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On orbit equivalence of Borel automorphisms

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Abstract. Let E and F be two Borel sets of the countable product Z of the two point space $\{0,1\}$. Assume that E and F are invariant sets for the odometer transformation R and that E and F are of measure zero with respect to the unique finite R-invariant measure on Z. We show that E and F are R-orbit equivalent in a strict sense.

Keywords. Odometer transformation; orbit equivalence of Borel automorphisms; compressibility.

1. Introduction

Let $Z = \{0,1\}^{\infty}$ be the countably infinite product of the two point space $\{0,1\}$. We equip Z with the product σ-algebra (denoted by $\mathscr{A}$) and the product probability measure (denoted by P), where the two point space $\{0,1\}$ is given the discrete σ-algebra and the uniform probability measure $p{:}p\{0\} = p\{1\} = 1/2$. The odometer transformation R on Z is defined as follows: If $z = (z_1, z_2, z_3 \ldots)$ be a point εZ and if n be the first positive integer for which $z_n = 0$, then the image $\omega = Rz$ of z is given by

$$\omega_k = \begin{cases} 0 & \text{if } k < n \\ 1 & \text{if } k = n \\ z_k & \text{if } k > n \end{cases}$$

where ω_k denotes the kth co-ordinate of ω. If $z = (1,1,1,\ldots)$, then $Rz = (0,0,0,\ldots)$. It is known that R is uniquely ergodic, the measure P being the unique R-invariant probability measure on Z. Let E and F be two R invariant Borel subsets of Z each of P-measure zero. Let U and V denote the restrictions of R to E and F respectively. The purpose of this paper is to prove theorem 1.

Theorem 1. *If the orbit spaces of U and V do not admit Borel cross-sections, then U and V are orbit equivalent, i.e., there exists a Borel isomorphism φ of E onto F such that for each $x \in E$,*

$$\{V^n \varphi(x)\}_{n=-\infty}^{\infty} = \varphi(\{U^n x\}_{n=-\infty}^{\infty}).$$

This result may be viewed as a contribution to the theme of descriptive ergodic theory. (See [1], [6], [7], [9], [10], [11]). The main contribution here is lemma 1 below, which when coupled with some known facts and Cantor-Bernstein theorem for orbit equivalence, yields theorem 1 above.

2. Compressibility of null invariant sets in Z

Let $(X,\mathscr{B})$ be a standard Borel space and T a Borel automorphism on X. For any $A\subseteq X$, we write sA to denote the T-invariant set $\bigcup_{n=-\infty}^{\infty} T^n A$ generated by A, and we call sA the saturation of A. Two sets $A, B\in\mathscr{B}$ are said to be equivalent by countable decomposition if (i) we can partition A into a countable number of pairwise disjoint sets $A_i\in\mathscr{B}$, $i\in\mathbb{N}$ (ii) we can partition B into a countable number of pairwise disjoint sets $B_i\in\mathscr{B}$, $i\in\mathbb{N}$ (iii) we can find integers n_i, $i\in\mathbb{N}$ such that for each i, $T^{n_i}A_i = B_i$. Here and in the sequel $\mathbb{N}$ will denote the set of natural numbers. We write $A\sim B$ whenever A and B are equivalent by countable decomposition. If $A\sim B$, then the map $\varphi\colon A\to B$ given by $\varphi = T^{n_i}$ on A_i, (where A_i, B_i, n_i are as above) is called a descriptive isomorphism between A and B. If $A\sim B$ we also say that A and B are descriptively isomorphic.

DEFINITION

We say that a set $A\in\mathscr{B}$ is compressible if A can be expressed as a disjoint union of two sets B and C in $\mathscr{B}$ such that

(i) $sA = sB = sC$, and (ii) $A\sim B$.

If m is a countably additive T-invariant measure on $\mathscr{B}$, then two descriptively isomorphic sets in $\mathscr{B}$ have the same m-measure. Further a compressible set of finite m-measure has necessarily zero m-measure. If T is uniquely ergodic and m is the unique T-invariant probability measure on $\mathscr{B}$, then any compressible T-invariant set in $\mathscr{B}$ has m-measure zero. It seems natural to conjecture that if T is uniquely ergodic then any T-invariant set in $\mathscr{B}$ of m-measure zero is compressible. We will verify this conjecture for odometer transformation.

DEFINITION

Given a standard Borel space $(X,\mathscr{B})$ and a Borel automorphism T on X, we say that T is set periodic with period k if there is a partition

$$\mathscr{D}_k = \{D_1, D_2, \ldots, D_k\}$$

of X associated with T such that

$$D_i = T^{i-1}D_1, \quad 1\leqslant i\leqslant k.$$

If for each $n\in\mathbb{N}$, T is set periodic with period 2^n and with associated partition $\mathscr{D}_n(T) = \{D_1^n,\ldots,D_{2^n}^n\}$, such that $D_i^n = D_i^{n+1}\cup D_{i+2^n}^{n+1}$, $i=1,2,\ldots,2^n$, $n\in\mathbb{N}$, then we call T a weak von-Neumann transformation. We call T a von-Neumann transformation if T is a weak von-Neumann transformation and the union $\bigcup_{n=1}^{\infty}\mathscr{D}_n(T)$ of the associated sequence of partitions generates the σ-algebra $\mathscr{B}$.

The odometer transformation R on Z and the restrictions of R to R-invariant Borel sets are all von-Neumann transformations. Conversely any von-Neumann transformation is isomorphic to the restriction of R to a suitable R-invariant Borel subset of Z. A von-Neumann transformation V on $(X, \mathscr{B})$ admits utmost one V-invariant countably additive probability measure on $\mathscr{B}$.

Lemma 1. *If a von-Neumann transformation V on a standard Borel space $(X, \mathscr{B})$ does not admit a V-invariant countably additive probability measure on $\mathscr{B}$, then X is compressible* (*with respect to V*).

Proof. Let $\mathscr{D}_n = \{D_1^n, \ldots, D_{2^n}^n\}$ be the sequence of partitions associated to V as per the definition of von-Neumann transformation. Let $\mathscr{P}_n$ aenote the algebra generated by $\mathscr{D}_n$. We have $\mathscr{P}_n \subseteq \mathscr{P}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathscr{P}_n = \mathscr{P}$ is again an algebra. On $\mathscr{P}$ we define a V-invariant finitely additive measure m by setting, for all n, $m(D_k^n) = 1/2^n$, $1 \leqslant k \leqslant 2^n$. We will need the following two observations:

(i) If A, $B \in \mathscr{P}_n$ and $m(A) \leqslant m(B)$ then there is a set $C \in \mathscr{P}_n$, $C \subseteq B$, such that $A \sim C$, $m(A) = m(C)$ and $m(B - C) = m(B) - m(A)$. The sets A and C are in fact equivalent by finite decomposition through sets in $\mathscr{P}_n$.
(ii) If $A, B \in \mathscr{P}$ and $m(A) \leqslant m(B)$, then there is a set $C \subseteq B$, $C \in \mathscr{P}$, such that $A \sim C$, $m(A) = m(C)$ and $m(B - C) = m(B) - m(A)$. The sets A and C are in fact equivalent by finite decomposition by sets in $\mathscr{P}$. This follows (i) because for large enough n, $A, B \in \mathscr{P}_n$.

Since there is no V-invariant countably additive probability measure on $\mathscr{B}$, the finitely additive measure m on $\mathscr{P}$ is not countably additive on $\mathscr{P}$. (For if m were countably additive on $\mathscr{P}$, it would extend to a V-invariant countably additive probability measure on the σ-algebra generated by $\mathscr{P}$, which is $\mathscr{B}$). Therefore there exist pairwise disjoint sets $\delta_1, \delta_2, \delta_3, \ldots$ in $\mathscr{P}$ such that $X = \bigcup_{n=1}^{\infty} \delta_n$ and $\sum_{n=1}^{\infty} m(\delta_n) \lneqq 1$. There is no loss of generality if we assume that $\delta_1 \in \mathscr{P}_1$, $\delta_2 \in \mathscr{P}_2, \ldots, \delta_n \in \mathscr{P}_n \ldots$. (Some of the δ_n's could be empty). Let $q = 1 - \sum_{n=1}^{\infty} m(\delta_n) > 0$ and choose a positive integer N such that $2^{-N} < \frac{1}{2}q$. Recalling that $\mathscr{D}_N = \{D_1^N, \ldots, D_{2^N}^N\}$ denotes the Nth partition associated to V, we set $B = X - D_{2^N}^N$ and $C = D_{2^N}^N$. The sets B, C belong to $\mathscr{P}_N$, hence to $\mathscr{P}$ and $\mathscr{B}$. We note that $sC = sB = sX = X$. We now show that $X \sim B_1 \subseteq B$, thus proving the compressibility of X. We have $m(\delta_1) \leqslant 1 - q < 1 - 1/2N = m(B)$. Hence there exists, by observation (ii), a $r_1 \subseteq B$ such that $r_1 \in \mathscr{P}$, $\delta_1 \sim r_1$ and

$$m(B - r_1) = m(B) - m(\delta_1) = 1 - 1/2^N - m(\delta_1) > 1 - q - m(\delta_1)$$
$$= \sum_{i=2}^{\infty} m(\delta_i) \geqslant m(\delta_2).$$

Again by observation (ii) there exists $r_2 \subseteq B - r_1$, $r_2 \in \mathscr{P}$, such that $\delta \sim r_2$ and

$$m(B - r_1 - r_2) = m(B - r_1) - m(\delta_2) = m(B) - m(\delta_1) - m(\delta_2)$$
$$= 1 - 1/2^N - m(\delta_1) - m(\delta_2) > 1 - q - m(\delta_1) - m(\delta_2)$$
$$= \sum_{i=3}^{\infty} m(\delta_i) \geqslant m(\delta_3).$$

Proceeding thus we can find $r_1, r_2, r_3, \ldots$, inside B such that for each i, $r_i \in \mathscr{P}$, $r_i \sim \delta_i$. Thus X is equivalent by countable decomposition to $\bigcup_{i=1}^{\infty} r_i \subseteq B$. Since $B \subseteq X$ we see that $X \sim B$. (See [7] 5.3). q.e.d.

Remark. The sets E and F of theorem 1 are compressible with respect to U and V respectively since U and V are von-Neumann transformations which do not admit countably additive invariant probability measures in view of unique ergodicity of R.

3. A Cantor-Bernstein theorem

We need Cantor-Bernstein theorem for orbit equivalence which is as follows.

Theorem 2. *Let $(X,\mathscr{B})$, $(Y,\mathscr{C})$ be standard Borel spaces. Let S and T be Borel automorphisms on X and Y respectively such that T is orbit equivalent to the restriction of S to an S invariant Borel set $A \subseteq X$ and S is orbit equivalent to the restriction of T to a T invariant Borel set $B \subseteq Y$. Then S and T are orbit equivalent.*

Proof. Let $f:X \to B \subseteq Y$ be a one-one onto Borel map such that for all $x \in X$

$$f(\{S^n x\}_{n=-\infty}^{\infty}) = \{T^n f(x)\}_{n=-\infty}^{\infty}.$$

Similarly, let $g: Y \to A \subseteq X$ be a one-one onto Borel map such that for all $y \in Y$,

$$g(\{T^n y\}_{n=-\infty}^{\infty}) = \{S^n g(y)\}_{n=-\infty}^{\infty}.$$

We now adapt one of the proofs of Cantor-Bernstein theorem. We have for $n \geqslant 0$

$$(g\circ f)^{n-1} g(Y) \supseteq (g\circ f)^n(X) \supseteq (g\circ f)^n g(Y).$$

The sets

$$X - g(Y),\ g(Y) - (g\circ f)(X), \ldots, (g\circ f)^{n-1} g(Y) - (g\circ f)^n(X),$$
$$(g\circ f)^n(X) - (g\circ f)^n g(Y), \ldots, \text{together with}$$
$$\bigcap_{n=0}^{\infty} (g\circ f)^n(X), \text{ form a countable partition of } X$$

into S-invariant Borel sets. We define $h: X \to Y$ as follows:

$$h = \begin{cases} f & \text{on } (g\circ f)^{n-1} g(Y) - (g\circ f)^n(X) \\ g^{-1} & \text{on } (g\circ f)^n(X) - (g\circ f)^n g(Y) \\ f & \text{on } \bigcap_{n=0}^{\infty} (g\circ f)^n(X) \end{cases}$$

where $n \geqslant 0$. The map h is clearly one-one, Borel and takes an S orbit onto a T-orbit. It remains to show that h is onto. For $n \geqslant 0$,

$$h((g\circ f)^{n-1} g(Y) - (g\circ f)^n(X)) = (f\circ g)^n(Y) - (f\circ g)^n f(X) \tag{1}$$

$$h((g\circ f)^n(X) - (g\circ f)^n g(Y)) = (f\circ g)^{n-1} f(X) - (f\circ g)^n(Y) \tag{2}$$

$$h\left(\bigcap_{n=0}^{\infty} (g\circ f)^n(X)\right) = \bigcap_{n=0}^{\infty} (f\circ g)^n f(X) = \bigcap_{n=0}^{\infty} (f\circ g)^n(Y) \tag{3}$$

The sets on the right hand side of (1) and (2) for all $n \geqslant 0$ together with the set on the right hand side of (3) give a partition of Y. This proves that h is onto, and completes the proof of the theorem.

4. Equivalence of tower with the base

Let T be a Borel automorphism on a standard Borel space $(X,\mathscr{B})$ and assume that T is free, i.e., T has no periodic points. A set W in $\mathscr{B}$ is said to be wandering if $T^n W \cap T^m W = \varnothing$ whenever $m \neq n$. The σ-ideal generated by wandering sets is denoted by $\mathbb{W}$ and called the Shelah-Weiss ideal (see [7]). If $A \in \mathscr{B}$, and, A_0 be the set of all those points $x \in A$ such that $T^n x$ return to A for infinitely many positive values of n and also for infinitely many negative values of n, then $A - A_0$ belongs to $\mathbb{W}$. If $A = A_0$ and A is compressible then the sets B and C needed in the definition of compressibility can be so chosen that $B = B_0$ and $C = C_0$.

Lemma 2. *If $A = A_0$ and A is compressible then $sA \sim A$.*

Proof. Since A is compressible and $A = A_0$, we can write $A = B \cup C$, $B \cap C = \varnothing$, $B = B_0$, $C = C_0$ where further $sA = sB = sC$ and $A \sim B$. Let $S: A \to B$ be a descriptive isomorphism. Then $C = A - B = A - SA$. The sets C, SC, $S^2C, \dots$ are all pairwise disjoint and contained in A. Further, since

$$C = C_0,\ sC = \bigcup_{n=0}^{\infty} T^n C.$$

Put $C_1 = TC_0 - C_0$, and inductively, $C_j = TC_{j-1} - C_0, j \in \mathbb{N}$. The sets $C_j, j = 0, 1, 2, \dots$, are pairwise disjoint and $\bigcup_{j=0}^{\infty} C_j = \bigcup_{n=0}^{\infty} T^n C_0 = sA$. Indeed C_j's are the levels of the Kakutani sky scraper construction with base $C = C_0$. (See [2]). We now define the map $S^*: sC_0 \to B = A - C$ as follows:

$$S^* x = S^{j+1} T^{-j} x, \quad x \in C_j, \quad j = 0, 1, 2, 3, \dots.$$

Then

$$S^* C_j \subseteq S^{j+1} C_0. \text{ Since } C_j, j = 0, 1, 2, \dots$$

are pairwise disjoint and make up sA, and, since $S^{j+1} C_0, j = 0, 1, 2, \dots$, are pairwise disjoint and contained in A, we have sA descriptively isomorphic to a subset of A. Since $A \subseteq sA$, we have $A \sim sA$. (See [7] Lemma 5.3 and [3] corollary 1.7).

If $A = A_0$ one can define a transformation T_A on A, the so-called induced transformation, by $T_A x = T^{n(x)} x$, where $n(x)$ is the first positive integer such that $T^{n(x)} x \in A$. Of course, x, to begin with is in A. If, for any two sets $A, B \in \mathscr{B}$, $A = A_0$, $B = B_0$ and A and B are descriptively isomorphic, then T_A and T_B are orbit equivalent, the transformation S which establishes the descriptive isomorphism between A and B also establishes the orbit equivalence between T_A and T_B. We see therefore, in view of lemma 2, that if $A = A_0$ and A is compressible then T_A and the restriction of T to sA are orbit equivalent.

5. Proof of theorem 1

We are now in a position to prove theorem 1. A theorem of Glimm and Effros coupled with a result of Ramsay and Mackey (see [10], [7]) permits one to conclude that if the orbit space of a Borel automorphism T on a standard Borel space $(X,\mathscr{B})$ does not admit a Borel cross section, then there exists a Borel set $A \subseteq X$ such that T_A is orbit equivalent to the odometer transformation R on $\{0,1\}^{\infty}$. Passing to a subset we conclude that if the orbit space of T does not admit a Borel cross section then there exists a set $A \subseteq X$, A Borel, such that T_A is orbit equivalent to U of theorem 1. Applying this fact with $T = V$ and $X = F$, we see that there is a Borel subset $F_1 \subseteq F$ such that V_{F_1} is orbit equivalent to U. Since E is compressible by lemma 1, we conclude that F_1 is compressible. Since F_1 is compressible, lemma 2 shows that V_{F_1} is orbit equivalent to the restriction of V to sF_1. Thus U is orbit equivalent to the restriction of V to a V invariant Borel subset of F. Similarly V is orbit equivalent to the restriction of U to a U-invariant Borel subset of E. The Cantor-Bernstein theorem of §3 yields now theorem 1.

Remarks. If m is an atom free probability measure on a standard Borel space $(X, \mathscr{B})$ and T a Borel automorphism on X which preserves m-null sets and is ergodic with respect to m, then it is known (see [8]) that the restriction of T to a suitable T-invariant Borel set (say Y) of full m-measure is orbit equivalent to the restriction of the odometer R to an R-invariant Borel set, say E. Further if there is no finite T-invariant measure on $\mathscr{B}$ with same null sets as m, then the R-invariant set E has P-measure zero. Since m is atom free and T is ergodic with respect to m, the orbit space of T (restricted to Y) does not admit a Borel cross section, a fortiriori, the orbit space of R restricted to E does not admit a Borel cross section. These facts together with theorem 1 permit us to conclude that if S and T are two non-singular ergodic automorphisms on $(X, \mathscr{B}, m)$ neither admitting finite invariant measure with same null sets as m, then there exists an S-invariant Borel set Y_1 of full m-measure and a T invariant Borel set Y of full m-measure such that $S|_{Y_1}$ and $T|_Y$ are orbit equivalent. This result does not contradict Krieger's work ([4], [5]) on weak equivalence via ratio sets because the Borel isomorphism between Y_1 and Y which implements the orbit equivalence is not claimed to preserve the m-null sets.

It seems natural to conjecture that any two compressible and free Borel automorphisms S and T (i.e. X is compressible with respect to both S and T) whose orbit spaces do not admit Borel cross sections are orbit equivalent. In [7] it is proved that any such Borel automorphism is orbit equivalent to a weak von-Neumann transformation. If the word "weak" in this result can be removed, the conjecture would follow from theorem 1. It seems natural to conjecture, also, that any two free Borel automorphism are orbit equivalent if and only if the cardinality of the ergodic invariant probability measures they admit is the same. In particular it seems that any two uniquely ergodic free Borel automorphism on a standard Borel space are orbit equivalent without discarding any null sets. Papers [1] and [7] are relevant for some of these questions, but it should be recorded here that there is a gap in the proof of theorem 2 of paper [1] and that a correct proof is available under the additional condition (*) stated in §3 of the same paper.

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