WEAKLY WANDERING SETS AND COMPRESSIBILITY IN A DESCRIPTIVE SETTING.

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1. INTRODUCTION.

The purpose of this note is to give an example of a Borel automorphism T on a standard Borel space $(X, \mathcal{B})$ such that (i) T does not admit any invariant probability measure, (ii) there is no weakly wandering set $W \in \mathcal{B}$ such that union of $T^n W$ over all integers n is all of X. This shows that a natural descriptive version of the Hajian-Kakutani Theorem on the existence of a finite equivalent invariant measure (explained below) is false. This answers a question raised in [10].

Let m be a probability measure on $\mathcal{B}$ and assume that null sets of m are preserved by T. Call a set $W \in \mathcal{B}$ weakly wandering if there exists a sequence of integers (n_i), $i = 1, 2, 3, ...$, such that $T^{n_i} W \cap T^{n_j} W = \emptyset$ whenever $i \neq j$. It is easy to see that if there is a weakly wandering set of positive m- measure then there is no probability measure on $\mathcal{B}$ invariant under T and having same null sets as m. A well known theorem of Hajian and Kakutani (see [4],[3]) states that the converse of this observation holds,i.e., if there is no weakly wandering set of positive m measure then there exists a probability measure on $\mathcal{B}$ invariant under T and having the same null sets as m. This result was further improved by Jones and Krengel [7] who showed that if there is no T-invariant probability measure on $\mathcal{B}$ absolutely continuous with respect to m then there exists a set W in $\mathcal{B}$ and a sequence of positive integers (n_i), $i = 1, 2, 3, ...$ such that $T^{n_i} W \cap T^{n_j} W = \emptyset$ whenever $i \neq j$ and $\bigcup_{i=1}^{\infty} T^{n_i} W = X$ (mod m). (The qualification mod m cannot in general be removed as the result of this paper will show.However, as pointed out to us by U. Krengel, if we take the completion $\mathcal{M}$ of the σ-algebra $\mathcal{B}$ with respect to m then the requirement mod m can be dispensed with. See remark end of section 2.)

Let us dispense with the measure m and consider only the standard Borel space $(X, \mathcal{B})$ and the Borel automorphism T on X. Say that a set $A \in \mathcal{B}$ is of full saturation if $\bigcup_{n=-\infty}^{\infty} T^n A = X$. We note that if T admits a weakly wandering set of full saturation then T does not admit a T-invariant probability measure. It is natural to ask, in analogy with the Hajian-Kakutani theorem mentioned above, whether the non-existence of a weakly wandering set of full saturation implies the existence of a T-invariant probability measure on $\mathcal{B}$. As stated in the beginning, the answer to this question is in the negative.

2. THE MAIN CONSTRUCTION

Definition. Given a non-singular system $(X, \mathcal{B}, m, T)$ a sequence of natural numbers $n_1, n_2, n_3, \cdots$ is said to be weakly wandering for T if there exists a set W in $\mathcal{B}$ of positive m measure such that $T^{n_k} W, k = 1, 2, 3, ...$ are pairwise disjoint.

Definition. A sequence $r_1, r_2, r_3, \cdots$ of natural numbers is said to be recurrent for T if its intersection with every weakly wandering sequence for T is finite.

Clearly a recurrent sequence for T can not be a weakly wandering sequence for T and vice-versa. The notion of weakly wandering, recurrent and other related sequences were investigated by Hajian and Ito [5].

Theorem 1. Let T be an ergodic measure preserving transformation on a σ-finite measure space $(X, \mathcal{B}, m)$. A sequence $r_1, r_2, r_3, ...$ of natural numbers is recurrent for T if and only if there is a set A of positive finite measure such that $\underline{\lim}(T^{r_n} A \cap A) > 0$.

For a proof of the above theorem we need the following two lemmas. We state and prove them for the sake of completeness; see [5].

Lemma 1: Let $\mathbf{W} = \{w_i\}$ be a weakly wandering sequence for the ergodic measure preserving transformation T. Then $\lim_{i \to \infty} m(T^{w_i} A \cap A) = 0$ for every set $A \in \mathcal{B}$ with $m(A) < \infty$.

Proof: Let A be a set with $m(A) < \infty$, and let $W \in \mathcal{B}$ be a weakly wandering set of positive measure under the sequence $\mathbf{W} = \{w_i\}$ for T. For any integer

$k > 0$ it is clear that $T^k W$ is again weakly wandering under both $\{w_i\}$ and $\{-w_i\}$. Since

$$m(A) \geq m(\bigcup_{i=1}^{\infty} T^{-w_i}(T^k W) \cap A) = \sum_{i=1}^{\infty} m(T^{-w_i}(T^k W) \cap A) = \sum_{i=1}^{\infty} m(T^k W \cap T^{w_i} A),$$

it follows that

$$\lim_{i \to \infty} m(T^{w_i} A \cap T^k W) = 0 \text{ for any integer } k > 0.$$

Since $m(A) < \infty$ and T is ergodic it follows that for any $\varepsilon > 0$ there exists an integer $N > 0$ and a set C with $m(C) < \varepsilon$ such that

$$A \subset C \cup \bigcup_{j=0}^{N} T^j W.$$

Then

$$m(T^{w_i} A \cap A) \leq m(T^{w_i} A \cap C) + \sum_{k=0}^{N} m(T^{w_i} A \cap T^k W),$$

and this implies

$$\overline{\lim}_{i \to \infty} m(T^{w_i} A \cap A) < \varepsilon.$$

Lemma 2: Let $\{n_i\}$ be a sequence of integers such that $\lim_{i \to \infty} m(T^{n_i} A \cap A) = 0$ for every set $A \in \mathcal{B}$ with $m(A) < \infty$. Then $\{n_i\}$ contains a weakly wandering subsequence.

Proof: Let C and D be two sets of finite measure. Since $\lim_{i \to \infty} m(T^{n_i} A \cap A) = 0$ for every set of finite measure and $m[T^{n_i}(C \cup D) \cap (C \cup D)] \geq m(T^{n_i} C \cap D)$ it follows that $\lim_{i \to \infty} m(T^{n_i} C \cap D) = 0$ for any two sets C and D of finite measure.

Let C be a set with $0 < m(C) < \infty$, and let $\varepsilon > 0$ be such that $\varepsilon < m(C)$. We show that C contains a subset $W \subset C$, with $m(C - W) < \varepsilon$, and such that W is weakly wandering under a subsequence $\{w_i\}$ of the sequence $\{n_i\}$.

Put $w_0 = 0$ and choose $\varepsilon_i > 0$ for $i = 1, 2, \cdots$ such that $\sum \varepsilon_i = \varepsilon$. Since $\lim_{i \to \infty} m(T^{n_i} C \cap C) = 0$ we choose $w_1 \in \{n_i\}$ such that $m(T^{w_1} C \cap C) < \varepsilon_1$. We let $D = C \cup T^{-w_1} C$, and since $\lim_{i \to \infty} m(T^{n_i} D \cap C) = 0$ we choose

$w_2 \in \{n_i\}$ such that $m[(T^{w_2}C \cup T^{w_2-w_1}C] \cap C) < \varepsilon_2$. We proceed inductively; having chosen the integers $w_0, w_1, \cdots, w_k$ satisfying

$$m[(\bigcup_{j=0}^{i-1} T^{w_i - w_j}C) \cap C] < \varepsilon_i, \qquad 1 \leq i \leq k$$

We put $D = \bigcup_{j=0}^{k} T^{-w_j}C$ and choose $w_{k+1} \in \{n_i\}$ such that $m(T^{w_{k+1}}D \cap C) < \varepsilon_{k+1}$ or

$$m[(\bigcup_{j=0}^{k} T^{w_{k+1} - w_j}C) \cap C] < \varepsilon_{k+1}.$$

Next we let

$$C' = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} T^{w_i - w_j}C \cap C.$$

It follows that the set $W = C - C'$ has positive measure, $m(C - W) < \varepsilon$ and since $W \subset C$ and $T^{w_i - w_j}W \subset T^{w_i - w_j}C \subseteq C' \cup (X - C)$ for $i > j$. We conclude $T^{w_i}W \cap T^{w_j}W = \emptyset$ for $i \neq j$. Thus $\{w_i\}$ is a subsequence of $\{n_i\}$ which is weakly wandering sequence for T. Q.E.D.

Proof of Theorem 1: Since a subset of a weakly wandering sequence is again weakly wandering, the sufficiency of the condition follows from Lemma 1; the necessity follows from Lemma 2. Q.E.D.

Let $I = [0, 1)$ and let V denote the von Neumann transformation (also known as the adding machine) acting on I. If $x = x_1x_2x_3...$ is the binary expansion of $x \in I$ then $V(x) = 000...01x_{k+2}x_{k+3}...$ where $k+1$ is the first integer i for which x_i is zero. V is unambiguously defined except for countably many points which admit two distinct binary expansions. V preserves Lebesgue measure on $[0, 1)$ and is uniquely ergodic.

For each sequence $\alpha = (k_1, k_2, k_3, ...)$ of non-negative integers construct a tower over V with ceiling

$$f_\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ k_1 & \text{if } 1/2 \leq x \leq 1/2 + 1/4 \\ k_1 + k_2 & \text{if } 1/2 + 1/4 \leq x < 1/2 + 1/4 + 1/8 \\ & \quad . \\ & \quad . \\ & \quad . \end{cases}$$

In other words let

$$X_\alpha = \{(x, n) : x \in [0, 1),\ 0 \le n < f_\alpha(x)\}$$

equipped with the natural Borel structure $\mathcal{B}_\alpha$ as a subset of $[0, 1)\times\{0, 1, 2, 3, \cdots\}$. Let m_α be the measure on $\mathcal{B}_\alpha$ which agrees with m on $[0, 1)\times\{0\}$ and which is invariant under T_α defined on X_α by:

$$T_\alpha(x, n) = \begin{cases} (x, n + 1) & \text{if } 0 \le n < f_\alpha(x) \\ (Vx, 0) & \text{if } n = f_\alpha(x) \end{cases}$$

The map T_α can also be viewed as a rank one transformation where at stage n of the construction the stack obtained at stage $(n-1)$ is divided into two equal columns and we add $k_1+k_2+k_3+\cdots+k_n$ spacers on the left hand column. Observe that if J is one of the intervals in the $(n)^{th}$ stack of height h_n and if J is divided into two equal intervals J_1 and J_2 , J_1 to the left of J_2, then $T_\alpha^{r_n} J_1 = J_2$ where $r_n = h_n + k_1 + ...k_{n+1}$. This at once shows that $m(T_\alpha^{r_n} I \cap I) \ge 1/2$. (Observe that I is partitioned into disjoint intervals in the n^{th} stack . The left half of each of these intervals coincides with the right half on application of $T_\alpha^{r_n}$.) By a slight refinement of this argument, if one chooses the sequence $\{k_n\}$ carefully, it is possible to construct an ergodic measure preserving transformation T which is of type 1/2. By a transformation of type α, $0 \le \alpha \le 1$, we mean an ergodic measure preserving transformation T with the property that $limsup_{n\to\infty} m(T^n A \cap A) = \alpha m(A)$ for every set $A \in \mathcal{B}$ with $m(A) < \infty$. We do not need this additional refinement.

Theorem 2. Given an increasing sequence of natural numbers $\{n_i\}$ there exists an increasing sequence $\alpha = \{k_n\}$ of natural numbers such that for the constructed space $m(X_\alpha) = \infty$ and further there exists a subsequence $\{r_i\}$ of $\{n_i\}$ which is recurrent for T_α

Proof: Let us suppose that an increasing sequence of positive integers $\{n_i\}$ is given. We choose inductively a subsequence $\{r_i\}$ of $\{n_i\}$ and construct a sequence $\{k_i\}$ of non-negative integers as follows. We choose $r_1 \in \{n_i\}$ such that $r_1 > 0$ and let $k_1 = r_1 - 1$. Next we choose $r_2 \in \{n_i\}$ such that $r_2 > 4r_1$ and let $k_2 = r_2 - r_1 - 1$. We continue this way; having chosen the integers $r_1, r_2, \cdots, r_{i-1}$ and constructed the integers $k_1, k_2, \cdots, k_{i-1}$, we choose $r_i \in \{n_j\}$ such that $r_i > 4(r_1 + r_2 + \cdots + r_{i-1})$ and let $k_i = r_i - (r_1 + r_2 + \cdots + r_{i-1}) - 1$. We note that the sequence of integers $\{k_i\}$ thus constructed satisfies $\sum k_n/2^n = \infty$. Thus, if we build the transformation T_α with $\alpha = \{k_n\}$ just chosen, then it follows from the above construction that $m(X_\alpha) = \infty$ and $m(T_\alpha^{r_i} I \cup I) \geq 1/2$ for all i. By Theorem 1 $\{r_i\}$ is a recurrent sequence for T_α Q.E.D.

We now give an example of a Borel automorphism T having properties (i) and (ii) stated in the beginning.

Let R denote all sequences of natural numbers equipped with its product Borel structure. The set α of points in R for which m_α is finite is given by $F = \{(k_1, k_2, k_3, ...) : \sum k_i 2^{-i} < \infty\}$ which is clearly a Borel subset of R. Let $E = R - F$. Let $\Omega = E \times [0, 1) \times \{0, 1, 2, 3, ...\}$ equipped with the product σ - algebra and set

$$X = \{(\alpha, x, n) \in \Omega : n \leq f_\alpha(x)\}.$$

Then X is a Borel subset of Ω since $(\alpha, x) \to f_\alpha(x)$ is jointly measurable on $E \times [0, 1)$. Define T on X by requiring it to be T_α on the α section of X

$$T(\alpha, x, n) = (\alpha, T_\alpha(x, n)).$$

T is Borel measurable and leaves each α section invariant.

Clearly T does not admit a weakly wandering set of full saturation since for any sequence of natural numbers there exists a T_α which does not admit the chosen sequence as a weakly wandering sequence. Also T does not admit a T-invariant probability measure, for if it did then the regular conditional probabilities (see K.R.Parthasarthy [11]) of this measure with respect to the measurable partition given by the α sections of X would be invariant with respect to T_α for almost every α.But no T_α admits an invariant probability measure on $\mathcal{B}_\alpha$ since α is in E and V is uniquely ergodic. This contradiction shows that T does not admit an invariant probability measure .

Remark. As mentioned in the introduction if we replace $\mathcal{B}$ by $\mathcal{M}$ the completion of $\mathcal{B}$ with respect to a finite or σ-finite measure μ then it is possible to find a set $W \in \mathcal{M}$ and an increasing sequence of positive integers $n_1, n_2, ...$ such that

$\bigcup_{i=0}^{\infty} T^{n_i}(W) = X$.We see this as follows.Since there is no invariant probability measure for T, by the theorem of Jones and Krengel mentioned in the introduction we can find a Borel set A and a sequence of positive integers $n_1 < n_2 < ...$ such that the union of iterates of A under this sequence is an invariant set Y with $\mu(X-Y)=0$. Moreover the sequence $n_1 < n_2 < ...$ can be so chosen that for a suitable subset E of integers the union

$$\bigcup_{i=0}^{\infty}(E+n_i)$$

is the set of all integers. Since we are in a completed σ-algebra every subset of $X-Y$ is in $\mathcal{M}$. We can find set B, if necessary by use of axiom of choice, such that every orbit of a point in $X-Y$ intersects B in exactly one point. Let

$C=\bigcup_{i\in E}T^i(B)$. The union of A and C is the required set W.

3. HOPF'S THEOREM AND RELATED PERSPECTIVE

An old expository paper of G.D. Birkhoff and P.A. Smith [2] sets forth nearly all the basic notions of classical ergodic theory (albeit in a slightly weak form) : wandering sets, decomposition into conservative and dissipative parts, weakly wandering sets and compressibility, ergodicity (under the name of metric transitivity), equivalence and singularity of ergodic measures etc. The question of existence of a finite invariant measure was discussed by Birkhoff and Smith in section 4 of their paper. They discussed the question for a continuous invertible mapping of a surface. The two notions, namely of weakly wandering sets and compressibility introduced there have remained important in subsequent discussions of this problem in the measure theoretic setting. Birkhoff and Smith begin by observing that even if a continuous transformation does not admit a non-empty wandering open set W, it may yet admit a non-empty open W whose iterates under T over a subsequence of integers are pairwise disjoint. This forbids the existence of a finite T-invariant measure which gives positive mass to every open set. No concrete example of such a situation is exhibited however. Instead they introduce their notion of compressibility. From this they are able to formulate a necessary and sufficient condition for the existence of a finite invariant measure. Their definition of compressibility was constrained by the requirement that partitions of the space into only finitely many sets were allowed. On the suggestion of Birkhoff, E. Hopf considered the question of existence of a finite invariant measure for a non-singular transformation [6]. He modified their definition of compressibility by allowing countable partitions of the space and by analyzing the problem in a measure theoretic setting. We report his work below.

Let $(X, \mathcal{B})$ be a standard Borel space and $T : X \leftrightarrow X$ a Borel automorphism. Two sets A, B in $\mathcal{B}$ are said to be equivalent by countable decomposition, $A \sim B$, if we can write (i) A as a countable union of pairwise disjoint sets A_i in $\mathcal{B}$, $i = 1, 2, 3, \cdots$, (ii) B as a countable union of pairwise disjoint sets $B_i \in \mathcal{B}$, $i = 1, 2, 3, \cdots$ and (iii) there exist integers n_i such that $T^{n_i} A_i = B_i$, $i = 1, 2, 3, \cdots$. This notion of equivalence by countable decomposition is basic to the theory of orbit equivalence. (Equivalence by countable decomposition of two sets essentially means that the induced transformations on the two sets are orbit equivalent.) Suppose that m is a finite measure on $\mathcal{B}$ whose null sets are preserved by T. We say that a set A in $\mathcal{B}$ is compressible in the sense of Hopf if there exists a set $B \in \mathcal{B}$, $B \subset A$ such that $m(A - B) > 0$ and A and B are equivalent by countable decomposition. It is easy to see that if there is a finite T invariant measure μ having the same null sets as m, then X is not compressible in the sense of Hopf. Hopf proved the non-trivial converse of this. He showed that if X is not compressible in his sense then there is a finite T invariant measure μ on $\mathcal{B}$ whose null sets agree with those of m.

Hopf's proof of his theorem was difficult. A simpler proof of his result and also simpler necessary and sufficient conditions for the existence of a finite equivalent invariant measure were therefore sought. The best known result in this connection is the theorem of Hajian and Kakutani mentioned above. Since there exists ergodic measure preserving transformations on a non-atomic infinite σ-finite measure space, such transformations always admit weakly wandering sets of positive measure. Thus Hajian and Kakutani proved not only the non-trivial converse of the observation due to Birkhoff and Smith but also that weakly wandering sets of positive measure exist in abundance in suitable non-dissipative systems.

There is a simple connection between weakly wandering sets and compressibility in the sense of Hopf. Suppose W in $\mathcal{B}$ is weakly wandering under T and of positive measure, $T^{n_k} W$, $k = 1, 2, 3, \cdots$ being pairwise disjoint. Then we can compress X into $X - W$ simply by mapping $W \rightarrow T^{n_1} W$, $T^{n_1} W \rightarrow T^{n_2} W \ \cdots$, $T^{n_k} W \rightarrow T^{n_{k+1}} W$, $\cdots$ and letting the identity map act on $X - \bigcup T^{n_k} W$. Thus we see that X is compressible in the sense of Hopf and a simple compression can be effected by means of iterates of a weakly wandering set of positive measure. Therefore, if X is not compressible in the sense of Hopf then T can not admit a weakly wandering set of positive measure and so by the Hajian-Kakutani Theorem admits a finite invariant measure having the same null sets as m. Hopf's Theorem follows.

G.W.Mackey has emphasized the importance of considering group actions on standard Borel spaces free of any measure on the space.(see [8],[9]). There has been renewed interest in this aspect of classical ergodic theory in recent years (see H.Baker and A.Kechris[1] and references therein). In this spirit, Hopf's theorem has the following descriptive formulation proved in Nadkarni[10]: Given a standard Borel space $(X, \mathcal{B})$ and a Borel automorphism T on X call X compressible if we can write X as a disjoint union of two sets C and D in $\mathcal{B}$ such that (a) C and D have the same saturation and (b) X is equivalent by countable decomposition to C. We note that if X is compressible then it is compressible in the sense of Hopf with respect to any probability measure quasi-invariant under T. Also it is easy to see that if X is compressible then there is no T-invariant probability measure on $\mathcal{B}$. The converse of this holds: if X is not compressible then T admits an invariant probability measure on $\mathcal{B}$. This is the descriptive version of Hopf's theorem proved in [10].

It is clear that if X admits a weakly wandering set in $\mathcal{B}$ of full saturation then X is compressible. The example of this paper shows that converse of this is not true: The T constructed above does not admit an invariant probability measure and so X is compressible under T, but T does not admit a weakly wandering set of full saturation.

Our analysis raises the following two problems of some interest: (i) When does a Borel automorphism T on a standard Borel space admit a weakly wandering set of full saturation? (ii) If X is descriptively compressible under T is there a countably generated partition of X into invariant sets on each of which T admits a weakly wandering set of full saturation?

Remark. The authors would like to take this opportunity to express their appreciation of the long forgotten paper of Birkhoff and Smith [2] which atonce sets forth, albeit in slightly weak form, nearly all the basic notions of classical ergodic theory: wandering sets, decompotition into conservative and dissipative parts,weakly wandering sets and compresibility, ergodicity (metric transitivity), equivalence or singularity of two ergodic measures, etc.

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