When is $f(x,y) = u(x) + v(y)$?

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Abstract. Let X and Y be arbitrary non-empty sets and let S a non-empty subset of $X \times Y$. We give necessary and sufficient conditions on S which ensure that every real valued function on S is the sum of a function on X and a function on Y.

Keywords. Subsets of cartesian products; good set.

1. Introduction

In this note we characterise subsets S of $X \times Y$ with the property that every complex valued function f on S can be expressed in the form

$$f(x,y) = u(x) + v(y), \quad (x,y) \in S, \tag{1}$$

where u and v are functions on X and Y respectively. The question is motivated by the preceding paper [1] where similar subsets occur as supports of measures associated of certain stochastic processes of multiplicity one.

2. Good sets

DEFINITION 2.1

We say that a subset $\emptyset \neq S \subseteq X \times Y$ is *good* if every complex valued function f on S can be expressed in the form (1).

It is obvious that any non-empty subset of a good set has also this property, but there exist sets which are not good and such that all proper subsets of it are good.

The purpose of this section is to describe good subsets of $X \times Y$, when X and Y are finite.

2.2. Let $\Pi_1 : X \times Y \longrightarrow X$ and $\Pi_2 : X \times Y \longrightarrow Y$ be the projections on X and Y respectively. If S is good, then any function $f : S \longrightarrow \mathbb{C}$, $f = u + v$, is completely determined by the values of u on $\Pi_1 S$ and v on $\Pi_2 S$. Therefore it is not a severe restriction on a good set S, if we assume in addition that $\Pi_1 S = X$ and $\Pi_2 S = Y$. This assumption will be made whenever necessary.

2.3. Assume that X and Y are *finite* with m and n elements respectively. We begin with the observation that a good set must be "thin" in the sense that it can have at most $m + n - 1$ points.

Let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and $S = \{s_1, s_2, \ldots, s_k\}$, where

$$s_1 = (x_{i_1}, y_{j_1}), \quad s_2 = (x_{i_2}, y_{j_2}), \ldots, s_k = (x_{i_k}, y_{j_k}).$$

We consider the $k \times (m+n)$-matrix M (called *the matrix of S*) with rows M_p, $1 \le p \le k$, given by

$$M_p = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 occurs at the places i_p and $m + j_p$, corresponding to the subscripts in the pair $s_p = (x_{i_p}, y_{j_p})$. Since S is good,

$$f(s_p) = f(x_{i_p}, y_{j_p}) = u(x_{i_p}) + v(y_{j_p}), \quad 1 \le p \le k.$$

We put

$$u(x_{i_1}) = \zeta_{i_1}, \ldots, u(x_{i_m}) = \zeta_{i_m}, \quad v(y_{j_1}) = \eta_{j_1}, \ldots, v(y_{j_n}) = \eta_{j_n}.$$

The relation (1) gives us k equalities

$$\zeta_{i_p} + \eta_{j_p} = f(s_p), \quad 1 \le p \le k.$$

In other words, the column vector $(\zeta_1, \ldots, \zeta_m, \eta_1, \ldots, \eta_n)^t \in \mathbb{C}^{m+n}$ is a solution of the matrix equation

$$M\vec{z} = \vec{\alpha}, \tag{2}$$

where $\vec{\alpha} = (f(s_1), f(s_2), \ldots, f(s_k))^t \in \mathbb{C}^k$.

Since S is good, we know that (2) has solution for every $\vec{\alpha}$. Since M has $m + n$ columns and since the vector $(\underbrace{1, 1, \ldots, 1}_{m \text{ times}}, \underbrace{-1, \ldots, -1}_{n \text{ times}})^t$ is a solution of the homogeneous equation $M\vec{z} = \vec{0}$, we see that its rank is at most $m + n - 1$. Clearly k cannot exceed the rank of M.

On the other hand the set $S = \{\{x_1\} \times Y\} \cup \{X \times \{y_1\}\}$, the union of two "axes", is a good subset of $X \times Y$ of cardinality $m + n - 1$. We have proved:

PROPOSITION 2.4

Let X, Y, $S \subseteq X \times Y$ be finite sets with m, n and k elements respectively; $\Pi_1 S = X$, $\Pi_2 S = Y$. Then S is good if and only if $k \le m + n - 1$ and the matrix M of S defined above has rank k. There always exists a good set of cardinality $k \le m + n - 1$.

DEFINITION 2.5

Let $S \subseteq X \times Y$. (We do not assume that X and Y are finite and S is not assumed to be good.) We say that a point $s = (x_0, y_0) \in S$ is *isolated in the vertical direction* (resp. *isolated in the horizontal direction*) if $(\{x_0\} \times Y) \cap S$ (resp. $(X \times \{y_0\}) \cap S$) is a singleton.

2.6. Let $S \subseteq X \times Y$ be an *arbitrary* subset of the cardinality $\le m + n - 1$ with $\Pi_1 S = X$ and $\Pi_2 S = Y$, $|X| = m$, $|Y| = n$, where $|A|$ denotes the cardinality of the set A. Every column of the matrix M associated to S is a non-zero vector. Each row has exactly two ones in it. Since the number of columns is $m + n$, there are at least two columns with exactly one 1 in each of them. Suppose that the jth column has exactly one 1 which occurs in the ith row. This means that s_i is isolated in the vertical direction if $1 \le j \le m$, and in the horizontal direction if $m + 1 \le j \le m + n$. We cancel from M the ith row and the jth column to obtain a matrix $\hat{M}$ and we drop from S the element s_i to

obtain a set S_1. We cancel from $\tilde{M}$ all columns which consist only of zero's and write M_1 for the matrix thus obtained. It is easy to see that M_1 is the matrix associated to S_1. If the number of rows in M_1 is greater or equal to the number of columns in M_1, then S_1 is not a good set and *a fortiori* S is not a good set. Otherwise, the number of rows in M_1 is smaller than the number of columns in M_1 and we can apply the above procedure to M_1. We obtain a reduced matrix M_2 and the smaller set S_2, of which M_2 is the associated matrix. If this process of reduction stops at a stage $l < k$, in the sense that the number of rows in M_l is greater or equal to the number of columns in M_l, then the set S is not good. If the process continues up to stage k (equal to the number of points in S), then S is good and the rank of the matrix M is equal to k. Thus we have obtained:

PROPOSITION 2.7

A subset $S \subseteq X \times Y$, where X, Y are two finite sets of cardinality m and n respectively, is good if and only if the process of reduction of the matrix M continues up to k steps, where k is the number of elements in S. Equivalently, if and only if the number of rows in M_i is smaller than the number of columns in M_i for each $1 \leq i \leq k$.

3. Graphs, couples and their unions

The sets X and Y are no more assumed to be finite in what follows, except when this assumption is explicitly stated.

PROPOSITION 3.1

If S is the graph of a function $g : E \longrightarrow Y$, where $E \subseteq X$, then S is good. Similarly, if S is the graph of a function $h : F \longrightarrow X$, where $F \subseteq Y$, then S is good.
   We have a more general result:

PROPOSITION 3.2

If $S = G \cup H$, where G is the graph of a function $g : E \longrightarrow Y$, $E \subseteq X$, H is the graph of a function $h : F \longrightarrow X \setminus E$, $F \subseteq Y \setminus g(E)$, then S is good.

Proof. For any complex valued function f on S, we define

$$u(x) = \begin{cases} f(x, g(x)) & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E, \end{cases}$$

$$v(y) = \begin{cases} f(h(y), y) & \text{if } y \in F, \\ 0 & \text{if } y \in Y \setminus F, \end{cases}$$

so that (1) is satisfied.
   This suggests the following:

DEFINITION 3.3

Let g be a function defined on a subset $E \subseteq X$ into Y and h be a function defined on a subset $F \subseteq Y$ into X. Let G and H be the graphs of g and h respectively. We say that the set $S = G \cup H$ is *a couple* if

$$g(E) \cap F = \emptyset, \quad h(F) \cap E = \emptyset.$$

Each couple is a good set. Let us observe also that not every union of two graphs is a couple; for example, if g and h are onto, then $G \cup H$ is not a couple. Moreover, a good set need not be a couple, for example the triplet $\{(0,0),(0,1),(1,0)\}$ is a good set in $\{0,1\} \times \{0,1\}$ which is not a couple.

3.4. We define

$$G = \{(x,y) \in S : (x,y) \text{ is isolated in the vertical direction}\},$$

$$H = \{(x,y) \in S : (x,y) \text{ is isolated in the horizontal direction}\}.$$

Note that $G \cup H = (G \setminus (G \cap H)) \cup H$ and the latter can be seen as a couple, since $\Pi_1(G \setminus (G \cap H)) \cap \Pi_1 H = \emptyset$ and $\Pi_2(G \setminus (G \cap H)) \cap \Pi_2 H = \emptyset$.
  Define

$$S_1 = S \setminus (G \cup H).$$

Let G_1, H_1 be obtained from S_1 in the same manner as G and H are obtained from S. Proceeding thus we get

$$S_2 = S_1 \setminus (G_1 \cup H_1), \ldots, S_{n+1} = S_n \setminus (G_n \cup H_n), \ldots$$

We note that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$. It is easy to see that each $G_i \cup H_i$ is a couple, being equal to $(G_i \setminus (G_i \cap H_i)) \cup H_i$.
  A natural generalisation of Proposition 2.7 is the following:

PROPOSITION 3.5

If $\bigcap_{n=1}^{\infty} S_n = \emptyset$, then S is good. If S is good and finite, then $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

DEFINITION 3.6

Two couples $S = G_1 \cup H_1$, $S_1 = G_2 \cup H_2$ are said to be *separated*, if the sets $\Pi_1 G_1$, $\Pi_1 H_1$, $\Pi_1 G_2$, $\Pi_1 H_2$, are mutually disjoint and the same is true for the sets $\Pi_2 G_1$, $\Pi_2 H_1$, $\Pi_2 G_2$, $\Pi_2 H_2$.
  In this case $S_2 = (G_1 \cup H_1) \cup (G_2 \cup H_2)$ is a couple too. More generally, it is clear that:

PROPOSITION 3.7

An arbitrary union of pairwise separated couples is a couple, hence a good set.

4. Links, linked and uniquely linked sets, loops

4.1. If S is finite and good, then at least one of the sets G or H defined in 3.4 is non-empty. The following example shows that this need not be true, if S is infinite.
  Let $X = Y = \mathbb{Z}$ and $S = \{(n, n-1) : n \in \mathbb{Z}\} \cup \{(n,n) : n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$. No point of S is isolated in either direction. However, S is good. For, let f be any complex valued function on S. We define $u(0) = c$, where c is an arbitrary constant. This forces $v(0) = f(0,0) - c$. Having defined $v(0)$, we see that $u(1) = f(1,0) - v(0)$, $v(1) = f(1,1) - u(1)$. Proceeding thus we see that u and v are uniquely determined as soon as we fix the value of $u(0)$.
  This example suggests a method of describing good subsets of $X \times Y$, which is valid also when X or Y or both are infinite.

DEFINITION 4.2

Consider two arbitrary points $(x, y), (z, w) \in S \subseteq X \times Y$ (S not necessarily good or finite). We say that (x, y), (z, w) are *linked*, if there exists a sequence $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of points in S such that:

(i) $(x_1, y_1) = (x, y)$, $(x_n, y_n) = (z, w)$;

(ii) for any $1 \leq i \leq n - 1$ *exactly one* of the following equalities holds:

$$x_i = x_{i+1}, \quad y_i = y_{i+1};$$

(iii) if $x_i = x_{i+1}$, then $y_{i+1} = y_{i+2}$, $1 \leq i \leq n - 2$, and if $y_i = y_{i+1}$, then $x_{i+1} = x_{i+2}$; equivalently, it is not possible to have $x_i = x_{i+1} = x_{i+2}$ or $y_i = y_{i+1} = y_{i+2}$ for some $1 \leq i \leq n - 2$.

The sequence $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is then called *a link* (of *length n*) joining (x, y) to (z, w) and we write $(x, y)L(z, w)$.

It is easy to see that the relation L is reflexive and symmetric and a verification shows that it is also transitive, so an equivalence relation.

DEFINITION 4.3

An equivalence class under the relation L is called *a linked component* of S. If $(x, y) \in S$, then the equivalence class to which (x, y) belongs is called the *linked component* of (x, y).

4.4. Let $(x_0, y_0) \in S \subseteq X \times Y$. The linked component of (x_0, y_0) is obtained as a union $\bigcup_{n=1}^{\infty} Q_n$, where

$$Q_1 = (X \times \{y_0\}) \cap S, \quad P_1 = \Pi_1 Q_1,$$
$$Q_2 = (\Pi_1^{-1} P_1) \cap S, \quad P_2 = \Pi_2 Q_2,$$
$$Q_3 = (\Pi_2^{-1} P_2) \cap S, \quad P_3 = \Pi_1 Q_3,$$

and so on. If n is odd, we have

$$P_n = \Pi_1 Q_n, \quad Q_{n+1} = (\Pi_1^{-1} P_n) \cap S, \quad P_{n+1} = \Pi_2 Q_{n+1}, \quad Q_{n+2} = (\Pi_2^{-1} P_{n+1}) \cap S, \ldots$$

A similar description is obtained if we start from the sets

$$\tilde{Q}_1 = (\{x_0\} \times B) \cap S, \quad \tilde{P}_1 = \Pi_2 \tilde{Q}_1.$$

4.5. Suppose that X and Y are standard Borel spaces and that $X \times Y$ is furnished with the product Borel structure. If $S \subseteq X \times Y$ is a Borel set, then each linked component of S is a countable union of analytic sets, hence the equivalence relation L decomposes S into at least analytic sets. We do not know whether the linked components are always Borel, or, if the partition into linked components is countably generated by Borel sets.

DEFINITION 4.6

Two points $(x, y), (z, w) \in S \subseteq X \times Y$ are said to be *uniquely linked*, if there is a unique link joining (x, y) to (z, w).

Theorem 4.7. *Let Q be linked component of S. Then the following properties are equivalent:*

(i) *any two points of Q are uniquely linked;*

(ii) *some two points of Q are uniquely linked;*

(iii) *for some $(x,y) \in Q$ the singleton $\{(x,y)\}$ is the only link joining (x,y) to itself.*

Proof. Left to the reader.

DEFINITION 4.8

A linked component of $S \subseteq X \times Y$ is said to be *uniquely linked* if any two points in it are uniquely linked.

The set S of 4.1 is uniquely linked.

DEFINITION 4.9

A non-trivial link joining (x,y) to itself is called *a loop*; by trivial link joining (x,y) to itself we mean the link consisting of the singleton $\{(x,y)\}$.

It is clear that a linked component is uniquely linked, if it has no loops. The four point set forming the vertices of a rectangle is a loop.

Theorem 4.10. *Assume that $S \subseteq X \times Y$ is linked. Then S is good if and only if it is uniquely linked.*

Proof. Assume that S is uniquely linked and let f be complex valued function on S. Let $(x_0, y_0) \in S$ and define $u(x_0) = c$, where c is a constant. This forces $v(y_0) = f(x_0, y_0) - u(x_0)$. We will now show that $u(x)$ and $v(y)$ can be defined unambiguously for all $(x,y) \in S$, so that (1) holds. Assume that we have defined $u(x)$ and $v(y)$ for all $(x,y) \in S$, which can be joined to (x_0, y_0) by a link of length n. Let $(z, w) \in S$ which is joined to (x_0, y_0) by a link of length $n + 1$ and let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})\}$ be this link. Since $(x_1, y_1) = (x_0, y_0)$ is joined to (x_n, y_n) by a link of length n, by the induction hypothesis $u(x_n)$ and $v(y_n)$ are correctly defined. If $x_n = x_{n+1}$, then $u(x_{n+1})$ is also defined and $v(y_{n+1}) = f(x_{n+1}, y_{n+1}) - u(x_{n+1})$. Note that $v(y_{n+1})$ is unambiguously defined, for (i) since S is uniquely linked, y_{n+1} cannot occur in $\{y_1, y_2, \ldots, y_n\}$, (ii) no point (x, y_{n+1}) can be joined to (x_0, y_0) by a link of length $\leq n$, for if $x \neq x_{n+1}$, then $\{(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1} y_{n+1}), (x, y_{n+1})\}$ is the unique link (of length $> n$) joining (x_1, y_1) to (x, y_{n+1}). On the other hand, if $y_n = y_{n+1}$, then $v(y_{n+1})$, hence also $u(x_{n+1})$ is correctly defined. We set u and v equal to zero on $X \setminus \Pi_1(S)$ and $Y \setminus \Pi_2(S)$ respectively and conclude (1).

We note that u and v are uniquely determined on $\Pi_1(S)$ and $\Pi_2(S)$ up to an additive constant, since the assignment of value to $u(x_0)$ completely determines u and v on these sets.

Assume now that the set S, which is good and linked, is not uniquely linked. Then S admits a loop $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, $(x_1, y_1) = (x_n, y_n)$, which we can assume to be of shortest length. Then either $x_1 = x_2 \neq x_{n-1}$ or $y_1 = y_2 \neq y_{n-1}$. In either case, since S is good, any function f on S satisfies

$$f(x_1, y_1) - f(x_2, y_2) + \cdots - f(x_{n-1}, y_{n-1}) = 0.$$

Since there are functions on S for which this fails, the theorem follows.

COROLLARY 4.11

A subset $S \subseteq X \times Y$ is good if and only if each linked component of S is uniquely linked, and also if and only if every finite subset of S is good.

Proof. Since S is good if and only if there is no link in S which is a loop, the corollary follows.

Remark 4.12. Assume that S is good. Then the decomposition $f = u + v$ is unique up to additive functions of x and y respectively, which are constant on the linked components. In other words, if $f = u + v = u_1 + v_1$, then $u - u_1$ and $v - v_1$ are constant on the linked components.

Theorem 4.13. *If a subset $S \subseteq X \times Y$ is uniquely linked, then S is of the form $G \cup H$, where G is the graph of a function g on a subset of X and H is the graph of a function h on a subset of Y.*

Proof. Fix $(x_0, y_0) \in S$. Assume that $(\{x_0\} \times Y) \cap S = \{(x_0, y_0)\}$, for simplicity. Let

$$G = \{(x,y): (x,y) \text{ is joined to } (x_0, y_0) \text{ by a link of even length}\},$$

$$H = \{(x,y): (x,y) \text{ is joined to } (x_0, y_0) \text{ by the link of odd length}\}.$$

We note that $S = G \cup H$ and $G \cap H = \emptyset$, since S is uniquely linked. We shall show that G is the graph of a function g on $\Pi_1 G$.

Let $(u,v), (w,z) \in G$, $(u,v) \neq (w,z)$. We show that $u \neq w$. Let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be a link joining (x_0, y_0) to (w, z). Note that y_{n-1} must be equal to y_n, since the link is of even length. If $u = w$, then $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)(= (w, z) = (u, z)), (u, v)\}$ is a link of odd length joining (x_0, y_0) to (u, v), contrary to the assumption that $(u, v) \in G$. Thus G is the graph of a function g on $\Pi_1 G$ defined by $g(x) = y$, if $(x, y) \in G$. Similarly H is the graph of a function h on $\Pi_2 H$ defined by $h(y) = x$, if $(x, y) \in H$.

We now remove the assumption that $(\{x_0\} \times Y) \cap S = \{(x_0, y_0)\}$. Let G_1 denote all those points $(x, y) \in S$, which can be joined to (x_0, y_0) by a link $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}, (x_1, y_1) = (x_0, y_0)$, of odd length and such that $x_1 = x_2$; let G_2 denote all those points $(x, y) \in S$, which can be joined to (x_0, y_0) by a link of even length and such that $y_1 = y_2$. Similarly we define H_1 and H_2. These four sets are mutually disjoint. If $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$, then $S = G \cup H$ and as before we can show that G and H are graphs of functions on subsets of X and Y respectively. The theorem is proved.

COROLLARY 4.14

If $S \subseteq X \times Y$ is good, then S is a union of two graphs G and H of functions defined on subsets of X and Y respectively.

Proof. Let $S = \cup_\alpha S_\alpha$ be the partition of S into uniquely linked components. Note that $\Pi_1 S_\alpha \cap \Pi_1 S_\beta = \emptyset$, $\Pi_2 S_\alpha \cap \Pi_2 S_\beta = \emptyset$, if $\alpha \neq \beta$. Since each $S_\alpha = G_\alpha \cup H_\alpha$, where G_α is the graph of a function g_α on $\Pi_1 G_\alpha$ and H_α is the graph of a function h_α on $\Pi_2 H_\alpha$, we see that $S = G \cup H$, $G = \cup_\alpha G_\alpha$, $H = \cup_\alpha H_\alpha$. Moreover, G and H are graphs of functions on $\Pi_1 G$ and $\Pi_2 H$ respectively.

PROPOSITION 4.15

Let $C_i = G_i \cup H_i, i \in I$, be an indexed family of couples, where the indexing set I is totally ordered such that for any $i \in I$, $C_i \cap (\Pi_1^{-1} \Pi_1 G_j \cup \Pi_2^{-1} \Pi_2 H_j) = \emptyset$ for all $j < i$. Then $\cup_{i \in I} C_i$ is a good set.

Proof. Assume, in order to arrive at a contradiction, that $S = \cup_{i \in I}(G_i \cup H_i)$ is not good. Then S admits a loop, say $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, which is of shortest possible length. Since $\cup_{i \in I} C_i = S$ and since there are only finitely many points in the loop, there is an index p such that $G_p \cup H_p$ contains a point from this loop, but no C_i, $i < p$, contains a point of this loop. Since $C_p \cap (\Pi_1^{-1}\Pi_1 G_j \cup \Pi_2^{-1}\Pi_2 H_j) = \emptyset$ for all $j < p$, we can replace X and Y by $X \setminus \cup_{j < p}\Pi_1 G_j$, and $Y \setminus \cup_{j < p}\Pi_2 H_j$. Without loss of generality assume that $(x_1, y_1) \in G_p$. Since G_p is the graph of a function on a subset of X, each point of it isolated in the vertical direction and so we conclude that $x_2 \neq x_1$, $y_1 = y_2$, $x_{n-1} \neq x_1, y_{n-1} = y_1$. But then $(x_2, y_2), (x_3, y_3), \ldots, (x_{n-1}, y_{n-1}), (x_2, y_2)$ is a loop in S of a smaller length if $x_{n-1} \neq x_2$; otherwise $(x_2, y_2), (x_3, y_3), \ldots, (x_{n-1}, y_{n-1})$ is a loop of smaller length in S. The result follows.

It is natural to ask if the good measure as defined in 2.1 of the preceding paper [1] is supported on a good set.

References

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