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it is shown that if the ordinary Coulomb forces between the ions is used, under certain assumptions, his treatment is valid. Owing to his use of a special generalized ensemble, a correction is necessary which is small for low-valency symmetrical electrolytes but which may be quite considerable for the non-symmetrical cases. The deviations from the inverse square law, due to the saturation and hydration effects on the water dipoles and to the polarization, van der Waals and exchange forces between two typical ions i and j is accounted for by means of a correction term E_{ij} in the expression for their energy of interaction. It is shown that the addition of this term is equivalent to a modification of the dielectric constant D to $D - \delta$, where δ depends on E_{ij}, and is a function of the concentration and temperature. The extension in Kramers's theory as a result of this new type of force is given, and it is seen that with the proper form for δ the method proposed here should satisfactorily describe the properties of electrolytic solutions at strong concentrations. Definite numerical results cannot be obtained until δ is known. The attempt has been made to avoid the essential difficulties in the original Debye-Hückel theory.

The Creation of Electron Pairs by Fast Charged Particles

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Introduction

We shall discuss in this paper the creation of electron-pairs by the collision of fast charged particles. This calculation goes farther than other calculations on this subject in considering the effect of screening, and in investigating the probability of the creation of a pair as a function of impact parameter, *i.e.*, the least distance of approach between the two colliding particles. We shall also treat certain other cases which have not been considered before, among them the creation of very slow pairs such that the kinetic energies of the electron and positron of the pair are small compared to their rest energy. When the energy of one of the colliding particles is large compared with its rest mass, we shall also show that to a certain approximation most of the formulae given by the direct calculation can be obtained quite simply by a method similar to that given by Weizsäcker for calculating the emission of radiation by fast electrons on colliding with nuclei.

The procedure consists in calculating the probability of the transition of an electron from its initial state of negative energy to a final state of positive energy under the perturbing influence of the two colliding particles, the electron and resulting hole then appearing as the electron and positron of the created air. We shall throughout use the Born approximation, in which the interaction between the particles is treated as a perturbation. The transition from the initial to the final state of the system can then happen in two ways. The electron *in the negative energy state* may either interact with one of the colliding particles and jump at once to its final state, the colliding particle going over into an intermediate state. This particle can then interact with the other colliding particle and both jump to their final states. Or, the electron in the negative energy state may interact with one of the colliding particles and jump to an intermediate state, after which its interaction with the other colliding particle causes it to jump to its final state. Both processes are strictly of the second order, but for brevity we shall call the former process a "first-order process," only in the sense that it involves just one matrix element of the interaction of the electron of the created pair with the colliding particles. The second process involves two matrix elements of the interaction of the electron of the created pair with the colliding particles, and we shall call it a "second-order process."

In the collision between two heavy particles, moving with kinetic energy *small compared with their rest energy*, the "first-order process" is much larger than the second. This is easily seen as follows. We consider the process in a system in which one of the colliding particles is initially at rest, and the other moving with kinetic energy $T = P^2/2M$, where P is its momentum and M its mass. Then its change of momentum ΔP is given by

$$\Delta P \sim \frac{M}{P}\,\Delta T \gtrsim \frac{M}{P}(E + E_+) \gtrsim \frac{Mc}{P}(p + p_+),$$

where E, p, E_+, p_+ are the total energies and momenta of the electron and positron respectively. Since

$$P \ll Mc,$$

$$\Delta P \gg (p + p_+).$$

The total change in momentum of all three particles must be zero, so that the change in momentum of each heavy particle is large compared with the change in momentum of the electron of the created pair. The matrix element for each jump is just the matrix element of the Born approximation for a Coulomb field, and is roughly proportional to $1/(\Delta P)^2$ where one heavy particle goes straight from its initial to its final state, and is proportional to $1/(\mathbf{p} + \mathbf{p}_+)^2$ where the electron jumps straight from its initial to its final state. In the "second-order process," both the heavy particles jump successively from their initial direct to their final states, so that this process is roughly proportional to $1/(\Delta P)^4$. In the "first-order process," the electron and only one of the heavy particles jump *direct* from their initial to their final states, so that this process is proportional to $1/(\Delta P)^2 . (\mathbf{p} + \mathbf{p}_+)^2$, and it follows directly from the inequality $\Delta P \gg p + p_+$ that the "first-order process" is much greater than the second.† For fast particles where $P \gtrsim Mc$, ΔP is no longer much greater than $(p + p_+)$ and the above considerations do not hold.

The "first-order process" depends on the acceleration of the colliding particles, and with increasing energy, the "second-order process" becomes larger than the first. This means that with increasing energy of the colliding particles those processes which depend on the deflection of one of the colliding particles due to its interaction with the other colliding particle eventually become negligible compared with those processes which depend on the interaction of the electron of the created pair with both the colliding particles at once. This is also shown by the results, since the "first-order process" calculated by Heitler and Nordheim for slow heavy particles *decreases* with increasing kinetic energy of the colliding particles, and is more than a thousand times smaller than the "second-order process" calculated by us for fast heavy particles. We may then neglect the interaction between the two colliding particles, and in consequence the deflection of either caused by the other.

If we restrict ourselves to the creation of pairs of total energy small compared with the relative kinetic energy of the colliding particles, we may neglect the deflection of the colliding particles caused by the reaction of the electron of the created pair. Further, one may treat the two colliding particles classically. One may regard one as fixed at the origin and the other as moving along a straight line with uniform velocity $V \simeq c$. The condition under which one may do this is as follows. If δP_1 be the uncertainty in the momentum of the fast particle and δX_1 the uncertainty in its position, then we must have

$$P_1 \gg \delta P_1 > \hbar/\delta X_1,$$

† Heitler and Nordheim, 'J. Phys. Rad.,' vol. 5, p. 451 (1934).

so that

$$\delta X_1 \gg \hbar/P_1, \tag{1}$$

and we cannot consider the particle localized in a region smaller than δX_1 given by (1). (For $cP_1 \sim 10^8$ e.v., $\delta X_1 \gg 10^{-13}$ cm.) This distance is much smaller than the order of the distances at which most of the processes of pair creation take place.

We therefore proceed in this paper as follows. The heavier of the two colliding particles, which we shall call the particle 2, and which may be a bare nucleus or a neutral atom of nuclear charge Z_2e, we shall consider as fixed at the origin of co-ordinates. The other colliding particle of charge Z_1e, which we shall call the particle 1, we shall describe as moving classically with uniform velocity V along a straight line, passing the particle 2 at minimum distance b. We now calculate the probability of the transition of an electron from its initial state of negative energy to a state of positive energy under the combined influence of the two colliding particles. To get the total effective cross-section we must integrate this probability over all values of the impact parameter b of the two colliding particles.

This procedure will give a good approximation under the two conditions stated above, firstly that the particle 1 has kinetic energy comparable with or larger than its rest energy, and secondly that the change in energy of this particle, which must be at least $2mc^2$, be small compared with its initial energy.

In the special case in which the particle 1 is an electron, our method implies a neglect of the effect of exchange of this electron with the electron of the created pair. For the region where our calculations are strictly valid, namely, where the energy of the created pair is small compared with the initial energy of the electron 1, it may easily be seen that the effect of exchange is small. Its effect, however, may not be small in the total integrated cross-section, and we shall discuss this point further with the final results.

In § 1 we give the general theory of the pair creation. In § 2 we give the differential and integral cross-sections. The effect of screening will be considered in § 3, and the dependence upon impact parameter investigated in § 4. In § 5 we shall show that to a certain approximation most of the formulae can be derived by a method analogous to that due to v. Weizsäcker for calculating the radiation emitted by fast electrons on colliding with nuclei. The results are discussed in § 6.

1—General Theory

We regard the heavier particle, of charge $Z_2 e$, as fixed at the origin of co-ordinates. Its field will be described by a scalar potential $G(r)$. (If it is an unscreened nucleus, this will be $Z_2 e/r$.)

The motion of the other particle, of charge $Z_1 e$, co-ordinates X, Y, Z, will be described by

$$X = b_x; \qquad Y = b_y; \qquad Z = Vt - l, \tag{2}$$

where b is the impact parameter, and l is large compared with atomic dimensions and to b. Its field in the system in which it itself is at rest will be described by a scalar potential $F(r) = Z_1 e/r$. In the system we are considering, its field will be described as usual by a scalar and vector potential

$$\left.\begin{aligned} \phi_F &= \frac{1}{\gamma} F\left(\sqrt{(x-b_x)^2 + (y-b_y)^2 + \frac{1}{\gamma^2}(z - Vt + l)^2}\right) \\ A_{Fx} &= A_{Fy} = 0 \\ A_{Fz} &= \frac{V}{c}\,\phi_F \end{aligned}\right\}, \tag{3}$$

where

$$\gamma = \sqrt{1 - V^2/c^2}.$$

We write the Dirac equation in the form

$$\left\{\frac{E + e\phi}{c} + \left(\alpha, \mathbf{p} + \frac{e}{c}\mathbf{A}\right) + \alpha_4 mc\right\}\psi = 0. \tag{4}$$

We are not interested in the spin of the electron or positron, and we shall have to sum over both directions of spin of the initial, intermediate, and final states. It is here convenient to use a method given by Dirac where this summation is performed automatically.† Our ψ will be a matrix of four rows and columns and a function of x, y, z, t, and hence may be expressed as a linear combination of the α's multiplied by functions of x, y, z, t.

The density will be given by

$$\text{spur}\,[\psi\,.\,\psi^*].$$

We expand ψ in a series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots,$$

† Dirac, 'Proc. Camb. Phil. Soc.,' vol. 26, p. 361 (1930).

where ψ_n then satisfies

$$\left\{\frac{E}{c} + (\boldsymbol{\alpha}, \mathbf{p}) + \alpha_4 mc\right\} \psi_n = -\frac{e}{c}\left\{G(r) + \left(1 + \frac{V}{c}\alpha_3\right)\phi_F\right\}\psi_{n-1}. \quad (5)$$

We take the unperturbed initial state ψ_0, which satisfies (5) with the right-hand side put equal to zero, to be of the form

$$\psi_0 = \frac{E_0 - H_0}{c} e^{i\{(\mathbf{p}_0, \mathbf{x}) - E_0 t\}/\hbar}, \quad (6)$$

where

$$H_0 \equiv c(\boldsymbol{\alpha}, \mathbf{p}_0) + \alpha_4 mc^2. \quad (7)$$

ψ_1 will now consist of two parts ψ_1^G and ψ_1^F, corresponding to the two perturbation terms G and ϕ_F in (5). We write

$$\psi_1^G = \iiint b^G(\mathbf{p}') \,.\, e^{i\{(\mathbf{p}', \mathbf{x}) - E_0 t\}/\hbar} \frac{d\mathbf{p}'}{h^3}. \quad (8\text{A})$$

Substituting this in (5), multiplying from the left by

$$(E_0 - H') \exp \frac{i}{\hbar}\{-(\mathbf{p}', \mathbf{x}) + E_0 t\}/(E_0^2 - H'^2),$$

and integrating over all space ($\mathbf{x}$), we easily find that

$$b^G(\mathbf{p}') = -\frac{(E_0 - H')(E_0 - H_0)}{E_0^2 - H'^2} \mathscr{G}(|\mathbf{p}' - \mathbf{p}_0|^2), \quad (8\text{B})$$

where

$$\mathscr{G}(|\mathbf{p}' - \mathbf{p}_0|^2) = \frac{e}{c}\iiint G(r)\, e^{i(\mathbf{p}_0 - \mathbf{p}', \mathbf{x})/\hbar} d\mathbf{x}. \quad (9)$$

H' is defined in a similar manner as H_0 in (7) and is a matrix. H'^2 is a number.

ψ_2 will now consist of four terms corresponding to the four terms $-\frac{e}{c}\left\{G(r) + \left(1 + \frac{V}{c}\alpha_3\right)\phi_F\right\}(\psi_1^F + \psi_1^G)$ on the right-hand side of (5). We are interested only in two of these, ψ_2^{FG} and ψ_2^{GF}, corresponding to the terms $-\frac{e}{c}\left(1 + \frac{V}{c}\alpha_3\right)\phi_F\psi_1^G$ and $-\frac{e}{c}G\psi_1^F$ in the perturbation. The other two contain the field of one particle only, and cannot lead to the creation of pairs.

We let

$$\psi_2^{FG} = \iiiint b^{FG}(\mathbf{p}, E'')\, e^{i\{(\mathbf{p}, x) - E''t\}/\hbar} \frac{d\mathbf{p}\, dE''}{h^4},$$

and get as before by substituting in (5), and using (8),

$$\frac{E'' + H}{c} b^{FG}(\mathbf{p}, E'') = \iiint \frac{\left(1 + \frac{V}{c}\alpha_3\right)(E_0 - H')(E_0 - H_0)}{E_0^2 - H'^2} \times \mathscr{G}(|\mathbf{p}' - \mathbf{p}_0|^2)\, F(\mathbf{p}, E'', \mathbf{p}', E_0)\frac{d\mathbf{p}'}{h^3}.$$

with

$$F(\mathbf{p}, E'', \mathbf{p}', E_0) = \frac{e}{c\gamma}\iiiint F\left(\sqrt{(x-X)^2 + (y-Y)^2 + \frac{1}{\gamma^2}(z-Z)^2}\right) \times e^{i\{(\mathbf{p}'-\mathbf{p}, \mathbf{x}) - (E_0 - E'')t\}/\hbar}\, d\mathbf{x}\, dt$$

$$= \frac{e}{c\gamma}\iiint_{-\infty}^{\infty} F\left(\sqrt{(x-X)^2 + (y-Y)^2 + \frac{1}{\gamma^2}(z-Z)^2}\right) \times e^{i(\mathbf{p}'-\mathbf{p}, \mathbf{x}-\mathbf{X})/\hbar}\, d(\mathbf{x}-\mathbf{X}) . \int_{-\infty}^{\infty} e^{i[(\mathbf{p}'-\mathbf{p}, \mathbf{X}) - (E_0 - E'')t]/\hbar}\, dt$$

$$= \mathscr{F}\{(p_x - p'_x)^2 + (p_y - p'_y)^2 + \gamma^2(p_z - p'_z)^2\} \times h\delta\{V(p'_z - p_z) - E_0 + E''\}\, e^{i/\hbar(\mathbf{p}'-\mathbf{p}, \mathbf{x}_0)},$$

using (2). Here

$$\mathscr{F}_\gamma(|\mathbf{p} - \mathbf{p}'|^2) \equiv \mathscr{F}\{(p_x - p'_x)^2 + (p'_y - p_y)^2 + \gamma^2(p'_z - p_z)^2\} = \frac{e}{c}\iiint F(r)\, e^{i[(p'_x - p_x)x + (p'_y - p_y)y + \gamma(p'_z - p_z)z]/\hbar}\, d\mathbf{x}, \quad (10)$$

and $\mathbf{X}_0$ denotes the value of $\mathbf{X}$ at $t = 0$ $(b_x, b_y, -l)$. Owing to the presence of the δ function in $F(\mathbf{p}, E'', \mathbf{p}', E_0)$ the integration with respect to p'_z may be carried out, and we obtain

$$b^{FG}(\mathbf{p}, E'') = \frac{c}{V}\iint_{-\infty}^{\infty} \frac{(E'' - H)\left(1 + \frac{V}{c}\alpha_3\right)(E_0 - H')(E_0 - H_0)}{(E''^2 - H^2)(E_0^2 - H'^2)} \times \mathscr{F}_\gamma(|\mathbf{p} - \mathbf{p}'|^2)\, \mathscr{G}(|\mathbf{p}' - \mathbf{p}_0|^2)\, e^{i(\mathbf{p}'-\mathbf{p}, \mathbf{X}_0)/\hbar}\, dp'_x\, dp'_y, \quad (11)$$

taken at the point $p'_z = P_0 \equiv -\left(\frac{E'' - E_0}{V} - p_z\right)$. We notice that b^{FG} has two singularities at $E''^2 = H^2$. In ψ_2^{FG} the E'' integration extends from $-\infty$ to ∞. We deflect the path of integration above the singularities, and for large t the integral then reduces to the residues at these points. Only the one with E'' positive is of interest to us here.

We finally get

$$\psi_2^{FG} = \frac{1}{E}\iiint J e^{i\{(\mathbf{p}, \mathbf{x}) - Et\}/\hbar}\frac{d\mathbf{p}}{h^3},$$

with

$$J = -\frac{\pi i}{h^3}\frac{c}{V}(E - H)\left(1 + \frac{V}{c}\alpha_3\right)\{L_0 - (\boldsymbol{\alpha}, \mathbf{L}) - \alpha_4 L_4\}(E_0 - H_0), \quad (11)$$

where

$$(L_0, L_1 \ldots L_4) \equiv -\frac{1}{c^2}\iint_{-\infty}^{\infty}(E_0, cp'_x, cp'_y, cP_0, mc^2)$$
$$\times \mathscr{F}\{(p_x - p'_x)^2 + (p_y - p'_y)^2 + \gamma^2(p_z - P_0)^2\}$$
$$\cdot\frac{\mathscr{G}(|\mathbf{p}' - \mathbf{p}_0|^2)}{p'^2_x + p'^2_y + P_0^2 - p_0^2} e^{i(\mathbf{p}'-\mathbf{p}_0, \mathbf{X}_0)/\hbar}\, dp'_x\, dp'_y,$$

and our notation means that E_0 is to be taken in the integral for L_0, cp'_x for L_1, etc. Writing $\mathbf{p}'_r$ for $\mathbf{p}'_r - \mathbf{p}_r$, we get

$$(L_0, L_1, \ldots, L_4) = -\frac{1}{c^2}\iint_{-\infty}^{\infty}\{E_0, c(p'_x + p_x), c(p'_y + p_y), cP_0, mc^2\}$$
$$\times \frac{\mathscr{G}\{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}\,\mathscr{F}(\mathbf{p}'^2_r + \varepsilon^2)}{(\mathbf{p}'_r + \mathbf{p}_r)^2 + P_0^2 - p_0^2} e^{i(\mathbf{p}'_r, \mathbf{b}_r)/\hbar}\, dp'_x\, dp'_y\, e^{i\varepsilon l/\hbar\gamma}. \quad (12)$$

(For definition of δ, ε, *see* (15).) The suffix r will be used to denote a two-vector with components along the x and y axes.

We similarly find for ψ_2^{GF}

$$\psi_2^{GF} = \frac{1}{E}\iiint I e^{i\{(\mathbf{p}, \mathbf{x}) - Et\}/\hbar}\frac{d\mathbf{p}}{h^3},$$

with

$$I = -\frac{\pi i}{h^3}\frac{c}{V}(E - H)\{K_0 - (\boldsymbol{\alpha}, \mathbf{K}) - \alpha_4 K_4\}\left(1 + \frac{V}{c}\alpha_3\right)(E_0 - H_0), \quad (13)$$

and

$$(K_0, K_1, K_2, K_3, K_4) = -\frac{1}{c^2}\iint_{-\infty}^{\infty}\{E, -c(p'_x - p_{0x}), -c(p'_y - p_{0y}), cP, mc^2\}$$
$$\cdot\frac{\mathscr{G}\{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}\,\mathscr{F}(\mathbf{p}'^2_r + \varepsilon^2)}{(\mathbf{p}'_r - \mathbf{p}_{0r})^2 + P^2 - p^2} \quad (14)$$
$$e^{i(\mathbf{p}'_r, \mathbf{b}_r)/\hbar}\, dp'_x\, dp'_y\, e^{i\varepsilon l/\hbar\gamma}.$$

$$\left.\begin{aligned}
P_0 &= -\left(\frac{E - E_0}{V} - p_z\right); \quad P = \frac{E - E_0}{V} + p_{0z} \\
|P - p_z| &= |P_0 - p_{0z}| = \frac{E - E_0}{V} + p_{0z} - p_z \equiv \delta \\
\gamma|P - p_{0z}| &= \gamma|P_0 - p_z| = \gamma\frac{E - E_0}{V} \equiv \varepsilon \\
P^2 - p^2 &= \left(\frac{E - E_0}{V} + p_{0z} - p\right)\left(\frac{E - E_0}{V} + p_{0z} + p\right) \\
P_0^2 - p_0^2 &= \left(\frac{E - E_0}{V} - p_z - p_0\right)\left(\frac{E - E_0}{V} - p_z + p_0\right)
\end{aligned}\right\}. \quad (15)$$

The initial state ψ_0 given by (6) corresponds to an electron density of $8E_0^2/c^2$ instead of 2, so that we must divide the density given by spur $[\psi_2 \cdot \psi_2^*]$ by $4E_0^2/c^2$ to get the final probability. We thus find that the probability of the creation of a pair, the electron of which has a momentum lying in an element $d\mathbf{p}$ and the positron a momentum lying in an element $d(-\mathbf{p}_0)$, is

$$\frac{c^2}{4E^2E_0^2}\,\text{spur}\,(I+J)(I^*+J^*)\,\frac{d\mathbf{p}\,d\mathbf{p}_0}{h^6}. \tag{16}$$

I, J are given by (11) and (13) and I + J may be written in the form

$$I+J = -\frac{\pi i}{h^3}\frac{c}{V}(E-H)\{M_0-(\alpha,\mathbf{M})-\alpha_4 M_4-(\sigma,\mathbf{N}) - \alpha_4\alpha_3 N_4\}(E_0-H_0), \tag{17}$$

where σ represents the matrices $\alpha_2\alpha_3$, $\alpha_3\alpha_1$, $\alpha_1\alpha_2$, and the M's and N's are given by

$$M_0, M_1, M_2, M_3, \ldots, N_4 = -\frac{1}{c^2}\iint_{-\infty}^{\infty}\mathcal{G}\{(\mathbf{p}'_r-\mathbf{p}_{0r}+\mathbf{p}_r)^2+\delta^2\}\,\mathcal{K}\{p'^2_r+\varepsilon^2\} \times [M'_0, M'_1, \ldots, N'_4]\, e^{i(\mathbf{p}'_r\cdot\mathbf{b}_r)/\hbar}\, dp'_x\, dp'_y\, e^{i/\hbar\,\varepsilon l/\gamma}, \tag{18}$$

with

$$\left.\begin{aligned}
M'_0 &= \frac{E_0-Vp_{0z}}{D}+\frac{E-Vp_z}{D_0}\\
M'_1 &= -\frac{c(p'_x-p'_{0x})}{D}+\frac{c(p'_x+p_x)}{D_0}\\
M'_2 &= -\frac{c(p'_y-p_{0y})}{D}+\frac{c(p'_y+p_y)}{D_0}\\
M'_3 &= \frac{c}{V}\left\{\frac{\gamma^2E-E_0+Vp_{0z}}{D}+\frac{\gamma^2E_0-E+Vp_z}{D_0}\right\}\\
M'_4 &= mc^2\left(\frac{1}{D}+\frac{1}{D_0}\right)\\
N'_1 &= -\frac{V}{c}\left\{\frac{c(p'_y-p_{0y})}{D}+\frac{c(p'_y+p_y)}{D_0}\right\}\\
N'_2 &= \frac{V}{c}\left\{\frac{c(p'_x-p_{0x})}{D}+\frac{c(p'_x+p_x)}{D_0}\right\}\\
N'_3 &= 0;\quad N'_4 = \frac{V}{c}mc^2\left(\frac{1}{D}-\frac{1}{D_0}\right)
\end{aligned}\right\}, \tag{19}$$

and

$$\left.\begin{aligned}
D &= (\mathbf{p}'_r-\mathbf{p}_{0r})^2+P^2-p^2\\
D_0 &= (\mathbf{p}'_r+\mathbf{p}_r)^2+P_0^2-p_0^2
\end{aligned}\right\}. \tag{20}$$

The spur in (16) may then be evaluated by the usual methods if one remembers that the spur of all the Dirac matrices is zero, and also of their products, excepting those giving the unit matrix, whose spur is 4. The final probability (16) may then be written in the form

$$\pi^2 \frac{c^4}{V^2} S \frac{d\mathbf{p}\, d\mathbf{p}_0}{h^{12}}, \tag{21}$$

with

$$\begin{aligned} S = \frac{4}{EE_0} \Big[& |M_0|^2 \{EE_0 + c^2 (\mathbf{p}, \mathbf{p}_0) + m^2c^4\} + |\mathbf{M}|^2 \{EE_0 - c^2 (\mathbf{p}, \mathbf{p}_0) - m^2c^4\} \\ & + \{|M_4|^2 + |\mathbf{N}|^2\} \{EE_0 - c^2 (\mathbf{p}, \mathbf{p}_0) + m^2c^4\} \\ & + |N_4|^2 \{EE_0 + c^2 (\mathbf{p}, \mathbf{p}_0) - 2c^2 p_z p_{0z} - m^2c^4\} \\ & + \{Emc^2 (M_0M_4^* + M_3N_4^*) + E_0mc^2 (M_0M_4^* - M_3N_4^*) \\ & + c (E\mathbf{p}_0 + E_0 p, \mathbf{M}) M_0^* + c (E\mathbf{p}_0 - E_0\mathbf{p}, \mathbf{M} \times \mathbf{N}^*) \\ & - c (Ep_{0z} - E_0 p_z) M_4N_4^* + c^2 (\mathbf{p}, \mathbf{M}) (\mathbf{p}_0, \mathbf{M}^*) \\ & + c^2 (\mathbf{p}, \mathbf{N}) (\mathbf{p}_0, \mathbf{N}^*) + c^2 (\mathbf{p}_0 \times \mathbf{p}, \mathbf{N}) M^*_0 \\ & + c (p_z - p_{0z}) M_0N_4^* mc^2 + c (\mathbf{p}_0 + \mathbf{p}, \mathbf{M}) M_4^* mc^2 \\ & - c (\mathbf{N} \times [\mathbf{p} + \mathbf{p}_0])_z N_4^* mc^2\} + \text{conj. complex} \Big]. \end{aligned} \tag{22}$$

We see at once that this expression is quite symmetrical in $\mathbf{p}$ and $\mathbf{p}_0$ if we remember that the M's go over into M* and the N's into $-$ N* on interchanging $\mathbf{p}$ and $\mathbf{p}_0$.

(21) gives the probability of the creation of a pair as function of the impact parameter $\mathbf{b}_r$. To get the effective cross-section for the production of such a pair, we must integrate over all $\mathbf{b}_r$. This is easily performed as follows. $\mathbf{b}_r$ only occurs in the M's and N's, and in S we have only products of two of these. Then, for example,

$$\begin{aligned} \iint_{-\infty}^{\infty} M_\alpha M^*_\beta \, db_x \, db_y = \iint_{-\infty}^{\infty} & \Big\{ -\frac{1}{c^2} \iint \mathcal{G} \{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\} \\ & \times \mathcal{K} (p'^2 + \varepsilon^2) M'_\alpha (\mathbf{p}'_r) e^{i(\mathbf{p}'_r, \mathbf{b}_r)/\hbar} dp'_x \, dp'_y \Big\} \\ \cdot \Big\{ -\frac{1}{c^2} \iint_{-\infty}^{\infty} & \mathcal{G} \{(\mathbf{p}''_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\} \\ & \times \mathcal{K} (p''^2_r + \varepsilon^2) M'_\beta (\mathbf{p}''_r) e^{-i(\mathbf{p}''_r, \mathbf{b}_r)/\hbar} dp''_x \, dp''_z \Big\} db_x \, db_y. \end{aligned} \tag{23}$$

We interchange the order of the integrations and carry out the $\mathbf{b}_r$ integration first. This immediately gives us two δ functions, $h^2 \delta (p'_x - p''_x)$

$\delta(p'_y - p''_y)$, so that the integrations with respect to p_x'', p''_y may be performed, and we are finally left with

$$\frac{h^2}{c^4}\iint_{-\infty}^{\infty} |\mathscr{G}\{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}|^2 \times |\mathscr{F}\{\mathbf{p}'^2_r + \varepsilon^2\}|^2 \,\mathrm{M}'_\alpha(\mathbf{p}'_r)\,\mathrm{M}'_\beta(\mathbf{p}_r')\, dp'_x\, dp'_y. \quad (23\text{A})$$

Thus the differential effective cross-section $d\mathrm{Q}$ for the creation of a pair, the electron of which has a momentum between $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$, and the positron a momentum between $\mathbf{p}_+$ and $\mathbf{p}_+ + d\mathbf{p}_+$, may be written

$$d\mathrm{Q} = \frac{\pi^2}{\mathrm{V}^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{S}'\, |\mathscr{G}\{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}|^2 \times |\mathscr{F}\{p'^2_r + \varepsilon^2\}|^2\, dp'_x\, dp'_y\right]\frac{d\mathbf{p}\, d\mathbf{p}_0}{h^{10}}, \quad (24)$$

where S′ is the same expression as (22) with the M's and N's replaced by the M″s and N″s respectively

When both colliding particles with charges Z_1 and Z_2 are unscreened, we get for $\mathscr{F}$ and $\mathscr{G}$

$$\left.\begin{aligned}\mathscr{F}(|\mathbf{p} - \mathbf{p}'|^2) &= -\frac{\mathrm{Z}_1 e^2 h^2}{c\pi}\frac{1}{|\mathbf{p} - \mathbf{p}'|^2}\\ \mathscr{G}(|\mathbf{p} - \mathbf{p}'|^2) &= -\frac{\mathrm{Z}_2 e^2 h^2}{c\pi}\frac{1}{|\mathbf{p} - \mathbf{p}'|^2}\end{aligned}\right\}. \quad (25)$$

Screening will be considered in § 5.

2—The Differential and Integral Cross-Sections

The integrals (24) can only be evaluated approximately in certain cases. The rather complicated calculations have been performed in another paper.† These calculations involve no further physical assumptions. We shall give only the results here, together with the conditions which lie at the bottom of the mathematical approximations.

1. We first consider the case where the electron and positron of the created pair have momenta small compared to mc, so that we may neglect terms of the order p^2, p_+^2 compared with m^2c^2

$$p, p_+ \ll mc. \quad (26)$$

We henceforth write $\mathrm{E}_+ = -\mathrm{E}_0$ for the energy of the positron, and $\mathbf{p}_+ = -\mathbf{p}_0$ for its momentum. The differential cross-section for the creation of a pair, the electron of which has a momentum between p and

† 'Proc. Camb. Phil. Soc.,' vol. 31, p. 394 (1935). Referred to here as B.

$p + dp$, lying in a small solid angle $d\omega$, and the positron a momentum between p_+ and $p_+ + dp_+$, lying in the solid angle $d\omega_+$, is then (B, § 1, formulae (12) and (13)).

$$dQ = \frac{1}{128\pi^3}\left(\frac{Z_1 Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2\left[(p_{+r}{}^2 + p_r{}^2)\left\{\log\frac{1}{\gamma^2} - \frac{17}{6} + C\right\}\right.$$
$$\left. + (\mathbf{p}_{+r} + \mathbf{p}_r)^2\left\{\frac{1}{10} + C_r\right\} + (p_{+z} + p_{+z})^2\left\{\frac{1}{20} + C_z\right\}\right]$$
$$\times\frac{p^2 p_+{}^2\, dp\, dp_+\, d\omega\, d\omega_+}{(mc)^8}, \qquad (27)$$

where

$$\left.\begin{aligned} C &= 4\frac{\gamma^2}{1-\gamma^2}\log\frac{1}{\gamma^2} - \frac{4}{3}\gamma^2 + \frac{1}{6}\gamma^4 \\ C_z &= 3\frac{\gamma^2}{1-\gamma^2}\left(1 - \frac{\gamma^2}{1-\gamma^2}\log\frac{1}{\gamma^2}\right) - \frac{13}{5}\gamma^2 + \frac{7}{4}\gamma^4 - \frac{9}{10}\gamma^6 + \frac{1}{5}\gamma^8 \\ C_r &= -\frac{3}{2}\frac{\gamma^2}{1-\gamma^2}\left(1 - \frac{\gamma^2}{1-\gamma^2}\log\frac{1}{\gamma^2}\right) + \frac{4}{5}\gamma^2 - \frac{1}{8}\gamma^4 - \frac{1}{20}\gamma^6 + \frac{1}{40}\gamma^8 \end{aligned}\right\}. \qquad (28)$$

This cross-section is accurate for all values of γ. The C's tend to zero as the energy $(1/\gamma)$ of the particle creating the pair increases, so that for large energies we may neglect them in (28). For $1/\gamma \gg 10$, the C's are already less than one-tenth of the corresponding terms in curly brackets in (28). It is also easily shown that each expression in curly brackets in (28), and hence dQ, tends to zero as $\gamma \to 1$, *i.e.*, as $V \to 0$. By a numerical calculation we find that the coefficients of $(p_r{}^2 + p_{+r}{}^2)$, $(\mathbf{p}_r + \mathbf{p}_{+r})^2$, and $(p_z + p_{+z})^2$ in curly brackets in (28) are 0·08, 0·02, and 0·03 for $1/\gamma = 2$, 0·87, 0·08, and 0·06 for $1/\gamma = 5$, and 1·95, 0·09, and 0·05 for $1/\gamma = 10$.

If we write E_K for the total kinetic energy of the pair, so that

$$E_K = \frac{1}{2m}(p_+{}^2 + p^2), \qquad (29)$$

then the integration of (27) gives the cross-section for the creation of a pair of kinetic energy E_K and total energy $2mc^2 + E_K$, namely,

$$\frac{1}{32}\left(\frac{Z_1 Z_2}{137}\right)\left(\frac{e^2}{mc^2}\right)^2\left\{\log\frac{1}{\gamma^2} - \frac{161}{60} + C + C_r + C_z\right\}\frac{E_K{}^3\, dE_K}{m^4 C^8}. \qquad (30)$$

2. We next consider the case where

$$mc^2 \ll E, E_+;\quad mc^2/\gamma \gg E, E_+, \qquad (31)$$

the fulfilment of which necessarily requires that $\gamma \ll 1$. In accordance with (31), we neglect terms of order $\gamma^2 E^2$, $\gamma^2 E_0{}^2$, $(m^2c^4/E)^2$ and $(m^2c^4/E_0)^2$

compared with m^2c^4. The differential cross-section for the creation of a pair, the electron of which has an energy between E and E + dE, and the positron an energy between E_+ and $E_+ + dE_+$, is now (B, § 2 (23))

$$dQ = \frac{8}{\pi}\left(\frac{Z_1Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2 \frac{E_+{}^2 + E^2 + \frac{2}{3}EE_+}{(E + E)^4} \log \frac{k\,EE_+}{(E + E_+)\,mc^2} \times \log \frac{k'mc^2}{\gamma\,(E + E_+)}\,dE\,dE_+. \quad (32)$$

We shall use k, k' to denote numbers of the order unity, but not always the same numbers. This formula disagrees with that given by Fury and Carlson in having an extra logarithmic term. It has also been obtained recently by Landau and Lifschitz† by a similar method to that given here.

If we write E_T for the total energy of the created pair, *i.e.*,

$$E_T = E + E_+, \quad (33)$$

then the cross-section for the creation of a pair whose total energy lies between E_T and $E_T + dE_T$ is

$$dQ = \frac{56}{9\pi}\left(\frac{Z_1Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2 \log \frac{kE_T}{mc^2} \log \frac{k'mc^2}{\gamma E_T}\frac{dE_T}{E_T}, \quad (34)$$

subject to the condition $E_T \gg mc^2$. In deriving (34) we have integrated (32) over all E from mc^2 to E_T, whereas (32) is accurate only subject to the condition (31). One may show, however, that the accurate formula in the region $E \sim mc^2$ gives a cross-section smaller than that given by (32), so that the error we make in carrying the E integration down to mc^2 is small if $E_T \gg mc^2$.

To get the total cross-section, we notice that the integral of (34) with respect to E_T varies as $\log^2 E_T$, so that it is not very sensitive to the limits of integration. We may then, without appreciable error, carry the integration from $2mc^2$ to mc^2/γ, and get

$$Q = \frac{28}{27\pi}\left(\frac{Z_1Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2 \log^3 \frac{k}{\gamma}. \quad (35)$$

The formula (35) will give the total cross-section accurately provided that in those regions where (34) fails, namely, when $E_T \sim 2mc^2$ and $E_T \sim mc^2/\gamma$, the accurate formulae do not give cross-sections larger than (34) by an order of magnitude. It can now be shown, provided that exchange effects play no part, which is certainly true if the particle 1 be not

† 'Phys. Z. Sowjet.,' vol. 6, p. 244 (1934). The formula (32) is the only result of this paper derived by Landau and Lifschitz.

an electron, that the accurate formulae in every case given results smaller than (34) in those regions where (34) is inaccurate (one can see this for the region $E_T \sim 2mc^2$ by a comparison of (34) and (30)). It is therefore justifiable to integrate (34) from $E_T \approx 2mc^2$ to $E_T \approx mc^2/\gamma$ when $\gamma \ll 1$. We shall discuss the effects of exchange in § 6.

In deriving (32), (34), and (35) we have neglected the effect of screening of the nucleus 2. It will be seen in § 3 that this is legitimate when

$$\frac{EE_+}{(E + E_+)\, mc^2} \ll \frac{137}{2} Z_2^{-\frac{1}{3}}. \tag{36}$$

The formula (32) is therefore accurate, subject to (36). It will appear in the next section that the effect of screening is only to change the expression under the first logarithm in (32) so that the deviations from (32) when (36) is not fulfilled are not very large. For lead, $137\, Z_2^{-\frac{1}{3}} \sim 31$, so that from (36) we must have $E \sim E_+ \ll 31\, mc^2$. Since (32) is further subject to the condition (31), the region of validity of (32) is small. In the next section we shall give the formulae which are not subject to the restriction (36).

For a given $E_T = E + E_+$, the left-hand side of (36) is greatest when $E = E_+$, and a large contribution to (34) comes from just these regions where $E \sim E_+$, so that the accuracy of (34) is subject to the condition $E_T \ll 2\,.\,137\, Z_2^{-\frac{1}{3}}$. Hence (35) will represent a fairly good approximation, provided

$$1/\gamma < 2\,.\,137\, Z_2^{-\frac{1}{3}}. \tag{44}$$

3. When the particle 1 is a heavy particle with rest mass large compared with that of the electron, a further case is of interest, namely,

$$mc^2/\gamma \ll E, E_+ ; \quad M_1 c^2/\gamma \gg E, E_+ . \tag{37}$$

We now neglect terms of order mc^2 compared with γE, γE_0, and consider γ^2 small compared with unity. The differential cross-section for the creation of a pair, the electron of which has an energy E, and the positron an energy E_+ is then (B, § 3 (33))

$$dQ = \frac{8}{\pi} \left(\frac{Z_1 Z_2}{137}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \frac{m^2 c^4}{(E + E_+)^4\, \gamma^2} \log \frac{2k}{\gamma}\, dE\, dE_+, \tag{38}$$

subject to the further condition $\gamma \ll 1$ and $|E - E_+| \ll (E + E_+)$. As regards the term γ^2 in the denominator of (38), we note that (38) is valid only when $E\gamma$, $E_+\gamma \gg mc^2$. If we write $E = E_+ = mc^2/\gamma$ in (32) and (38), *i.e.*, extrapolate both formulae into regions where the conditions for their validity (31) and (37) respectively are not fulfilled, we find that the two formulae go over into each other except that (32) is larger than (38) by a factor 8/3. This means that when (38) is extrapolated down to the region $E \sim E_+ \sim mc^2/\gamma$ it is not in error by more than a factor three.

To get the total cross-section for the creation of a pair with $E, E_+ > mc^2/\gamma$, we integrate (38) over all E, E_+ from mc^2/γ to M_1c^2/γ. We get

$$Q_3 \sim \frac{1}{3\pi}\left(\frac{Z_1Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2 \log\frac{2k}{\gamma}. \tag{39}$$

The largest contribution to (39) comes from the region $E, E_+ \sim mc^2/\gamma$, so that from what has already been said above regarding the error in extrapolating (38), (39) represents no more than the correct order of magnitude. It should not be in error by more than a factor three. We notice that Q_3 increases with the logarithm of the energy of the particle 1.

3—The Effect of Screening

We shall now suppose that the fixed nucleus of charge Z_2 is surrounded by a distribution of electrons so as to form a neutral atom. The screening effect of this distribution of electrons may easily be taken into account if we write for the matrix element $\mathscr{G}(q^2)$ instead of (25)

$$G(\mathbf{q}^2) = -\frac{Z_2e^2h^2}{c\pi}\cdot\frac{1-F(\mathbf{q}^2)}{|\mathbf{q}|^2} \tag{40}$$

by a usual transformation. $F(q^2)$ is the atomic form factor defined by

$$F(\mathbf{q}^2) = \frac{1}{Z_2}\int \rho e^{i(\mathbf{q},\mathbf{x})/\hbar}\,d\mathbf{x} \tag{41}$$

integrated over the whole of space. ρ is the density of electrons at a distance r from the nucleus. We shall assume that the electron density is that given by the statistical method of Fermi. Now ρ is considerable only inside a region of about $137\,Z_2^{-\frac{1}{3}}\hbar/mc$, which is roughly the radius of the Fermi atom, so that by (41) $F(q^2)$ will be small compared with unity if

$$q \gg \delta' \equiv \frac{\hbar}{137\,Z_2^{-\frac{1}{3}}\hbar/mc} = \frac{Z_2^{\frac{1}{3}}}{137}\cdot mc. \tag{42}$$

By (18) $q^2 = \{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}$, so that the smallest value of q that occurs is δ. Screening is therefore negligible if

$$\delta \gg \delta' \equiv \frac{Z_2^{\frac{1}{3}}}{137}\,mc, \tag{43}$$

in which case (40) goes over into (25).

1. In the creation of slow pairs, given by (27) $\delta \gtrsim 2mc$ by (15), so that screening has no effect even for the heaviest atoms.

2. For the case of fast pairs given by (32), the particles of the pair are ejected within small angles of the order mc^2/E, mc^2/E_+ about the Z axis, so that by (15)

$$\delta \sim \frac{mc}{2} \frac{(E + E_+)\, mc^2}{E\, E_+}.$$

Screening is therefore negligible if

$$\frac{EE_+}{(E + E_+)} \ll \frac{137}{2} Z_2^{-\frac{1}{3}} mc^2, \tag{36}$$

as already stated.

As E, E_+ increase, δ decreases, and eventually becomes less than δ'. Screening has now an appreciable effect. It may then be shown (B, § 4 (45)) that for complete screening, namely,

$$\frac{EE_+}{(E + E_+)} \gg \frac{137}{2} Z_2^{-\frac{1}{3}} mc^2, \tag{45}$$

the differential cross-section is given instead of (32) by

$$dQ = \frac{8}{\pi} \left(\frac{Z_1 Z_2}{137}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \frac{E^2 + E_+{}^2 + \frac{2}{3} EE_+}{(E_+ + E)^4}$$

$$\log (k 137\, Z_2^{-\frac{1}{3}}) \log \frac{k' mc^2}{\gamma\, (E_+ + E)}\, dE dE_+. \tag{46}$$

To get the total cross-section roughly, we integrate (34) over all E_T up to $2\,.\,137\, Z_2^{-\frac{1}{3}} mc^2$, and then integrate (46) from $E_T \sim 2\,.\,137\, Z_2^{-\frac{1}{3}} mc^2$ to mc^2/γ. The result may be written†

$$\frac{28}{27\pi} \left(\frac{Z_1 Z_2}{137}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \log (k\, 2\,.\,137\, Z_2^{-\frac{1}{3}}) \left\{3 \log \frac{k'}{\gamma}\right.$$

$$\left.\log \frac{k''}{2\,.\,137\, Z_2^{-\frac{1}{3}} \gamma} + \log^2 (k\, 2\,.\,137\, Z_2^{-}\ )\right\}, \tag{47}$$

which we may use for $1/\gamma > 2\,.\,137\ Z_2^{-\frac{1}{3}}$ We have neglected terms like $\log k$ compared with $\log (2\,.\,137\, Z_2^{-\frac{1}{3}})$ consistently with our approximation.

† [*Note added in proof, October* 15, 1935. In a recent paper Nordheim (' J. Phys. Rad.,' vol. 6, p. 135 (1935)) has given a formula for the pair creation cross-section with screening, where the term $\log^3 k/\gamma$ of (35) is replaced by $\log^3 (k\ 137\ z_2^{-\frac{1}{3}})$. This is incorrect, as the considerations of § 5 make clear, since only one of the two colliding particles is screened. The correct formula should continue to increase with increasing $1/\gamma$ as $\log^2 k/\gamma$ for very large $1/\gamma$ (as, indeed, (47) does). I have Dr. Nordheim's authority for stating that he agrees with this.]

Since k is a number of the order unity, and $2 \cdot 137\, Z_2^{-\frac{1}{3}} > 60$ for all atoms, the error is about 25%. The formula (47) goes over into (35) for $1/\gamma = 2 \cdot 137\, Z_2^{\frac{1}{3}}$ to this approximation.

3. In the case when (37) is fulfilled, it may be shown that screening is effective only when

$$\frac{E_+ + E}{c}\frac{\gamma^2}{2} \lesssim \frac{Z_2^{\frac{1}{3}}}{137} mc, \tag{48}$$

which by (37) becomes for lead, $Z_2 = 82$,

$$\frac{1}{\gamma} \gtrsim 137\, Z_2^{-\frac{1}{3}} \frac{(E + E_+)\gamma}{2mc^2} > 30. \tag{48A}$$

Screening is therefore appreciable only for very high energies of the particles 1, and its effect is to replace the logarithmic term in (38) by $\log \{k\, 137\, Z_2^{-\frac{1}{3}} (E_+ + E)\, \gamma/mc^2\}$ which by (48A) is smaller than $\log k/\gamma$.

For a given γ, therefore, such that for the smallest $(E_+ + E)$ subject to (37) the condition (48A) is satisfied, the logarithmic term increases as $\log \{k\, 137\, Z_2^{-\frac{1}{3}} (E_+ + E)\, \gamma/mc^2\}$ with increasing $(E + E_+)$ until it becomes equal to $\log k/\gamma$, and then remains constant. For larger $(E_+ + E)$, the pair is produced too near the nucleus for screening to have any effect. (*See* §4. 2.) The total cross-section for pairs with $E, E_+ > mc^2/\gamma$ is now

$$Q_3 \sim \frac{1}{3\pi}\left(\frac{Z_1 Z_2}{137}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \log (k\, 2 \cdot 137\, Z_2^{-\frac{1}{3}}), \tag{49}$$

which should be used instead of (39) for $1/\gamma > 137\, Z_2^{-\frac{1}{3}}$.

4—Probability as Function of Impact Parameter

The probability of the creation of a pair, the electrons of which have definite momenta, when the particle 1 of charge Z_1 passes at a distance $\mathbf{b}_r$ from the fixed nucleus, is given by (21) with the M's and N's given by (18). The term $\exp \{i\, (\mathbf{p}'_r, \mathbf{b}_r)/\hbar\}$ in the integrals (18) gives the dependence on $\mathbf{b}_r$. For our present discussion it is convenient to choose our axes so that $b_y = 0$. Using (20) and (25), all the integrals (18) are of the form

$$\iint_{-\infty}^{\infty} \frac{1}{\{(\mathbf{p}'_r - \mathbf{p}_{0r} + \mathbf{p}_r)^2 + \delta^2\}\{\mathbf{p}'^2_r + \varepsilon^2\}} \times \left[\frac{A}{(\mathbf{p}'_r - \mathbf{p}_{0r})^2 + P^2 - p^2} + \frac{B}{(\mathbf{p}'_r + \mathbf{p}_r)^2 + P_0^2 - p_0^2}\right] e^{ip_x' b/\hbar}\, dp'_x\, dp'_y, \tag{50}$$

where A and B are the corresponding numerators of the quantities M or N given by (19).

1. We consider first the case (31), namely,

$$mc^2 \ll E, E_+ \; ; \; mc^2/\gamma \gg E, E_+.$$

It may now be shown (B, § 2 (16)) that the electron and positron emerge at angles of the order mc^2/E, and mc^2/E_+ respectively, so that by (15)

$$\left.\begin{aligned} \delta &\approx \frac{(E_+ + E)\, m^2c^3}{2\, EE_+} \\ \varepsilon &\approx \gamma\, \frac{E_+ + E}{c} \\ P^2 - p^2, P_0^2 - p_0^2 &\sim m^2c^2 \end{aligned}\right\}, \qquad (51)$$

and hence

$$\delta^2, \varepsilon^2 \ll P^2 - p^2, P_0^2 - p_0^2.$$

Most of the integral (50) therefore comes from two regions of the order δ and ε round the points $\mathbf{p}_{0r} - \mathbf{p}_r$ and 0 respectively. Suppose $\varepsilon \ll \delta$, then the important region is of the order ε round the point 0, and as b increases from 0 to ∞, the integral will start decreasing rapidly due to the oscillation of the exponential term when

$$b_{\max} \sim \hbar/\varepsilon,$$

i.e., by (51) when

$$b_{\max} \sim \frac{\hbar c}{\gamma\,(E_+ + E)}. \qquad (52)$$

This is for the case $\varepsilon \ll \delta$, *i.e.*,

$$\frac{1}{\gamma} \gg \frac{2\, EE_+}{m^2c^4}. \qquad (52\text{A})$$

Thus the creation of pairs of total energy $(E_+ + E)$ takes place up to distances of the order (52) when (52A) holds.

When $\varepsilon \gg \delta$, *i.e.*,

$$\frac{1}{\gamma} \ll \frac{2\, EE_+}{m^2c^4}, \qquad (53\text{A})$$

the production of pairs of energy $(E_+ + E)$ takes place up to distance given by

$$b_{\max} \sim \frac{\hbar}{\delta} \sim \frac{\hbar}{mc}\, \frac{2\, EE_+}{(E_+ + E)\, mc^2}. \qquad (53)$$

When $\varepsilon \sim \delta$, $1/\gamma \sim 2\, EE_+/m^2c^4$, and (52) and (53) become the same. We shall see in the next paragraph that these results have a direct physical significance.

2. We now consider the case (37), namely,

$$\frac{mc^2}{\gamma} \ll \mathrm{E}, \mathrm{E}_+ ; \quad \frac{\mathrm{M}_1 c^2}{\gamma} \gg \mathrm{E}, \mathrm{E}_+ .$$

It may now be shown that the particles of the pair emerge at angles of the order γ (B, § 3 (28)), so that from (15)

$$\delta \approx \frac{\mathrm{E}_+ + \mathrm{E}}{c} \cdot \frac{\gamma^2}{2}$$

and

$$\delta^2 \ll \varepsilon^2, \mathrm{P}^2 - p^2, \mathrm{P}_0{}^2 - p_0{}^2.$$

The important contribution to the integral (50) now comes from a region of order δ round the point $\mathbf{p}_{0r} - \mathbf{p}_r$. The production of a pair, the particles of which have energies E, E_+, therefore takes places at a maximum distance given by

$$b_{\max} \sim \frac{\hbar}{\delta} \sim \frac{\hbar c}{\gamma^2 (\mathrm{E}_+ + \mathrm{E})} . \tag{54}$$

From this we see that if $1/\gamma^2 < (\mathrm{E}_+ + \mathrm{E})/mc^2$, $b_{\max} < \hbar/mc$. In the reverse case it is greater than $\hbar/mc$.

5—Connexion with Pair Creation by γ-Rays

We shall now show that there is a very close connexion between the creation of pairs by one charged particle (1) passing through the field of another (2), the process we have been considering above, and the pair creation by γ-rays in the field of the latter particle (2). This is due to the following reason. When the particle 1 (of charge Z_1) moves with a velocity very near that of light, its field suffers a Lorenz contraction, and the electric force perpendicular to its path is $1/\gamma$ times larger than its value when the particle is at rest. To a certain approximation, therefore, the electric field may be considered as perpendicular to the path of the particle. Further, the field has now a magnetic force perpendicular and nearly equal to the electric force, and perpendicular to the direction of motion. In some region, then, small compared with its distance from the path of the particle, the field of the particle is very similar to that of electromagnetic waves, so that we may make a Fourier analysis of the field of the particle and consider each component as a γ-ray. The error in doing this is of the order γ, as has been shown by v. Weizsäcker.† We now use the appropriate formulae, giving the pair production by this γ-radiation in the field of the

† *Cf.* v. Weizsäcker, 'Z. Physik,' vol. 88, p. 612 (1934).

stationary particle (2) to get the total pair production by the moving particle (1) in the field of the stationary one.

The analysis of the field of the moving particle into monochromatic waves may be schematized according to v. Weizsäcker† thus. If n_ν be the number of quanta of frequency ν per unit frequency range which cross unit area at a distance r from the path of the moving particle 1, then

$$\left.\begin{aligned} n_\nu d\nu &= \frac{2}{\pi}\frac{Z_1^2 e^2}{r^2 c}\frac{d\nu}{h\nu} && \text{if } \nu < k\frac{c}{2\pi r\gamma} \\ n_\nu d\nu &= 0 && \text{if } \nu > k\frac{c}{2\pi r\gamma} \end{aligned}\right\}, \tag{55}$$

where k is a number of the order unity. To get the total number of quanta N_ν of frequency ν due to a state in which one particle crosses unit area per unit time, we must integrate $n_\nu d\nu$ over all r from $r_{\min}$ to $r_{\max}$. $r_{\max}$ is given by (55), because for larger distances the frequency ν no longer occurs. $r_{\min}$ we take as usual to be the Compton wave-length. We find

$$N_\nu d\nu = 2\pi\int_{r_{\min}}^{r_{\max}} (n_\nu d\nu)\, dr = \frac{Z_1^2}{137}\frac{2}{\pi}\log\left(k\frac{mc^2}{\gamma h\nu}\right)\cdot\frac{d(h\nu)}{h\nu}. \tag{56}$$

To get the effective cross-section for the creation by the colliding particles of a pair, the electron of which has energy E, and the positron an energy E_+, we must multiply $N_\nu d\nu$ by Q (E) dE, the effective cross-section for the creation of a pair, the electron of which has an energy E by a photon of energy $h\nu = (E_+ + E)$ in the field of the stationary particle 2.

We consider first the case (31). Then, according to Bethe and Heitler,‡

$$Q(E)\, dE = 4\frac{Z_2^2}{137}\left(\frac{e}{mc^2}\right)^2 \frac{E^2 + E_+^2 + \frac{2}{3}EE_+}{(E_+ + E)^3}\left\{\log\frac{2EE_+}{mc^2(E_+ + E)} - \tfrac{1}{2}\right\} dE. \tag{57}$$

If we multiply this by (56) we get

$$dQ = \frac{8}{\pi}\cdot\left(\frac{Z_1 Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2\frac{E^2 + E_+^2 + \frac{2}{3}EE_+}{(E_+ + E)^4}$$
$$\times\left\{\log\frac{2EE_+}{mc^2(E_+ + E)} - \tfrac{1}{2}\right\}\log\frac{kmc^2}{\gamma(E_+ + E)}\, dE\, dE_+.$$

This is just formula (32), if we remember that there and here the coefficients of order unity inside the logarithms are indeterminate.

† *Cf.* v. Weizsäcker, 'Z. Physik,' vol. 88, p. 612 (1934).
‡ 'Proc. Roy. Soc.,' A, vol. 146, p. 83 (1934).

Formula (57) holds according to Bethe and Heitler if

$$\delta = \frac{(mc^2)^2 (E_+ + E)}{2EE_+} \cdot \frac{1}{c} \gg \frac{mc}{137} Z_2^{\frac{1}{3}}. \tag{58}$$

If E, E_+ are larger, so that the inequality (58) is reversed, screening becomes important, and we must use for Q (E/dE)

$$Q(E)\,dE = 4\frac{Z_2^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \frac{\left\{(E^2 + E_+^2 + \frac{2}{3}EE_+)\log 183Z_2^{\frac{1}{3}} - \frac{EE^+}{9}\right\}}{(E_+ + E)^3}\,dE. \tag{59}$$

Multiplying this by (56) gives just (46) if we remember that $EE_+/9$ in (59) is a term of the order of the error we make in this approximation.

To get the variation with impact parameter b, we remark that the maximum distance r_{max} at which the frequency ν occurs is given by (55)

$$r_{max} \sim k \cdot \frac{c}{2\pi\nu\gamma}. \tag{60A}$$

Further, according to Bethe and Heitler, a quantum of frequency ν can produce a pair at distances from the fixed nucleus of the order d given by

$$d \sim \frac{\hbar}{\delta} = \frac{\hbar}{mc}\frac{2EE_+}{(E_+ + E)\,mc^2}, \tag{60B}$$

for E, $E_+ \gg mc^2$. Therefore b_{max} will be of the order r_{max} if $r_{max} \gg d$, *i.e.*,

$$\left.\begin{aligned} b_{max} &\sim k\frac{\hbar c}{(E_+ + E)\gamma} \\ \text{if} \quad \frac{1}{\gamma} &\gg \frac{2\,EE_+}{m^2c^4} \end{aligned}\right\}, \tag{61}$$

which is identical with (52). Similarly, we get (53) when $r_{max} \ll d$.

We now consider the creation of slow pairs under the condition (26). According to Nishina and others, one may write $Q(E_+)\,dE_+$ for E_+, E $\approx mc^2$ in the form ($\alpha \equiv e^2/\hbar c$)

$$\begin{aligned} Q(E_+)\,dE_+ = \frac{1}{64\pi}\frac{e^2h}{m^2c^3}(\alpha Z_2)^4 & \frac{1}{(e^{2\pi\alpha Z_2/\beta_+} - 1)(1 - e^{-2\pi\alpha Z_2/\beta})} \\ & \cdot\Big[2(\beta_+^2\sin^2\theta_+ + \beta^2\sin^2\theta) \\ & + (\alpha Z_2)^2\left(\frac{\pi^2}{8} + 1 + \cos\theta_+\cos\theta\right)\Big]\beta_+\,d\beta_+\,d\omega_+\,d\omega, \end{aligned} \tag{62}$$

which is valid for $\beta_+ \equiv v_+/c \ll 1$, $\beta \equiv v/c \ll 1$, $\alpha Z_2 \ll 1$.

θ_+, θ are the angles of ejection of the positron and electron with respect to the direction of motion of the light quantum.

If, further,

$$\frac{\alpha Z_2}{\beta_+} \ll 1, \frac{\alpha Z_2}{\beta} \ll 1, \tag{63}$$

which is also the condition for the validity of the Born approximation, (62) reduces to

$$\frac{1}{128\pi^3}\frac{e^2 h}{m^2 c^3}(\alpha Z_2)^2 (\beta^2_+ \sin^2\theta_+ + \beta^2 \sin^2\theta)\, \beta_+{}^2\, \beta\, d\beta_+\, d\omega_+\, d\omega. \tag{64}$$

This is the formula of Bethe and Heitler calculated by the Born approximation for the case when $\beta_+, \beta \ll 1$. If we multiply (64) by (56) we get the effective cross-section for the production of slow pairs by the collision of the two particles,

$$\frac{1}{128\pi^3}\cdot\left(\frac{Z_1 Z_2}{137}\right)^2\left(\frac{e^2}{mc^2}\right)^2 \log\left(\frac{k}{\gamma}\right)^2 \cdot \frac{(p_{+r}{}^2 + p_r{}^2)\, p_+{}^2 p^2\, dp\, dp_+\, d\omega\, d\omega_+}{(mc)^8}. \tag{65}$$

This is just formula (27), keeping only the term which predominates as $1/\gamma \to \infty$, *i.e.*, the logarithmic term. Since (27) is correct for all γ, a comparison of (65) with (27) gives us the error we make by using the Weizsäcker method. For example, the neglect of the term 17/6 in (27) which does not occur in (65) causes (65) to be nearly three times larger than (27), even for $1/\gamma = 10$. For $1/\gamma = 100$, the error is about 20%.

We have thus shown that an adaptation of the Weizsäcker method will to a certain approximation give the same results as the direct calculation. We may then use this method to supplement our results in those regions where the direct calculation cannot be carried out, *i.e.*, for the case $E \sim E_+ \sim 2mc^2$. In this way we may establish that for $E \sim E_+ \sim 2mc^2$, the accurate formula gives a result smaller than that given by (32), which is what was required to justify our method of deriving (34) and (35). In general, one can say that the effective cross-section for the production of a pair, the electron and positron of which have an energy E and E_+ respectively, is got by multiplying the formula (21) of Bethe and Heitler's paper by (56) above. (*Cf.* also their fig. 5.)

In passing, we may remark that a comparison of (62) and (64) shows us the error in the formulae of Bethe and Heitler for slow pairs due to the use of the Born approximation. Integrating (62) and (64) over all angles we have

$$\frac{Q_{\text{Nishina}}}{Q_{\text{Bethe and Heitler}}} = \frac{4\pi^2(\alpha Z_2)^2/\beta_+\beta}{(e^{2\pi\alpha Z_2/\beta_+} - 1)(1 - e^{-2\pi\alpha Z_2/\beta})}\left[1 + \frac{(\alpha Z_2)^2}{\beta_+{}^2 + \beta^2}\left(\frac{\pi^2}{8} + 1\right)\right].$$

This ratio is less than $\frac{1}{2}$ for

$$\frac{\alpha Z_2}{\beta_+} = \frac{\alpha Z_2}{\beta} = 0 \cdot 6. \qquad (66)$$

This means that when the condition (63) is not fulfilled, the Born approximation gives a cross-section too large by a factor two. For fast pairs, $\beta_+ \approx \beta \approx 1$, but (63) may still not hold if Z_2 is too large. For lead, instead of (63) we have (66), so that the above analogy seems to indicate that even for fast pairs, where the above argument does not strictly hold, the Born approximation may have given a result too large by a factor two. This would reduce somewhat the discrepancy between theory and experiment found for the stopping of fast electrons. The good agreement between theory and experiment for the production of pairs by γ-rays is, perhaps, to be regarded as fortuitous.

All our results have also been based on the Born approximation, so that our formulae, if applied to lead, may also be too large by about a factor two on this account.

6—Results

The Magnitude of the Cross-Sections—(*a*) We start by giving the cross-section calculated by Heitler and Nordheim† for the creation of pairs by a particle of charge Z_1, mass M_1, moving with velocity small compared with c, on colliding with a particle of charge Z_2 mass M_2, initially at rest. They get

$$Q \approx \left(\frac{1}{137}\right)^2 \left(\frac{e^2}{mc^2}\right)^2 \frac{m^2c^2}{M_1 T_1} Z_1^4 Z_2^2 \left(1 - \frac{M_1 Z_2}{M_2 Z_1}\right)^2, \qquad (67)$$

where T_1 is the kinetic energy of the particle 1. For slow heavy particles, the "first-order process" is far greater than the "second-order," and it is the former which is given by (67). From (67) we see that it *decreases* inversely as the kinetic energy T_1 of the particle creating the pair. If the particle 1 is a proton with $T \sim 10^6$ e.v., and the particle 2 a lead nucleus, $Q \sim 2 \cdot 8 \times 10^{-30}$ cm^2. If the particle 1 is an α-particle of the same energy, $Q \sim 1 \cdot 3 \times 10^{-30}$ cm^2, the decrease being due to the term $(1 - M_1 Z_2/M_2 Z_1)^2$. The pair production by slow heavy particles is therefore negligible.

(*b*) We now consider the creation of slow pairs by fast particles subject to the condition (26). The differential cross-section is given by (27). The positron and electron emerge at angles independent of one another, and the probability is greatest for their being ejected perpendicular to the path of the particle 1. The angular distribution is also symmetrical about a plane

† 'J. Phys. Rad.,' vol. 5, p. 451 (1934).

perpendicular to this path. The integration of this cross-section leads to (34), which increases as the cube of the kinetic energy E_K of the pair. The formula (30) is valid only for $E_K \ll mc^2$. We cannot get an estimate of the partial cross-section for the production of a pair with any kinetic energy less than mc^2 by just integrating (30) from $E_K = 0$ to $E_K = mc^2$, for we make a considerable error in extrapolating (30). We may see this by writing $E_K = mc^2$ in (30) and $E_T = 3mc^2$ in (34), when it appears that (34) is nearly ten times larger than (30). We can, however, use (30) in conjunction with the method of the last section to get a good estimate of the partial cross-section for the creation of a pair with any total energy E_T less than $3\,mc^2$. We find that this partial cross-section is in lead of the order 10^{-28} cm², and 10^{-27} cm², for a proton with $1/\gamma = 2$ and $1/\gamma = 10$ respectively. It increases like (30) logarithmically with the energy of the colliding particle. The cross-section is still extremely small, but is nevertheless about a hundred times larger than the cross-section for pair creation by slow heavy particles.

(*c*) The cross-section for the creation of pairs with energy $E, E_+ \gg mc^2/\gamma$ by heavy particles is given by (38). One sees from this that roughly the probability varies inversely as the fourth power of the total energy of the pair. The mean angles at which the particles of the pair are ejected are of order γ. Also, by integrating (38) over all E_+, we see that the probability of the electron having an energy E, irrespective of that of the positron is

TABLE I—PARTIAL CROSS-SECTION FOR THE CREATION OF PAIRS WITH $E, E_+ > mc^2/\gamma$ BY PROTONS IN LEAD

Energy	$1/\gamma$	Q_3
$\sim 10^9$ e.v.	2	$\sim 2 \times 10^{-27}$ cm²
$\sim 10^{10}$ e.v.	10	$\sim 7 \times 10^{-27}$ cm²
$\sim 10^{11}$ e.v.	100	$\sim 1 \cdot 2 \times 10^{-26}$ cm².

proportional to E^{-3}. The total cross-section for the creation of a pair, each particle of which has an energy greater than mc^2/γ, is given by (39), which gives only the order of magnitude. When the particle 1 is a proton and 2 a lead nucleus we get the values given in Table I. The value given in the third row is obtained by using (49) instead of (39), for screening is now effective.

These cross-sections are again about ten times larger than the corresponding cross-sections for the creation of slow pairs by protons of the same energy, so that if a proton produces a pair at all it will be a fast pair. The effect is still very small.

(*d*) The cross-section for the creation of pairs subject to the condition (31) is given by (32) and (46). The distribution of energy between the particles of the pair, and the dependence of the probability on the total energy of the pair is almost exactly the same as for the creation of the same pair by a γ-ray, multiplied by the factor $\log\{mc^2/\gamma(E_+ + E)\}/(E_+ + E)$, as indeed the considerations of § 5 make clear. The term $\log\{mc^2/\gamma(E_+ + E)\}$ varies but little. For pairs with $E_T \equiv (E_+ + E)$ large, *i.e.*, greater than 100 mc^2, there is a tendency for one particle to get more energy than the other. This tendency increases as the total energy E_T increases. For $E_T \sim 20\, mc^2$ there is a very broad maximum when both the particles get about the same amount of energy, and this maximum becomes more pronounced as E_T decreases. The distribution curves are like those given in the paper by Bethe and Heitler (p. 107). The mean angles at which the electron and positron appear are roughly mc^2/E and mc^2/E_+ respectively, and the two particles tend to emerge on opposite sides of the z axis. The effect of screening is also exactly the same as for γ-rays, and has been considered in § 3.

The integration of (32) over all energies of the positron E_+ gives, under the condition $mc^2 \ll E \ll mc^2/\gamma$, just the expression (34) with E substituted for E_T. This shows that the probability of the electron having an energy E, independently of the energy of the positron, is roughly proportional to $1/E$.

The total cross-section is roughly given by (35) for $1/\gamma < 2 . 137\, Z_2^{-\frac{1}{3}}$, and by (47) for $1/\gamma > 2 . 137\, Z_2^{-\frac{1}{3}}$. The resulting values of Q for different energies are given in Table II. Another quantity of interest is Q/Q_γ where Q is the cross-section for the creation of a pair by an electron, and Q_γ is the cross-section for the creation of a pair by a γ-ray of the same energy. This ratio is approximately

$$\frac{Q}{Q_\gamma} \sim \frac{1}{3\pi} \cdot \frac{1}{137}\, Y,$$

where

$$Y = \begin{cases} \log^2 1/\gamma & \text{for } 1/\gamma < 2 . 137\, Z_2^{-\frac{1}{3}} \\ \left\{3 \log \frac{1}{\gamma} \log \frac{1}{2 . 137\, Z_2^{-\frac{1}{3}} \gamma} + \log^2 2 . 137\, Z_2^{-\frac{1}{3}}\right\} & \text{for } 1/\gamma > 2 . 137\, Z_2^{-\frac{1}{3}}. \end{cases}$$

It is given in the third row of Table II.

TABLE II—CROSS-SECTIONS IN CM² FOR THE CREATION OF PAIRS IN LEAD

$1/\gamma$	10	50	100	500	1000
Q	11×10^{-26}	56×10^{-26}	$0{\cdot}9 \times 10^{-24}$	$2{\cdot}2 \times 10^{-24}$	$2{\cdot}9 \times 10^{-24}$
$Q/Q_\gamma \sim$	(0·004)	0·01	0·02	0·05	0·06

The cross-sections up to $1/\gamma = 100$ apply to protons as well as electrons, and will be relatively accurate for them. $1/\gamma = 100$ corresponds to a proton of $\sim 10^{11}$ e.v. For electrons of energy $10mc^2$ the value of Q given is likely to be too large, since for such low energies the deflection of the electron by the nucleus as well as the reaction of the electrons of the created pair may no longer be neglected. The values become progressively more accurate with increasing energy.

The Effect of Exchange—We shall now consider qualitatively the effect of exchange, when the particle creating the pair is an electron. Our calculations are accurate only in the region $E, E_+ \ll mc^2/\gamma \equiv E_i$ and $E_i \gg mc^2$.

E_i denotes the initial energy of the electron 1. The electron after the pair creation is left with an energy E_f very nearly given by $E_f \approx E_i - E_T$, since in all important cases the energy communicated to the particle 2 is small compared with mc^2. Since $E_T \ll E_i$ by our assumptions, $E_f \sim E_i \gg E$. Let $P(p_i \rightarrow p_f, p_0 \rightarrow p)$ be the matrix element without exchange for the transition in which the electron 1 jumps from a state of momentum p_i to a state of momentum p_f, and the electron of the created pair jumps from its initial state p_0 to a final state p. Then $|P(p_i \rightarrow p_f, p_0 \rightarrow p)|^2$ integrated over all final states p_f of the electron 1 is just the process we have calculated, and is given by (32) under the conditions (31). Let $P(p_i \rightarrow p, p_0 \rightarrow p_f)$ denote the matrix element for the process in which the electron 1 jumps from its initial state p_i to a final state p, and the electron of the created pair jumps from an initial state p_0 to a final state p_f. Then $|P(p_i \rightarrow p, p_0 \rightarrow p_f)|^2$ integrated over all final states p is given very roughly by (32) if we there write E_f instead of E, since (32) is not accurate when the electron creating the pair loses a large fraction of its initial energy. More accurately, $|P(p_i \rightarrow p, p_0 \rightarrow p_f)|^2$ would be still smaller. Hence $|P(p_i \rightarrow p, p_0 \rightarrow p_f)| \ll |P(p_i \rightarrow p_f, p_0 \rightarrow p)|$ if $E_i \sim E_f \gg E$. If we had taken exchange into account accurately, then the cross-section for the process in which one electron was in a final state p_f, and the other in a final state p would have been proportional to

$$|P(p_i \rightarrow p_f, p_0 \rightarrow p) - P(p_i \rightarrow p, p_0 \rightarrow p_f)|^2,$$

which, on account of the inequality given above, is very nearly $|P(p_i \rightarrow p_f, p_0 \rightarrow p)|^2$. For those cases where $E_f \sim E$, both matrix elements are of the same order, and the effects of exchange are not so simple. We may thus sum up by saying that exchange has but a small effect on formulae (27), (30), (32), and (34) in the region where they are valid, *i.e.*, $E_T \ll mc^2/\gamma$.

Cross-Section as Function of the Total Energy of the Pair—For a given large energy of the particle creating the pair, the differential cross-section as function of the energies of the particles of the pair behaves as follows. For $p, p_+ \ll mc$, the cross-section for the creation of a pair of total kinetic energy E_k increases as E_k^3, as shown by (30), reaches a maximum for some total energy E_T of the order $3mc^2$, and then for $E_T \gg mc^2$ decreases nearly as E_T^{-1}, as seen from (34). If the particle creating the pair be a heavy particle, then the further case $E_T \gg mc^2/\gamma$ is possible, and in this region the cross-section decreases approximately as E_T^{-3}, as one may see by integrating (38).

Comparison of Pair Creation by Protons and Electrons—One sees from the above cross-sections that if a particle of energy 10^{10} e.v. is observed to create a pair in its passage through a plate, the chance of its having been a heavy particle, *i.e.*, with rest mass comparable to that of the proton, is less than 4%, whereas for a particle of 10^9 e.v. the chance is less than 0·2%. Further, the angles at which the particles of the pair emerge are much larger for creation by protons than by electrons. For example, for a particle of $\sim 10^9$ e.v. γ, if it be a proton, is 1/2. For a pair with $E, E_+ \gg mc^2/\gamma \sim 2mc^2$, the mean angles at which the electrons of the pair emerge are roughly γ. If the particle creating the pair had been an electron, the mean angles would have been of the order $\frac{mc^2}{E}, \frac{mc^2}{E_+}$ respectively, which are much smaller than γ.

In conclusion, we remark that the errors which may be caused by the use of the Born approximation are exactly the same here as in the creation of pairs by γ-rays, calculated by Bethe and Heitler. For the reasons given in § 5, the results for lead for slow pairs are liable to be too large roughly by a factor two on this account.

Summary

The creation of electron pairs in the collision of particles moving with relative velocity very near the velocity of light is calculated. The effect of screening is considered, and the variation of probability of pair creation as function of impact parameter is investigated. It is shown that to a certain approximation most of the formulae can be derived by a method similar to one due to v. Weizsäcker, where the field of the moving particle is considered as a superposition of γ-rays.

The effective cross-section for the pair creation by fast protons in lead is

more than a thousand times larger than the cross-section for pair creation by slow protons calculated by Heitler and Nordheim. For the collision of electrons of 10^8 e.v., with a lead nucleus, the cross-section is of the order 10^{-24}cm^2, and is about one-fifteenth of the cross-section for the creation of a pair by a γ-ray of the same energy. It increases with increasing energy of the electron.

On the Instability of a Fluid when Heated from Below

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[Plate 15]

Introduction

It has been known for some time that when a horizontal layer of fluid is heated from below the fluid remains stationary, if initially at rest, until a certain temperature difference, depending on the physical constants of the fluid and the depth of the fluid layer, is reached. The equilibrium, which becomes less and less stable as the temperature rises, then becomes unstable for infinitely small disturbances, so that the fluid begins to move.

Rayleigh,* in 1916, put forward a theory which gave the temperature at which motion first occurs when the top and bottom layers are free surfaces. The results agreed qualitatively with previous experiments due to Bénard.† Later Jeffreys‡ calculated a critical temperature for the stability of a fluid between two rigid horizontal conducting planes. The investigation has since been revised by Jeffreys§ and also by Low.¶

The mathematical work involves certain not yet fully established assumptions, and it appeared desirable to investigate the problem experimentally to test the theoretical result. The mathematical investigations have not so far shown whether any sudden increase occurs in the heat transfer when motion begins, but this appeared probable and is also investigated in the present work. It was decided to suspend two plates

* 'Phil. Mag.,' vol. 32, p. 529 (1916).
† 'Ann. Chim. Phys.,' vol. 23, p. 62 (1901).
‡ 'Phil. Mag.,' vol. 2, p. 833 (1926).
§ 'Proc. Roy. Soc.,' A, vol. 118, p. 195 (1928).
¶ 'Proc. Roy. Soc.,' A, vol. 125, p. 180 (1929).