β be taken as the unit of energy in every substance just as l has been taken as the unit of length, then the equations and the solutions are the same in all substances for initial energies

which are the same multiples of β in each case. Just as there is a transformation of the scale of length in going from one substance to another, so there is a transformation of the scale of energy. In mathematical terms *the number of particles with an energy between $\beta\epsilon$ and $\beta(\epsilon+d\epsilon)$ produced by a primary of energy $\beta\epsilon_0$ at any given thickness t is the same in all substances.*

Since the right-hand side of (17) is smaller than the left by a factor of the order β/E_0 it would appear possible to solve (17) by a perturbation method when $E_0 \gg \beta$. This would give for $p(s,t)$ a solution of the form

$$p(s,t) = E_0^{s-1}\left[\sum_{n=0}^{\infty}\left(\frac{\beta}{E_0}\right)^n \{v_n(s)\, e^{-\lambda_{s-n}t} + w_n(s)\, e^{-\mu_{s-n}t}\}\right], \tag{40}$$

where $v_n(s)$ and $w_n(s)$ are functions of s which must also depend on β in order that the boundary conditions (21a) and (39) may be satisfied. A solution of the type (40) is nevertheless useless for getting numerical results, for in calculating $P(E,t)$ from it by equation (14a), the nth term of (40) leads to an integral of the type

$$\frac{1}{2\pi i E_0}\beta^n \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_0^{s-n}}{E^s}\{v_n(s)\, e^{-\lambda_{s-n}t} + w_n(s)\, e^{-\mu_{s-n}t}\}\, ds.$$

Writing s for $s-n$ this can be written as

$$\frac{1}{2\pi i E_0}\left(\frac{\beta}{E}\right)^n \int_{\sigma-n-i\infty}^{\sigma-n+i\infty}\left(\frac{E_0}{E}\right)^s \{v_n(s+n)e^{-\lambda_s t} + w_n(s+n)\, e^{-\mu_s t}\}\, ds, \tag{41}$$

and since the path of integration can be moved to the right it follows that we can replace $\sigma-n$ by σ in the limits of the above integral. In this integral n only appears in the functions v_n and w_n which do not alter the order of magnitude of the integral. Thus in reality (41) leads to a solution for $P(E,t)$ which is an infinite series in powers of $(\beta/E)^n$ in which the coefficients do not vary rapidly with n. Indeed, it can easily be shown that

$$\begin{aligned} P(E,t) = \sum_{n=0}^{\infty}\left(-\frac{\beta}{E}\right)^n \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} E^{-s}\frac{\Gamma(s+n)}{\Gamma(s)} \\ \times\left\{\frac{(D-\lambda_s)^n v(s)\, e^{-\lambda_s t}}{(\lambda_s-\lambda_{s+1})\ldots(\lambda_s-\lambda_{s+n})(\lambda_s-\mu_{s+1})\ldots(\lambda_s-\mu_{s+n})}\right. \\ \left. + \frac{(D-\mu_s)^n w(s)\, e^{-\mu_s t}}{(\mu_s-\lambda_{s+1})\ldots(\mu_s-\lambda_{s+n})(\mu_s-\mu_{s+1})\ldots(\mu_s-\mu_{s+n})}\right\} ds \end{aligned} \tag{42}$$

is an exact formal solution of equation (17) where $v(s)$ and $w(s)$ are arbitrary functions of s which have to be chosen to satisfy the boundary conditions (21a) and (39). They can be given as series in powers of β/E_0, the first terms of the series being respectively $E_0^{s-1}(D-\lambda_s)/(\mu_s-\lambda_s)$ and $E_0^{s-1}(\mu_s-D)/(\mu_s-\lambda_s)$, agreeing with the form they have in (24). If the series for $v(s)$ and $w(s)$ be inserted in (42) and the terms properly bracketed to give a strict expansion in powers of β, then, as has been shown by Iyengar, the ensuing series is absolutely convergent. The series (42) as written

is however divergent *for all values of* E due to the appearance of $\Gamma(s+n)$, and as a result the series for $v(s)$ and $w(s)$ are also divergent. Thus the series (42) is strictly not a proper solution at all. Nor is it possible to look upon it as an asymptotic solution of equation (17) valid for small β, for the successive integrals actually increase as one goes from one term to the next when t is large. The solution (42) is therefore quite useless for obtaining any numerical results especially for $E \lesssim \beta$. The methods of solution followed by Snyder and Serber also lead for purposes of calculating the spectrum to series of the type (42) and indeed, Serber's solution can be transformed exactly into (42).

We shall therefore try and find a solution of the equation (17) by generalizing (24), which is an exact solution of this equation when $\beta = 0$. Assume that an exact solution of (17) when $\beta \neq 0$ can be written in the form

$$P(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_0}{E+\beta g(s,t)}\right\}^s f(s,t,\beta)\,ds, \tag{43}$$

where $g(s,t)$ and $f(s,t,\beta)$ are functions of s and t, but not of E. Assume g to be independent of β. Then f must be a function of β, and as will appear presently, is expressible as a power series in β. We assume that σ is such that for $R(s) > \sigma$, $g(s,t)$ and $f(s,t,\beta)$ have no singularities.

Assume further that $f(s,t,\beta)$ satisfies the same boundary conditions at $t = 0$ as the corresponding function in curly brackets of (24), namely

$$f(s,0,\beta) = 1, \quad \left\{\frac{\partial}{\partial t} f(s,t,\beta)\right\}_{t=0} = -A_s. \tag{44}$$

It can then be shown that in order that (43) should satisfy the boundary conditions (20), $g(s,t)$ must satisfy at $t = 0$ the boundary conditions

$$g(s,0) = 0, \quad \left\{\frac{\partial}{\partial t} g(s,t)\right\}_{t=0} = 1. \tag{45}$$

To see that (43) satisfies the correct boundary conditions at $t = 0$, differentiate (43) and write η for s. Then

$$\begin{aligned}\left[\frac{\partial}{\partial t} P(E,t)\right]_{t=0} &= \left[\frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} d\eta \left\{\frac{E_0}{E+\beta g(\eta,t)}\right\}^\eta \right.\\ &\qquad \left. \times\left\{-\frac{\beta\eta f(\eta,t,\beta)}{\{E+\beta g(\eta,t)\}}\frac{\partial g(\eta,t)}{\partial t} + \frac{\partial f(\eta,t,\beta)}{\partial t}\right\}\right]_{t=0}\\ &= \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} d\eta \left(\frac{E_0}{E}\right)^n \left\{-\frac{\eta\beta}{E} - A_s\right\}.\end{aligned}$$

Multiplying this equation by E^{s-1} and integrating from 0 to ∞, we at once get (39). Hence the function $p(s,t)$ connected with (43) by (13a) rigorously satisfies the correct boundary conditions (21a) and (39), provided $g(s,t)$ and $f(s,t,\beta)$ satisfy (45) and (44) respectively, so that (43) rigorously satisfies the boundary conditions (20) at $t = 0$.

We now proceed to find the equations that $g(s,t)$ and $f(s,t,\beta)$ must satisfy in order that (43) should be an exact solution of (17). Assume for the moment that asymptotically for large s

$$g(s,t) \approx \frac{2}{\alpha' \log s}, \quad \text{when} \quad \alpha' t > 2, \tag{46a}$$

and

$$g(s,t) \approx t, \quad \text{when} \quad \alpha' t < 2. \tag{46b}$$

It will be shown in the appendix that $g(s,t)$, as determined by the equations given below, has in fact this asymptotic form for large s. It will also appear that for $R(s)$ greater than some finite positive number, that is in the whole complex plane to the right of some line parallel to the imaginary axis and to the right of it, neither $g(s,t)$ nor $f(s,t,\beta)$ have any singularities.

If $\alpha' t > 2$, then by (46a), $g(s,t) \to 0$ as $|s| \to \infty$. Since $g(s,t)$ and $f(s,t,\beta)$ have no singularities for $R(s) > \sigma$, we can always move the contour of the integration in (43) to the right and hence in view of (46) by making σ large enough the path of integration can be changed into another line parallel to the imaginary axis such that

$$E > |\beta g(s,t)| \tag{47}$$

at every point of it. We can then expand $(E+\beta g)^{-s}$ at every point of this contour as a series in ascending powers of $\beta g(s,t)/E$. If $\alpha' t < 2$, then it follows from (46) that a similar contour can be chosen such that on it $(E+\beta g)^{-s}$ can be expanded as a power series in $\beta g(s,t)/E$, provided $E > \beta t$. In order to find $g(s,t)$ and $f(s,t,\beta)$, the expansion may be carried out formally even for $E < \beta t$. (A rigorous proof of the correctness of the solution (43) is given in the paper by Iyengar.) If $f(s,t,\beta)$ satisfies certain conditions at infinity, then the series can be integrated term by term, and

$$P(E,t) = \sum_{n=0}^{\infty} \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s \frac{\Gamma(s+n)}{\Gamma(s)\,\Gamma(n+1)} \left\{-\frac{\beta g(s,t)}{E}\right\}^n f(s,t,\beta)\,ds. \tag{48}$$

Writing $\eta = s+n$ in this integral and shifting the path of integration again to the left so that $\sigma+n$ is replaced by σ, the $(n+1)$th term becomes

$$\frac{1}{2\pi i E_0}\left(-\frac{\beta}{E_0}\right)^n \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{\eta} \frac{(\eta-1)\ldots(\eta-n)}{n!} \{g(\eta-n,t)\}^n f(\eta-n,t,\beta)\,d\eta.$$

Now calculate the function $p(s,t)$ connected with $P(E,t)$ by the transformation (13a). It at once follows from the property of the Mellin transform that this is

$$E_0^{s-1}\left(-\frac{\beta}{E_0}\right)^n \frac{(s-1)\ldots(s-n)}{n!} \{g(s-n,t)\}^n f(s-n,t,\beta).$$

Hence

$$p(s,t) = E_0^{s-1}\left[f(s,t,\beta) - (s-1)\frac{\beta}{E_0} g(s-1,t) f(s-1,t,\beta) + \frac{(s-1)(s-2)}{2!}\left\{\frac{\beta g(s-2,t)}{E_0}\right\}^2 f(s-2,t,\beta) + \ldots\right]. \tag{49}$$

From (49) it can be easily deduced that if $f(s,t,\beta)$ satisfies the boundary conditions (44), then $g(s,t)$ must satisfy (45) in order that $p(s,t)$ should satisfy (21a) and (39). Conversely if $g(s,t)$ satisfies (45) then $f(s,t,\beta)$ must satisfy (44).

Now assume that $f(s,t,\beta)$ can be expressed as a power series in β thus

$$f(s,t,\beta) = f_0(s,t) + \left(\frac{\beta}{E_0}\right)^2 f_2(s,t) + \left(\frac{\beta}{E_0}\right)^3 f_3(s,t) + \ldots, \tag{50}$$

and that the coefficient of β in this series is zero. It will appear presently that this is possible. There are now sufficient conditions to determine g and f uniquely. Substituting (50) in (44) and equating the different powers of β to zero, it follows that $f_0(s,t)$ satisfies the boundary conditions (44), while $f_n(s,t)$ for $n>0$ satisfies the boundary conditions

$$f_n(s,0) = 0, \quad \left\{\frac{\partial}{\partial t} f_n(s,t)\right\}_{t=0} = 0. \tag{51}$$

Introducing the series (50) into (49) and rearranging the terms to form a power series in ascending powers of β we get

$$p(s,t) = E_0^{s-1}\Big[f_0(s,t) - \frac{\beta}{E_0}(s-1)\, g(s-1,t) f_0(s-1,t) + \left(\frac{\beta}{E_0}\right)^2 \left\{ \frac{(s-1)(s-2)}{2!} \{g(s-2,t)\}^2 f_0(s-2,t) + f_2(s,t) \right\} + \ldots \Big]. \tag{52}$$

Write for brevity
$$\Delta_s \equiv \frac{\partial^2}{\partial t^2} + (A_s + D)\frac{\partial}{\partial t} + (A_s D - B_s C_s). \tag{53}$$

Introducing (52) into (17) and equating the coefficients of the different powers of β to zero, we get a set of differential equations which successively determine $g(s,t)$ and $f_n(s,t)$, thus

$$\Delta_s f_0(s,t) = 0, \tag{54a}$$

$$\Delta_s \{g(s-1,t) f_0(s-1,t)\} = \left(\frac{\partial}{\partial t} + D\right) f_0(s-1,t), \tag{54b}$$

$$\Delta_s \left\{ f_2(s,t) + \frac{(s-1)(s-2)}{2!} \{g(s-2,t)\}^2 f_0(s-2,t) \right\} = \left(\frac{\partial}{\partial t} + D\right) \{(s-1)(s-2)\, g(s-2,t) f_0(s-2,t)\}, \tag{54c}$$

.....

It follows from (54a) and the boundary conditions (44) that $f_0(s,t)$ is precisely the function we had in the case $\beta = 0$, namely,

$$f_0(s,t) = \frac{D-\lambda_s}{\mu_s - \lambda_s} e^{-\lambda_s t} + \frac{\mu_s - D}{\mu_s - \lambda_s} e^{-\mu_s t}. \tag{55}$$

Substituting this into (54*b*) we get at once

$$g(s-1,t)f_0(s-1,t) = h_s e^{-\lambda_s t} + j_s e^{-\mu_s t} + \frac{(D-\lambda_{s-1})}{(\lambda_{s-1}-\lambda_s)(\lambda_{s-1}-\mu_s)} \frac{(D-\lambda_{s-1})}{(\mu_{s-1}-\lambda_{s-1})} e^{-\lambda_{s-1}t} + \frac{(D-\mu_{s-1})}{(\mu_{s-1}-\lambda_s)(\mu_{s-1}-\mu_s)} \frac{\mu_{s-1}-D}{\mu_{s-1}-\lambda_{s-1}} e^{-\mu_{s-1}t}, \tag{56}$$

where we have made use of relations of the type

$$\lambda_{s-1}^2 - (A_s + D)\lambda_{s-1} + A_s D - B_s C_s = (\lambda_{s-1}-\lambda_s)(\lambda_{s-1}-\mu_s).$$

λ_s and μ_s are the functions of s defined by (19), and h_s and j_s are arbitrary functions of s independent of t which have to be determined so that g may satisfy the boundary conditions (45). After some calculation we find

$$g(s,t) = \left[\frac{(D-\lambda_s)^2 e^{-\lambda_s t}}{(\mu_s-\lambda_s)(\lambda_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_s)} + \frac{(D-\lambda_{s+1})^2 e^{-\lambda_{s+1}t}}{(\mu_{s+1}-\lambda_{s+1})(\lambda_s-\lambda_{s+1})(\mu_s-\lambda_{s+1})} - \frac{(D-\mu_s)^2 e^{-\mu_s t}}{(\mu_s-\lambda_s)(\lambda_{s+1}-\mu_s)(\mu_{s+1}-\mu_s)} - \frac{(D-\mu_{s+1})^2 e^{-\mu_{s+1}t}}{(\mu_{s+1}-\lambda_{s+1})(\lambda_s-\mu_{s+1})(\mu_s-\mu_{s+1})}\right] \times \left\{\frac{D-\lambda_s}{\mu_s-\lambda_s} e^{-\lambda_s t} + \frac{\mu_s-D}{\mu_s-\lambda_s} e^{-\mu_s t}\right\}^{-1}. \tag{57}$$

Since $\mu_s > \lambda_s$ for real s greater than 1, the terms proportional to $e^{-\mu t}$ are negligible for all except very small t, and then to a good approximation

$$g(s,t) \approx \frac{D-\lambda_s}{(\lambda_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_s)} - \frac{(D-\lambda_{s+1})^2(\mu_s-\lambda_s)e^{-(\lambda_{s+1}-\lambda_s)t}}{(D-\lambda_s)(\mu_{s+1}-\lambda_{s+1})(\mu_s-\lambda_{s+1})(\lambda_{s+1}-\lambda_s)}. \tag{58}$$

Since, further, $\lambda_{s+1} > \lambda_s$ for real s greater than 1, the second term tends to zero for large t, and $g(s,t)$ becomes independent of t. For all except small t the second term gives a small correction. The values of $g(s,t)$ for $t = 2, 4, 10$ and ∞ as calculated from (58) are given in table 4A. For $t = 2$ and 4 and large s the figures are in error by 10 % as (58) is then not a good enough approximation to (57). This can be seen by comparing these figures with those of table 4B, which are calculated from (57). But just for $t \lesssim 2$ the saddle-point s_0 usually lies near or below 2, and here the differences between the figures of tables 4A and 4B are negligible. The figures for $t = 10$ and ∞ in table 4A are quite accurate. The general dependence of $g(s,t)$ on t for fixed s is easily seen. From the boundary conditions (45) it follows that for small t

$$g(s,t) \sim t, \tag{59}$$

and it rapidly tends to some constant limiting value depending on s as $t \to \infty$. This limiting value is about 0·5 for $s = 1{\cdot}5$, about 0·8 for $s = 2$, and about 1 for s between 2·5 and 3·0. This behaviour of $g(s,t)$ is of physical significance, as will appear in the next section. From table 4B and (46) it is clear that for large s the curves have a 'dip' near $t \sim 2/\alpha' \approx \frac{3}{2}$, which becomes more pronounced as s increases, until for $s \to \infty$ it becomes a sharp discontinuity at $t = 2/\alpha'$.

TABLE 4A. VALUES OF $g(s, t)$ FROM FORMULA (58)

s	$g(s, 2)$	$g(s, 4)$	$g(s, 10)$	$g(s, \infty)$
1·1	—	—	—	—
1·2	0·0000	—	—	0·3099
1·3	—	—	—	0·3922
1·4	—	—	—	0·4673
1·5	0·5252	0·5353	0·5364	0·5364
1·6	—	—	—	0·6008
1·7	—	—	—	0·6609
1·8	—	—	—	0·7167
1·9	—	—	—	0·7695
2·0	0·7599	0·7981	0·8177	0·8186
2·1	—	—	—	0·8641
2·2	—	—	—	0·9061
2·3	—	—	—	0·9450
2·4	—	—	—	0·9807
2·5	0·895	0·944	0·9993	1·013
2·6	0·913	0·962	—	1·043
2·7	0·927	0·976	—	1·070
2·8	0·939	0·987	—	1·095
2·9	0·950	0·997	—	1·120
3·0	0·958	1·004	1·0835	1·141
3·1	0·962	1·006	1·0882	1·160
3·2	0·967	1·011	1·0946	1·179
3·3	0·972	1·014	1·0984	1·198
3·4	0·972	1·010	1·0944	1·209
3·5	0·971	1·010	1·0944	1·226
3·6	0·968	1·005	1·0895	1·240
3·7	0·966	1·000	1·0835	1·253
3·8	0·965	1·000	1·0817	1·268
3·9	0·965	0·999	1·0782	1·283
4·0	0·958	0·990	1·0675	1·292
4·1	—	—	1·0633	1·304
4·2	—	—	1·0559	1·315
4·3	—	—	1·0453	1·326
4·4	—	—	1·0406	1·339
4·5	—	—	1·0272	1·350
4·6	—	—	1·0198	1·360
4·7	—	—	1·0111	1·375
4·8	—	—	1·0021	1·378
4·9	—	—	0·9968	1·383
5·0	—	—	0·9876	1·386

TABLE 4B. ACCURATE VALUES OF $g(s, t)$ FROM FORMULA (57)

s \ t	1·0	1·3	1·5	2·0	4·0	10·0
1·5	0·4892	0·5113	0·5174	0·5269	0·5353	0·5364
2·0	0·6309	0·7071	0·7155	0·7705	0·8009	0·8186
2·5	0·7264	0·8298	0·8759	0·9339	0·9564	0·9993
3·0	0·7738	0·9035	0·9659	1·048	1·037	1·083
3·5	0·8054	0·9545	1·029	1·129	1·066	1·094
4·0	0·8317	0·9949	1·079	1·189	1·064	1·067
4·5	0·8475	1·023	1·116	1·239	1·047	1·027
5·0	0·8630	1·050	1·149	1·287	1·027	0·9876

If we wish to calculate the limiting form of $g(s,t)$ for large s but finite t, it is necessary to consider all the terms in (57) and (58), for the first and second terms and the third and fourth terms in the numerator of (57) largely compensate each other as $s\to\infty$, as also do the two terms in (58). The limiting form of $g(s,t)$ as $s\to\infty$ is calculated in the appendix and the result has already been given in (46) above.

The expression (57) for $g(s,t)$ like $f_0(s,t)$ given by (55) and indeed, like $f_n(s,t)$ in general, is symmetrical in λ and μ. Hence, although λ and μ are themselves not one-valued functions of s in the whole complex plane, $g(s,t)$, $f_0(s,t)$ and $f_n(s,t)$ are one-valued functions of s throughout. Moreover, since they contain $e^{-\lambda t}$ and $e^{-\mu t}$, it is clear that the points in the s plane where λ and μ have singularities are essential singularities of $g(s,t)$, $f_0(s,t)$ and $f_n(s,t)$. Now λ and μ are singular whenever A_s, B_s or C_s defined by (16) are singular, and this happens when s is equal to 1, 0 or a negative integer.

The functions $f_2(s,t)$, $f_3(s,t)$, etc. can be calculated at once by solving (54c) and the succeeding equations of the set. All these equations are of the same type, in which the right-hand side is known, and the unknown function is on the left-hand side. The general solution of equations of this type is given in the appendix. But (54c) can also be solved directly. We get

$$\begin{aligned} f_2(s,t) = &-\frac{(s-1)(s-2)}{2}\{g(s-2,t)\}^2 f_0(s-2,t) \\ &+\frac{\frac{\partial}{\partial t}+D}{A_s}\{(s-1)(s-2)\,g(s-2,t)\,f_0(s-2,t)\} \\ &+k_s e^{-\lambda_s t}+l_s e^{-\mu_s t}. \end{aligned} \tag{60}$$

(56) shows that the term in curly brackets in (60) just contains t in four exponentials, and the operator outside the bracket is to be taken in the sense

$$\frac{\frac{\partial}{\partial t}+D}{A_s}e^{-\lambda_{s-1}t} = \frac{D-\lambda_{s-1}}{(\lambda_{s-1}-\lambda_s)(\lambda_{s-1}-\mu_s)}e^{-\lambda_{s-1}t}.$$

The arbitrary functions k_s and l_s which are independent of t, have to be determined to make $f_2(s,t)$ satisfy the boundary conditions (51) at $t=0$. It is clear from (60) that $f_2(s,t)$ will contain terms proportional to $e^{-\lambda_{s-2}t}$, $e^{-\lambda_{s-1}t}$, $e^{-\lambda_s t}$, and a similar set of three terms with μ instead of λ. Now it will be shown below that for $E_0 \gg \beta$, an appreciable contribution to $P_2(E,t)$ only comes from the term proportional to $e^{-\lambda_{s-2}t}$, except for very small t. Hence it is not necessary to determine k_s and l_s explicitly. The effect of the two terms containing them is largely to cancel the effect of the $e^{-\lambda_{s-2}t}$ term at small t, and thus to make the total contribution of f_2 to the series (50) quite negligible for small t, as will be seen in the next section. The term

proportional to $e^{-\lambda_{s-2}t}$ in $f_2(s,t)$ can be calculated quite easily, remembering (56) and (58). We find

$$f_2(s,t) = (s-1)(s-2)\frac{(D-\lambda_{s-2})^3}{(\mu_{s-2}-\lambda_{s-2})(\lambda_{s-1}-\lambda_{s-2})^2(\mu_{s-1}-\lambda_{s-2})^2},$$
$$\times\left\{\frac{(\lambda_{s-1}-\lambda_{s-2})(\mu_{s-1}-\lambda_{s-2})}{(\lambda_s-\lambda_{s-2})(\mu_s-\lambda_{s-2})}-\frac{1}{2}\right\}e^{-\lambda_{s-2}t}$$
$$+n_s e^{-\lambda_{s-1}t}+\text{terms containing } e^{-\lambda_s t} \text{ and } e^{-\mu t}. \tag{61}$$

The coefficients of the other terms can be calculated easily.

The functions $f_3(s,t)$, etc., can be calculated explicitly by the method given in the appendix. It is therefore clear that an exact formal solution of equation (17) with the boundary conditions (21a) and (39) can be written in the form (43) with the function $g(s,t)$ given by (57) and $f(s,t,\beta)$ given by the series (50). Now corresponding to the series (50) for $f(s,t,\beta)$, $P(E,t)$ given by (43) can be written as a series

$$P(E,t) = P_0(E,t)+P_2(E,t)+P_3(E,t)+\ldots \tag{62}$$

where
$$P_0(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_0}{E+\beta g(s,t)}\right\}^s f_0(s,t)\,ds, \tag{63}$$

$$P_2(E,t) = \frac{1}{2\pi i E_0}\left(\frac{\beta}{E_0}\right)^2\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_0}{E+\beta g(s,t)}\right\}^s f_2(s,t)\,ds, \tag{64}$$

and so on. It should be noted that the series (62) is not a simple series in powers of β since, owing to the appearances of β in the denominator of the integrand of each term, each term by itself is a function of β which may be expanded as a power series only in certain circumstances.

We now proceed to calculate (63). The function $f_0(s,t)$ given by (55) is the same as the function in curly brackets of (24). The only difference between (63) and (24) is the appearance of $E+\beta g(s,t)$ in place of E. As in (24) the contribution to (63) which comes from the term proportional to $e^{-\mu_s t}$ in $f_0(s,t)$ is negligible compared with the contribution which comes from the term proportional to $e^{-\lambda_s t}$ for all except very small t. Thus to a very good approximation for all except small t, we may write

$$P_0(E,t)\approx\frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_0}{E+\beta g(s,t)}\right\}^s\frac{D-\lambda_s}{\mu_s-\lambda_s}e^{-\lambda_s t}\,ds. \tag{65}$$

In view of the properties of $g(s,t)$ discussed above and its asymptotic behaviour for large s, it is clear that (65) can again be evaluated by the saddle-point method and indeed with the same degree of accuracy. The effect of replacing E in (27) by $E+\beta g(s,t)$ in (65) is therefore to give E a larger effective value depending on the position of the saddle-point s_0. The position of the saddle-point s_0 which depends on E and t is shifted to somewhat greater values by the presence of $g(s,t)$, as compared

with the corresponding values it had in (27). In calculating (65) it is convenient to introduce a variable y_0 defined by

$$y_0 = \log \frac{E_0}{\beta}. \tag{66}$$

Writing the integrand of (65) again in the form $e^{\psi(s)}$ we find

$$\psi(s) = y_0 s - \lambda_s t + \log \frac{D-\lambda_s}{\mu_s - \lambda_s} - s \log \left\{ \frac{E}{\beta} + g(s,t) \right\}. \tag{67}$$

This differs from the ψ of (29) only by the addition of $g(s,t)$ in the last term. The rest of the procedure of finding the saddle-point s_0 and then evaluating (65) follows the method of the last section closely. In fact, by using tables 2 and 4 it is no more difficult to evaluate (65) than (27).

The qualitative effect of $\beta g(s,t)$ in the denominator of (63) can be seen at once. It follows from (59) that $E + \beta g(s,t) = E + \beta t$ for small t, so that the effect of this term is to shift the whole spectrum to lower energies by an amount βt corresponding to the energy lost by collision by each particle in the thickness t. For all but very small t, however, $g(s,t) \sim 1$, as shown by table 4A and hence $E + \beta g(s,t) \sim E + \beta$. In other words, instead of the number of particles tending to infinity as $E \to 0$, as it did in (24), the spectrum becomes flat for $E < \beta$ and tends to a finite value as $E \to 0$. Thus for all but small t the effect of collision loss is to flatten out the spectrum for energies below the critical energy. We see that our present calculations justify the original assumption of Bhabha & Heitler concerning the effect of collision loss on the energy spectrum of a shower. The finer features of the energy spectrum will be discussed in the next section.

We now proceed to calculate $P_2(E,t)$ inserting in (64) only the term proportional to $e^{-\lambda_{s-2} t}$ of the expression (61) for $f_2(s,t)$. Writing $s+2$ in the integrand in place of s and moving the resulting contour again to the right, i.e. bringing it back from $\sigma - 2$ to σ, we get

$$P_2(E,t) = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left\{ \frac{E_0}{E + \beta g(s+2,t)} \right\}^s \frac{\beta^2}{\{E + \beta g(s+2,t)\}^2} m_{s+2} e^{-\lambda_s t} ds. \tag{68}$$

m_s denotes the coefficient of $\exp(-d_{s-2} t)$ in (61). E_0 appears here in precisely the same way as in (65) and t also appears in an exponential multiplied by the same coefficient λ_s. (68) now differs from (65) first in having $g(s+2,t)$ in place of $g(s,t)$. Table 4 shows that for s between 1·5 and 3·0, $g(s+2,t) \sim g(s,t)$, so that the effect of this difference alone would be to make (68) only slightly different from (65). The other difference between (68) and (65) is that the integrand of the former is

$$\left\{ \frac{\beta}{E + \beta g(s+2,t)} \right\}^2 \frac{\mu_s - \lambda_s}{D - \lambda_s} m_{s+2} \tag{69}$$

times the integrand of the latter. Even for E as small as β/e and s less than 2·5 this factor is less than $\frac{1}{2}$. This extra factor decreases as s increases and hence always shifts the saddle-point of (68) to somewhat smaller values compared with (65).

But since the saddle-point is precisely the point at which the value of the integrand is stationary, this shift does not greatly affect the value of the integral. Thus, (68) will be smaller than (65) roughly by the factor (69). We now see the great advantage of putting $E+\beta g(s,t)$ instead of E in (43) for it multiplies the nth term of the series (62) by a factor $\beta^n\{E+\beta g(s+n,t)\}^{-n}$, instead of the factor $(\beta/E)^n$ which occurs in (42). This factor is always less than or of the order unity however small E may be, whereas β/E becomes very large for small E, and makes the successive terms in (42) increase rapidly.

It is also clear now why the term proportional to $e^{-\lambda_{s-1}t}$ in (61) would make a contribution to $P_2(E,t)$ small compared with (68). For on introducing this in place of $f_2(s,t)$ in (64), one would have to write $s+1$ instead of s to bring the exponential to the form $e^{-\lambda_s t}$ as in (68) and this would give an integral like (65) but with the extra factor

$$\frac{\beta}{E_0}\frac{\beta}{E+\beta g(s+1,t)}\frac{\mu_s-\lambda_s}{D-\lambda_s}n_{s+1}$$

in place of (69). Since $E_0 \gg \beta$ in the cascade theory this is small compared with unity and hence the contribution of the $e^{-\lambda_{s-1}t}$ terms in (61) to $P_2(E,t)$ is quite negligible provided $E_0 \gg \beta$. For smaller E_0 the effect of these terms is actually to compensate (68) and make the total contribution of $P_2(E,t)$ to the series (62) still smaller.

In order to compare the contribution of the second term $P_2(E,t)$ of the series (62) compared with the first $P_0(E,t)$ we have worked out (68) accurately by the saddle-point method for $t = 10$, and y_0 ranging from 4 to 10 at intervals of 1, and for the two cases where E is equal to β and β/e respectively. The results are given in table 5. The first and second rows in each case give the values of $\beta P_0(E,t)$ and $\beta P_2(E,t)$ with the corresponding positions of the saddle-point. The third row gives the corresponding figures for $\beta P_\lambda(E,t)$ given by (27) of the previous section, in which collision loss is neglected completely. The table shows clearly that the contribution of P_2 is always considerably smaller than P_0, and for large y_0 it is about one-fifth. Now the thickness $t = 10$ is approximately that at which a particle of initial energy corresponding to $y_0 \sim 12$ produces the greatest number of particles, while for $y_0 = 4$ it corresponds to a position far beyond the maximum, when the shower is getting absorbed by collision loss. We should therefore expect P_2 to give a greater relative contribution for the smaller values of y_0 in table 5, as indeed the figures show. Even so, the table shows that the figures calculated by using the first term $P_0(E,t)$ alone of the series (62) will not differ from the true figures by more than about 30 % except when $E \sim \beta/e$. On the other hand, the figures for the number of particles calculated by neglecting collision loss altogether are three to five times too large for electrons of the critical energy ($E \doteq \beta$), and eight to seventeen times too large for $E = \beta/e$.

When $E \ll \beta/e$, the factor (69) becomes practically equal to unity and then $P_2(E,t)$ becomes nearly as great as $P_0(E,t)$. We should then expect that the higher terms of the series (62) would make an appreciable contribution and that it would be quite insufficient to stop at the first term. The range of energy concerned is small com-

pared with the critical energy, so that the region is not important as far as the total number of particles is concerned.

TABLE 5. COMPARISON OF $P_0(E, t)$ AND $P_2(E, t)$ FOR $t = 10$

	y_0	4	5	6	7	8	9	10
$E = \beta$	s_0	3·19	2·86	2·63	2·46	2·32	2·21	2·12
	$\beta P_0(E, t)$	0·0819	0·456	2·155	8·822	32·37	100·5	306·0
	s_0	2·78	2·53	2·36	2·24	2·15	2·06	2·00
	$\beta P_2(E, t)$	0·0374	0·1614	0·585	1·944	5·842	15·00	40·72
	$\beta P_\lambda(E, t)$	0·591	2·84	11·4	40·7	130·0	377·0	1012
$E = \beta/e$	s_0	3·10	2·80	2·59	2·43	2·30	2·19	2·10
	$\beta P_0(E, t)$	0·2560	1·209	5·472	21·90	78·97	247·2	715·8
	s_0	2·66	2·45	2·30	2·19	2·11	2·02	1·95
	$\beta P_2(E, t)$	0·1994	0·7820	2·789	8·782	24·99	65·17	157·5
	$\beta P_\lambda(E, t)$	7·72	30·9	110	354	1024	2751	6912

Since the second term P_2 of the series (62) is of the order β^2, it might appear that stopping at the first term means a total neglect of terms of the order β^2. This is, however, not so, since the integrand of the first term given by (63) itself contains β in an expression which can be expanded as a power series in β, as has been done in (48). To compare the contributions to the term of order β^2 in $P(E, t)$ which come from P_0 and P_2 respectively, calculate $P(E, t)$ by using the transformation (14*a*) on the series (52).* We get*

$$P(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^s \left[f_0(s,t) - \frac{\beta}{E_0}(s-1)\,g(s-1,t)f_0(s-1,t) + \left(\frac{\beta}{E_0}\right)^2 \left\{ \frac{(s-1)(s-2)}{2}\{g(s-2,t)\}^2 f_0(s-2,t) + f_2(s,t) \right\} + \ldots \right]. \quad (70)$$

In this series the term independent of β, and the term of order β are entirely due to the first term $P_0(E, t)$. Of the contributions to the term of order β^2 the first part in curly brackets in (70) is due to $P_0(E, t)$ while the second is due to $P_2(E, t)$. To get an idea of the relative magnitude of the two terms in curly brackets in (70) consider that part of both which has $e^{-\lambda_s - \lambda^t}$ as a factor. Using (58), (57) and (61), and writing s for $s-2$ for convenience, this part becomes

$$\left[1 + \left\{ 2\frac{(\lambda_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_s)}{(\lambda_{s+2}-\lambda_s)(\mu_{s+2}-\lambda_s)} - 1 \right\}\right],$$

multiplied by a function of s which we have not written explicitly for brevity. The term in curly brackets is the contribution of $f_2(s, t)$. The expression in curly brackets

* The same result could be obtained after some elementary manipulation by introducing the series (50) into (48).

is 0·52 for $s = 1{\cdot}5$, 0·40 for $s = 2$ and 0·32 for $s = 2{\cdot}5$. Using (25) it can be shown that it tends to zero as $s \to \infty$. It is clear from this that *at least* two-thirds of the term proportional to β^2 in the strict expansion of $P(E,t)$ as a power series in β comes from P_0, while P_2 only contributes less than 35 % to this term. We see therefore that the first term P_0 of the series (62) already contains the whole contributions to $P(E,t)$ of the terms independent of β, and proportional to β, and the major part of the contribution of the term proportional to β^2 and probably the important part of the contribution of higher powers of β. The terms P_2, P_3, etc., then only give small corrections to the terms proportional to β^2 and higher powers of β.

We have therefore established that (62) is an accurate formal solution of the equations of the cascade theory taking collision loss into account. The convergence of the series (62) has not been proved, but this is not necessary since we have shown that *in general it is sufficient to take only the first term*, the second being small compared with the first, *even for energies below the critical energy*, so that it is possible to look upon (62) as an asymptotic solution of the problem should the series (62) not converge.*

Values of $P_0(E,t)$ have been calculated by the saddle-point method for y_0 between 3 and 10 at intervals of 1, and three values of E equal to $e\beta$, β and β/e respectively. Table 6 gives the figures for $t = 2$, table 7 for $t = 4$, and table 8 for $t = 10$. The corresponding values of s_0 are given in table 7 as an indication. The figures for $P_\lambda(E,t)$ are those given by formula (27) where collision loss is neglected completely. *The tables clearly show that for large t and small y_0 the figures given hitherto, neglecting collision loss altogether, are too large by a factor two even when E is e times the critical energy, and for E equal to the critical energy, the old figures neglecting collision loss are sometimes too large by a factor seven.* This is as we should expect, for it is just for relatively large t and small y_0, that is after the shower has passed its maximum, that collision loss has the greatest effect. In this region our calculations show that the previous theories in which collision loss was neglected completely cannot claim accuracy even for energies which are two or three times the critical energy.

Energy distribution of the cascade electrons

Small thicknesses. First consider the energy distribution of cascade electrons in a shower as it appears after passing through a very thin layer of substance such that $t \ll 1$. (62) is still a correct solution, but in $f_0(s,t), f_2(s,t)$, etc., it is no longer possible to neglect the terms containing $e^{-\mu t}$. For $t \ll 1$ it is therefore simpler to start directly from the expression (43) and to insert in it for g and f the expressions they assume for very small t. Since f_0 satisfies the boundary conditions (44) it follows from (54a) that

$$\left\{\frac{\partial^2}{\partial t^2} f_0(s,t)\right\}_{t=0} = A_s^2 + B_s C_s.$$

* The convergence has been proved in the paper by Iyengar (1942).

TABLE 6. $t = 2$

	y_0	3	4	5	6	7	8	9	10
$E = e\beta$	$\beta P_0(E, t)$	0·259	0·503	0·753	1·12	1·62	2·18	2·86	3·84
	$\beta P_\lambda(E, t)$	0·391	0·619	0·955	1·39	1·91	2·59	3·45	4·53
$E = \beta$	$\beta P_0(E, t)$	0·831	1·519	2·27	3·34	4·56	6·19	8·16	10·7
	$\beta P_\lambda(E, t)$	1·68	2·60	3·78	5·19	7·05	9·37	12·3	15·3
$E = \beta/e$	$\beta P_0(E, t)$	1·87	3·14	4·97	7·14	9·89	13·6	17·9	23·5
	$\beta P_\lambda(E, t)$	7·05	10·3	14·1	19·2	25·5	33·5	41·6	49·7

TABLE 7. $t = 4$

	y_0	3	4	5	6	7	8	9	10
$E = \beta e$	s_0	—	2·63	2·24	2·01	1·86	1·75	1·67	1·61
	$\beta P_0(E, t)$	—	0·333	0·989	2·39	4·90	9·41	16·6	28·3
	s_0	2·96	2·39	2·10	1·92	1·79	1·71	1·64	1·58
	$\beta P_\lambda(E, t)$	0·171	0·609	1·60	3·52	6·88	13·0	21·1	36·0
$E = \beta$	s_0	3·00	2·43	2·13	1·94	1·81	1·72	1·64	1·59
	$\beta P_0(E, t)$	0·385	1·29	3·61	8·00	16·3	30·3	52·0	87·9
	s_0	2·39	2·10	1·92	1·79	1·71	1·64	1·58	1·54
	$\beta P_\lambda(E, t)$	1·65	4·34	9·57	18·7	35·2	57·3	97·7	160·0
$E = \beta/e$	s_0	2·84	2·37	2·09	1·92	1·79	1·71	1·64	1·58
	$\beta P_0(E, t)$	0·931	3·11	8·13	18·5	35·8	67·4	116·0	189·0
	s_0	2·10	1·92	1·79	1·71	1·64	1·58	1·54	1·50
	$\beta P_\lambda(E, t)$	11·8	26·0	50·8	95·8	155·8	266·0	435·0	671·0

TABLE 8. $t = 10$

	y_0	4	5	6	7	8	9	10
$E = \beta e$	$\beta P_0(E, t)$	0·0138	0·0870	0·471	2·04	7·89	26·1	80·6
	$\beta P_\lambda(E, t)$	0·0379	0·217	1·05	4·18	15·0	47·9	139·0
$E = \beta$	$\beta P_0(E, t)$	0·0819	0·456	2·15	8·82	32·4	100·0	306·0
	$\beta P_\lambda(E, t)$	0·591	2·84	11·4	40·7	130·0	377·0	1012
$E = \beta/e$	$\beta P_0(E, t)$	0·256	1·21	5·47	21·9	79·0	247·0	716·0
	$\beta P_\lambda(E, t)$	7·72	30·9	110·0	354·0	1024	2751	6912

This together with (44) shows that for small t

$$f_0(s, t) \approx 1 - A_s t + \tfrac{1}{2}(A_s^2 + B_s C_s) t^2 + 0(t^3) \tag{71}$$

as could be derived after some calculation from (55). Further, in view of the boundary conditions (44) and (45), it follows from (54*b*) that

$$\left\{\frac{\partial^2}{\partial t^2} g(s, t)\right\}_{t=0} = A_s - A_{s+1}.$$

Hence for small t $$g(s, t) \approx t - \tfrac{1}{2}(A_{s+1} - A_s) t^2 + 0(t^3). \tag{72}$$

By using the boundary conditions (44), (45) and (51) it can then be deduced from (54*c*) and the subsequent equations that $\partial^2/\partial t^2 f_n(s, t) = 0$ for all $n > 0$, so that $f_n(s, t)$ is of order t^3. All the following terms in (50) are therefore negligible compared with

f_0 for small t. It follows from this that for small t all the terms in (62) except the first are of order t^3 and negligible compared with $P_0(E,t)$. Hence, inserting (71) and (72) into (43),

$$P(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_0}{E+\beta[t-\frac{1}{2}(A_{s+1}-A_s)t^2]}\right\}^s [1-A_s t+\tfrac{1}{2}(A_s^2+B_sC_s)t^2]\,ds.$$

Correct to terms of order t^2 this may equally well be written after some elementary transformations

$$P(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E+\beta t}\right)^s\left[1-A_s t+\tfrac{1}{2}(A_s^2+B_sC_s)t^2+\frac{\beta}{E_0}\frac{s-1}{2}(A_s-A_{s-1})t^2\right]ds. \tag{73}$$

After inserting the expressions (16) for A_s, B_s and C_s in (73) the resulting integral can be evaluated exactly. We give only the result here. It is convenient to introduce a quantity y' defined by

$$y' = \log\frac{E_0}{E+\beta t}. \tag{74}$$

The 1 in the square brackets in (73) gives as usual

$$\delta(E_0-E-\beta t). \tag{75}$$

This, of course, just represents the original electron which has lost an energy βt by collision.

The term proportional to t in the square brackets in (73) only contains A_s. It is obvious from the way that equation (15a) was derived from (12a) that A_s represents the effects of radiation loss only, so that the spectrum of electrons created in a very small thickness is due to one radiation process only. The term proportional to t in the square brackets in (73) gives

$$-(\alpha'\kappa-\alpha'+\tfrac{1}{2})t\delta(E_0-E-\beta t)+\frac{t}{E_0}\left(\frac{\alpha'}{e^{y'}-1}+1-e^{-y'}\right), \tag{76}$$

where

$$\kappa = \int_0^1 \frac{dZ}{Z}.$$

κ is therefore an infinite constant which multiplies the δ function. The spectrum reaches a finite value as $E\to 0$, increases with increasing E, and tends to infinity as $E\to E_0-\beta t$, that is as $y'\to 0$. This form of the spectrum is quite understandable, for the spectrum of low energy quanta given by formula (1) is of the form dE'/E', so that the probability is greatest for the electron losing a very small part of its energy. The δ function in (76) diminishes the δ function in (75) corresponding to the removal of the initial electron by radiation loss. The diminution is, however, infinite, just because the total chance of a quantum being emitted is infinite according to (1). Both the singularities in (76) are due entirely to a failure of the Bethe-Heitler formula (1) for quanta of very low energy, and here, as has been shown by Bloch & Nordsieck

(1937), we have to consider the simultaneous emission of a large number of quanta involved in a transition to the classical theory. (76) could of course be derived by a direct integration of (12a). The resulting electron spectrum is given in the third column of table 9 for $t = 0{\cdot}1$.

TABLE 9. ENERGY SPECTRUM OF ELECTRONS AT DIFFERENT THICKNESSES FOR $y_0 \equiv \log E_0/\beta = 5$. $y = \log(E/\beta)$

		(76)	(77)	(78)	Total				
E/β	y \ t	0·1	0·1	0·1	0·1	0·3	2·0	4·0	10·0
0·37	−1	0·1001	0·0274	3·259	3·387	20·94	736·95 (2094)	1206·0 (7539)	179·4 (4582)
1	0	0·1003	0·0231	1·356	1·479	11·74	337·3 (560·3)	535·4 (1420)	67·67 (421·7)
2·72	1	0·1007	0·0185	0·5035	0·6227	4·669	111·7 (141·7)	146·8 (237·0)	12·91 (32·20)
7·39	2	0·1022	0·0137	0·1709	0·2868	1·920	33·83	33·24	2·063
20·1	3	0·1078	0·0090	0·0512	0·1680	0·8583	7·865	3·443	—
54·7	4	0·1424	0·0042	0·0121	0·1587	0·5739	—	—	—
—	4·8	0·6315	−0·0862	0·0019	0·5472	1·139	—	—	—

The figures in brackets are those neglecting collision loss. All the other figures represent $E_0 P_0(E, t)$.

The first term proportional t^2 in the square brackets in (73) has a part proportional to A_s^2, representing the spectrum created by two successive radiation processes, which merely results in a spectrum of the general form (76) but still more flattened out. It gives

$$(\alpha'\kappa - \alpha' + \tfrac{1}{2})^2 t^2 \delta(E_0 - E - \beta t) + \frac{t^2}{E_0}\left[\alpha'\left(\frac{\alpha'}{e^{y'}-1} + 1 - e^{-y'}\right)\log(1-e^{-y'}) + \frac{1}{2}\left\{\frac{\alpha'^2}{e^{y'}-1} + 1 + (1-2\alpha')\,e^{-y'}\right\} y' + \frac{\alpha'(\alpha'-\frac{1}{2})}{e^{y'}-1} + (\alpha'-\tfrac{3}{2})(1-e^{-y'})\right]. \quad (77)$$

The resulting spectrum is given in the fourth column of table 9 for $t = 0{\cdot}1$. The other part is proportional to $B_s C_s$. It is obvious from the way these quantities were derived that C_s represents the creation of quanta by radiation, while B_s represents the creation of pairs by quanta, so that this part represents the electron spectrum produced by the original electron through the intermediary of one quantum. The resulting spectrum due to this part is

$$\frac{t^2}{E_0}\left[\alpha' D(e^{y'} - e^{-2y'}) + \left(1 - \alpha' + \frac{\alpha'^2}{2}\right)(1 - e^{-y'}) - \alpha' y'(1 + e^{-y'})\right]. \quad (78)$$

It tends to zero as $y' \to 0$, i.e. $E \to E_0 - \beta t$, and increases monotonically as E decreases, reaching its maximum value when $E \to 0$. This is again what we should expect, for the spectrum of low energy quanta being of the form dE'/E', the spectrum of electrons created by these quanta rises to a maximum as $E \to 0$. It should be noticed that owing to the appearance of $e^{y'}$ in (78) its order of magnitude is in general much

larger than (76) or (77). The spectrum due to (78) is given by the fifth column of table 4 for $t = 0{\cdot}1$.

The last term in square brackets in (73) is proportional to β. It can be written

$$-\frac{\beta t}{2}\frac{\partial}{\partial E}\left[(1-e^{-y'})\left\{-\frac{t}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{y's}A_s\,ds\right\}\right] = \frac{\beta t^2}{E_0^2}\left(1-\frac{\alpha'}{2}-e^{-y'}\right). \tag{79}$$

The expression in curly brackets is just equal to (76) and is the spectrum created by one radiation process only. The meaning of the term (79) is then the following. In (76) the spectrum proportional to t has been shifted down to smaller energies by an amount βt, as if all the electrons had lost this amount of energy by collision. This is obviously incorrect, since some of the electrons are created at some intermediate point of the layer and hence lose less energy than βt by collisions. The purpose of the expression (79) is to correct for this error. Moreover, the energy of the primary particle is not E_0, as it appears in (76), at every point of the layer, but less than this by the amount lost by collision. This also introduces a slight correction to the spectrum (76), and this correction is included in (79). It should be noticed that since $E_0 > \beta t$ always, (79) is of order t, and is hence small compared with 1 for small t. It is always small compared with the sum of (76), (77) and (78). In the fifth and sixth columns of table 9 we have given E_0 times the resulting spectrum, namely, E_0 times the sum of (76), (77), (78) and (79) for $t = 0{\cdot}1$ and $t = 0{\cdot}3$, and $y_0 = 5$. The spectrum is quite different from what has been generally supposed. There is, as has been shown above, a δ-function at $E = E_0 - \beta t$, the height of which decreases as t increases. A more detailed investigation by K. S. K. Iyengar in which terms of all orders in t have been considered shows that for $t > 1/\alpha'$ this δ-function at $E = E_0 - \beta t$ completely disappears and the spectrum then decreases monotonically as E increases from zero. In particular it shows that the spectrum given by Arley for small thicknesses is not even qualitatively correct. This is mainly due to his neglect of the electron spectrum produced through the intermediary of one quantum. But quite apart from this, his assumptions about the collision and radiation loss are too artificial to have any claim to physical reality.

Formulae (75) to (79) could be derived for small t from a direct integration of the equations (12). Thus every part of the expression (73) has a direct physical meaning and this affords another verification that the solution (43) is correct for small t. In deriving expression (73) from (43), no assumption was made about the magnitude of E_0. Indeed, (43) is also valid when $E_0 < \beta$. As mentioned in the first section, collision loss alone would prevent the electron from penetrating to a thickness t greater than E_0/β, which in this case is small compared with 1. Hence, for $E_0 \ll \beta$, (43) automatically reduces to (73), and this is therefore the complete solution of the problem of the absorption of a low-energy electron by collision loss and cascade production.

Large thicknesses. For large thicknesses the solution (43) can be calculated in the form (62), where as was shown in the last section, the major part of the contribution comes from the first term alone, and it is possible in general to restrict oneself to this term. The form of the spectrum can be obtained from tables 6–8. In table 9 the

spectra for $t = 2, 4$ and 10 have again been given for $y_0 = 5$. The figures in brackets give the spectra that would have been obtained if collision loss had been neglected. Even for energies near the critical energy, the figures neglecting collision loss are sometimes six times too large. It is clear from the correct figures that the effect of collision loss is to flatten the spectrum for $E < \beta$ provided t is not too small. The spectrum of electrons in a shower never falls off for $E < \beta$ as has been suggested by Arley.

The calculation of the total number of particles in a cascade as a function of E_0 and t has been carried out by us on the basis of (62) in another paper (1942).

Appendix

Consider the function

$$X(s,t) = \int_0^t f_0(s, t-t')\, Y(s,t')\, dt', \tag{80}$$

where $f_0(s,t)$ is the function defined by (55) which satisfies the equation (54a) and the boundary conditions (44) at $t = 0$. Then

$$\frac{\partial}{\partial t} X(s,t) = Y(s,t) + \int_0^t \frac{\partial}{\partial t} f_0(s, t-t')\, Y(s,t')\, dt',$$

$$\frac{\partial^2}{\partial t^2} X(s,t) = \frac{\partial}{\partial t} Y(s,t) - \Delta_s\, Y(s,t) + \int_0^t \frac{\partial^2}{\partial t^2} f_0(s, t-t')\, Y(s,t')\, dt'.$$

Hence, remembering that f_0 satisfies (45a),

$$\Delta_s X(s,t) = \left(\frac{\partial}{\partial t} + D\right) Y(s,t). \tag{81}$$

Hence, if a function $X(s,t)$ is to be found satisfying an equation of the type (81) in which the right-hand side is given, the particular integral is at once given by (80). To it may be added the complementary function satisfying the left-hand side of (81) equated to zero, in order that X may satisfy the given boundary conditions at $t = 0$.

Now all the equations (54) are of the form (81). Hence, from (54b),

$$g(s-1,t) f_0(s-1,t) = \int_0^t f_0(s, t-t') f_0(s-1, t')\, dt'. \tag{82}$$

This already satisfies the correct boundary conditions (45) at $t = 0$. It leads at once to (57). The subsequent equations may be solved in the same way, where the direct method given in the text fails for $n > 2$.

To calculate the asymptotic forms of $f_0(s,t)$ and $g(s,t)$ for large s we notice that according to (25), as $s \to \infty$,

$$D - \lambda_s \to \frac{2}{\alpha' s^2 \log s}, \tag{83a}$$

while

$$\lambda_{s+1} - \lambda_s \to \frac{4}{\alpha' s^3 \log s}. \tag{83b}$$

Hence, together with (25), for large s

$$f_0(s,t) \to \frac{2}{\alpha'^2 s^2 (\log s)^2} e^{-Dt} + \frac{1}{s^{\alpha' t}} e^{-(\alpha'\gamma - \alpha' + \frac{1}{2})t}. \tag{84}$$

f_0 therefore tends to zero quite differently depending on whether $\alpha' t$ is less than or greater than 2, for in the former case the second term, in the latter the first term, of (84) becomes the dominant one as $s \to \infty$.

Now calculate the asymptotic form of $g(s,t)$ given by (57). The first two terms in the numerator must be taken together, and in view of (83b) writing

$$e^{-(\lambda_{s+1}-\lambda_s)t} \approx 1 - (\lambda_{s+1} - \lambda_s)\, t,$$

these then give

$$\frac{e^{-\lambda_s t}}{\lambda_{s+1}-\lambda_s}\left[\frac{(D-\lambda_s)^2}{(\mu_s-\lambda_s)(\mu_{s+1}-\lambda_s)} - \frac{(D-\lambda_{s+1})^2}{(\mu_{s+1}-\lambda_{s+1})(\mu_s-\lambda_{s+1})} + \frac{(D-\lambda_{s+1})^2(\lambda_{s+1}-\lambda_s)t}{(\mu_s-\lambda_{s+1})(\mu_{s+1}-\lambda_{s+1})}\right].$$

The last term in the above square brackets is of higher order, and the first two give, neglecting terms of a higher order,

$$e^{-Dt}\left[\frac{2D-\lambda_s-\lambda_{s+1}}{(\mu_s-\lambda_s)(\mu_{s+1}-\lambda_s)}\right] \approx e^{-Dt}\left[\frac{4}{\alpha'^3 s^2 (\log s)^3}\right]. \tag{85}$$

The third and fourth terms in the numerator of (57) must also be taken together. In view of (25)

$$\mu_{s+1} - \mu_s \to \frac{\alpha'}{s}$$

for large s, so that writing

$$e^{-(\mu_{s+1}-\mu_s)t} \to 1 - (\mu_{s+1} - \mu_s)\, t,$$

the third and fourth terms of the numerator of (60) then give

$$\frac{e^{-\mu_s t}}{\mu_{s+1}-\mu_s}\left[\frac{(\mu_s-D)^2}{(\mu_s-\lambda_s)(\mu_s-\lambda_{s+1})} - \frac{(\mu_{s+1}-D)^2}{(\mu_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_{s+1})} + \frac{(\mu_{s+1}-D)^2(\mu_{s+1}-\mu_s)t}{(\mu_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_{s+1})}\right].$$

The last term is of order $(\mu_{s+1}-\mu_s)t$, while the first two combine together to give a term of higher order. Hence, as $s \to \infty$ the above expression reduces to

$$te^{-\mu_s t} \to \frac{t}{s^{\alpha' t}} e^{-\{\alpha'(\gamma-1)+\frac{1}{2}\}t}.$$

Adding this to (85) and dividing by (84) the asymptotic form of $g(s,t)$ for large s is

$$g(s,t) \to \frac{\dfrac{4}{\alpha'^3 s^2 (\log s)^3} e^{-Dt} + \dfrac{t}{s^{\alpha' t}} e^{-\{\alpha'(\gamma-1)+\frac{1}{2}\}t}}{\dfrac{2}{\alpha'^2 s^2 (\log s)^2} e^{-Dt} + \dfrac{1}{s^{\alpha' t}} e^{-\{\alpha'(\gamma-1)+\frac{1}{2}\}t}}. \tag{86}$$

Depending on whether $\alpha' t$ is greater or less than 2, the first or the second terms in the numerator and denominator of (86) dominate, leading to (46a) and (46b) respectively.

References

Arley 1938 *Proc. Roy. Soc.* A, **168**, 519–45.
Bethe & Heitler 1934 *Proc. Roy. Soc.* A, **146**, 83–112.
Bhabha & Chakrabarty 1942 *Proc. Indian Acad. Sci.* A, **15**, 464–476.
Bhabha & Heitler 1937 *Proc. Roy. Soc.* A, **159**, 432–58.
Bloch & Nordsieck 1937 *Phys. Rev.* **52**, 54–9.
Carlson & Oppenheimer 1937 *Phys. Rev.* **51**, 220–31.
Chakrabarty 1942 *Proc. Nat. Inst. Sci. India*, **8** (in press).
Corben 1941 *Phys. Rev.* **60**, 435–9.
Iyengar 1942 *Proc. Indian Acad. Sci.* A, **15**, 195–229.
Landau & Rumer 1938 *Proc. Roy. Soc.* A, **166**, 213–28.
Serber 1938 *Phys. Rev.* **54**, 317–20.
Snyder 1938 *Phys. Rev.* **53**, 960–5.
Whittaker & Watson 1927 *Modern analysis*, 4th ed. Camb. Univ. Press.

A magnetic study of the two-phase iron-nickel alloys. II

By K. Hoselitz and W. Sucksmith, F.R.S.

H. H. Wills Physical Laboratory, University of Bristol

(*Received* 1 *April* 1942)

The method of using measurements of magnetic saturation intensity of annealed iron-nickel alloys for the determination of the equilibrium phase boundaries, as demonstrated by Pickles & Sucksmith, has been extended. The phase diagram of the system has been determined accurately between 525 and 365° C. The mechanism of phase segregation from the single-phase α-state has been studied, where it was found that contrary to the usual case, one of the phases crystallizes out in its equilibrium concentration whilst the residue of the alloy progressively and uniformly approaches equilibrium composition. It was possible to study and express quantitatively the rate of attainment of equilibrium, and on evidence obtained in this way the view is based that the lower practical limit of temperature where the equilibrium diagram can be studied by annealing experiments has been reached.

Introduction

In a recent paper Pickles & Sucksmith (1940) described an investigation of the magnetic properties of the two-phase iron-nickel alloys. Magnetic measurements showed the existence of a two-phase field and the phase diagram above 450° C was determined. The relations between the magnetization temperature curves and phase changes suggested that the magnetic method of investigating the phase diagram was capable of being extended to lower temperatures, if a method could be evolved which allowed the extrapolation to equilibrium conditions from intermediate stages of phase segregation. It was therefore the object of the present work to study the mechanism and kinetics of phase changes in the iron-nickel system and to extend the phase diagram so far as possible to lower temperatures.