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On the existence of a finite invariant measure

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Abstract. A necessary and sufficient condition is given for a Borel automorphism on a standard Borel space to admit an invariant probability measure.

Keywords. Borel space; Borel automorphism; compressible sets; invariant measures.

1. Introduction

The purpose of this paper is to seek conditions under which a Borel automorphism T on a Borel space $(X,\mathscr{B})$ admits a non-trivial finite T-invariant measure. For simplicity we will assume that $\mathscr{B}$ contains singleton sets. It is easy to see that if T admits periodic points, such a measure always exists. Hence we assume that $T^n x \neq x$ for all $n \in Z - \{0\}$. Going back to classical ergodic theory, especially the work of Hopf [5] on the existence of a finite invariant measure equivalent to a given quasi-invariant measure, we are able to define the notion of incompressible set intrinsically (i.e., without reference to any measure) as follows:

An invariant set $A \in \mathscr{B}$ is said to be compressible if we can write A as a disjoint union of two sets C and D in $\mathscr{B}$ such that

$$\bigcup_{n=-\infty}^{\infty} T^n C = \bigcup_{n=-\infty}^{\infty} T^n D = A$$

and A is equivalent by countable decomposition to either C or D. An invariant set in $\mathscr{B}$ is said to be incompressible if it is not compressible.

If there is a finite T-invariant probability measure on $\mathscr{B}$ then X is incompressible. In the converse direction we are able to prove, under a mild condition which is satisfied whenever $\mathscr{B}$ is countably generated, that if X is incompressible then there exists a function

$$m\colon \mathscr{B} x X \to [0,1] \text{ such that}$$

(i) $m(A,.)$ is measurable for all $A, m(X,x)=1$
(ii) $m(A,x)=m(TA,x)=m(A,\ Tx)(\bmod \mathscr{H})$
(iii) $m(A,x)\equiv 0(\bmod \mathscr{H})$ if and only if $A\in\mathscr{H}$
(iv) $m(\cup_{i=1}^{\infty} A_i,x)=\Sigma_{i=1}^{\infty} m(A_i,x)(\bmod \mathscr{H})$ whenever $A_1,A_2,A_3,\ldots$ are pairwise disjoint sets in $\mathscr{B}$.
(v) If $E\in\mathscr{B}$ is an invariant set (i.e., $TE=E$) then for $x\in E, m(A\cap E,x)=m(A,x)$ $(\bmod \mathscr{H})$.

Here $\mathscr{H}$ is the σ-ideal of all sets $A \in \mathscr{B}$ such that $\cup_{n=-\infty}^{\infty} T^n A$ is compressible. $\mathscr{H}$ is called the Hopf ideal.

The classical theorem of Hopf can be recovered by integrating the function m with respect to the quasi-invariant measure in the statement of Hopf's theorem ([3], [5]). We mention, however, that the purpose here is not to give yet another proof of Hopf's theorem, but rather to isolate the descriptive content in Hopf's definition of incompressibility. When $(X, \mathscr{B})$ is a standard Borel space the function m can be so chosen that (i) – (iv) hold for every x (i.e., not modulo $\mathscr{H}$) and the measures $(m(.,x)$ yield all the probability measures on $\mathscr{B}$ invariant and ergodic under T. The method of proof, which we give in sufficient detail, relies on elementary ideas of ergodic theory [3], [4] and the method of Murray and von-Neumann [6] wherein they obtain a measure on the class of projections in a factor. This paper may be viewed as a contribution to the theme of descriptive ergodic theory [1], [2], [7]–[12]. The paper considers the problem of the existence of a finite invariant measure formulated in [8]. The result here differs from the one in [7] in that the σ-ideal with respect to which incompressibility is defined is described intrinsically and no assumption is made about the ergodicity of T with respect to the σ-ideal.

2. Preliminaries and the main theorem

2.1 Let $(X, \mathscr{B})$ be a Borel space, i.e., a non-empty set X together with a σ-algebra $\mathscr{B}$ of subsets of X. Let $T: X \to X$ be a one–one onto map such that $T\mathscr{B} = T^{-1}\mathscr{B} = \mathscr{B}$. Such a map is called a Borel automorphism of $(X, \mathscr{B})$. We will assume that T has no periodic points. For any $A \subseteq X$ we write $sA = \cup_{n=-\infty}^{\infty} T^n A$ and call sA the saturation of A. It is the smallest T-invariant subset of X containing A. We write $s^+ A = \cup_{n=0}^{\infty} T^n A$.

2.2 A subset $A \subseteq X$ is said to be wandering if $T^n A \cap T^m A = \emptyset$ wherever $m \neq n$. The σ-ideal of $\mathscr{B}$ generated by all wandering sets in $\mathscr{B}$ is denoted by $\mathscr{W}$ and called the Shelah-Weiss ideal of T (see [8], [9], [11]).

2.3 Two sets $A, B \in \mathscr{B}$ are said to be equivalent by countable decomposition or descriptively isomorphic if (i) we can partition A into a countable number of pairwise disjoint sets $A_i \in \mathscr{B}$, $i \in \mathbb{N}$ (ii) we can partition B into a countable number of pairwise disjoint sets $B_i \in \mathscr{B}$, $i \in \mathbb{N}$ (iii) we can find integers n_i, $i \in \mathbb{N}$ such that for each i, $T^{n_i} A_i = B_i$. Here and in the sequel $\mathbb{N}$ denotes the set of natural numbers. If A and B are equivalent by countable decomposition then the map from A to B which is T^{n_i} on A_i, $i = 1, 2, 3, \ldots$ is called a descriptive isomorphism between A and B. It can be shown that equivalence by countable decomposition is indeed an equivalence relation.

2.4 A set $A \in \mathscr{B}$ is said to be compressible if we can write A as a disjoint union of two sets $C, D \in \mathscr{B}$ such that $sA = sC = sD$ and A is equivalent to C by countable decomposition. A is said to be incompressible if A is not compressible. If X is compressible then T is called compressing Borel automorphism.

2.5 Orbit of a point is compressible and so is the saturation of any wandering set in $\mathscr{B}$. It is not true, however, that every compressible invariant set in $\mathscr{B}$ is the saturation of a wandering set in $\mathscr{B}$ except in special cases.

2.6 A finite non-empty set is not compressible nor is a set A compressible if the orbit of a point intersects A in a finite non-empty set. If there is a probability measure on $\mathscr{B}$ which is invariant under T, then no set of positive measure is compressible. In particular, in such a situation, X is incompressible. The purpose of this paper is to investigate the extent to which the incompressibility of X implies the existence of a finite T-invariant probability measure on $\mathscr{B}$. At least when $(X, \mathscr{B})$ is a standard Borel space we will show that incompressibility of X implies the existence of a T-invariant probability measure on $\mathscr{B}$.

2.7 Since finite non-empty sets are incompressible, it is clear that a subset of compressible set need not be compressible.

2.8 A set $E \in \mathscr{B}$ is said to the T-invariant or simply invariant if $TE = E$. If $E \in \mathscr{B}$ is invariant and compressible and $F \in \mathscr{B}$ is an invariant subset of E, then F is compressible since a compression of E when restricted to F yields a compression of F. Countable pairwise disjoint union of compressible invariant sets in $\mathscr{B}$ is again compressible. Finally any countable union of compressible invariant sets in $\mathscr{B}$ is compressible since such a union can be expressed as a countable pairwise disjoint union of compressible invariant sets in $\mathscr{B}$.

2.9 Compressible sets in $\mathscr{B}$ do not form a σ-ideal but compressible invariant sets in $\mathscr{B}$ are closed under countable union and taking of invariant subsets in $\mathscr{B}$. Consequently the collection $\mathscr{H}$ of sets in $\mathscr{B}$ whose saturations are compressible form a σ-ideal in $\mathscr{B}$ which we call the Hopf ideal. We note that $\mathscr{W} \subseteq \mathscr{H}$ since the saturation of any set in $\mathscr{W}$ is the union of all the iterates under T of a suitable wandering set in $\mathscr{B}$.

2.10 A set A in $\mathscr{B}$ is said to be decomposable if we can write A as a disjiont union of two sets C and D in $\mathscr{B}$ such that $sA = sC = sD$.

2.11 Not every set in $\mathscr{B}$ is decomposable. For example a singleton set in $\mathscr{B}$ is not decomposable nor can a set in $\mathscr{B}$ be decomposable if it intersects an orbit in exactly one point. In the sequal we will need decomposability $(\text{mod}\, \mathscr{W})$. A set $A \in \mathscr{B}$ is said to be decomposable $(\text{mod}\, \mathscr{W})$ if there is a set $W \in \mathscr{W}$ such that $A \Delta W$ is decomposable. It can be shown that when $\mathscr{B}$ is countably generated every non-empty set in $\mathscr{B}$ is decomposable $(\text{mod}\, \mathscr{W})$ see [8].

2.12 Let $\mathscr{N} \subseteq \mathscr{B}$ be a σ-ideal such that (i) $T\mathscr{N} = T^{-1}\mathscr{N} = \mathscr{N}$ (ii) $\mathscr{W} \subseteq \mathscr{N}$. We call such a system $(X, \mathscr{B}, \mathscr{N}, T)$ a descriptive dynamical system. The quadruplets $(X, \mathscr{B}, \mathscr{W}, T)$, $(X, \mathscr{B}, \mathscr{H}, T)$ are examples of such systems. If μ is a σ-finite measure on $\mathscr{B}$ whose null sets in $\mathscr{B}$ are preserved under T and T is conservative, then $(X, \mathscr{B}, \mathscr{N}_\mu, T)$ is such a system where $\mathscr{N}_\mu = \mu$ null sets in $\mathscr{B}$. If two sets $A, B \in \mathscr{B}$ are such that $A \Delta B \in \mathscr{N}$ then we say that A and B are equal $(\text{mod}\, \mathscr{N})$ and write $A = B$ $(\text{mod}\, \mathscr{N})$. It is well known and easy to prove that $(\text{mod}\, \mathscr{W})$ the sets sA and s_+A are equal and more generally the sets sA and $T^n s_+ A$ are equal $(\text{mod}\, \mathscr{W})$ (see [8] 5·4). Since $\mathscr{W} \subseteq \mathscr{N}$ we also have $sA = T^n s_+ A$ $(\text{mod}\, \mathscr{N})$ for all $A \in \mathscr{B}$ and $n \in \mathbb{Z}$.

2.13 Two sets $A, B \in \mathscr{B}$ are said to be equivalent by countable decomposition $(\text{mod}\, \mathscr{N})$ or descriptively isomorphic $(\text{mod}\, \mathscr{N})$ if we can find sets $N, M \in \mathscr{N}$ such that $A - N$ and

$B-M$ are equivalent by countable decomposition, and, we then write $A \sim B \pmod{n}$. We note that if $A \sim B \pmod{n}$ then $sA = sB \pmod{n}$. We say that A and B are strictly equivalent by countable decomposition if $A \sim B \pmod{\mathscr{W}}$. A set $A \in \mathscr{B}$ is said to be compressible (mod n) if there is a set $N \in n$ such that $A-N$ is compressible. In case A is compressible (mod $\mathscr{W}$) we say that A is strictly compressible. For a T-invariant set in $\mathscr{B}$ compressibility (mod $\mathscr{W}$), compressibility (mod $\mathscr{H}$) and compressibility are all equivalent. A set $A \in \mathscr{B}$ is said to be decomposable (mod n) if there exists $N \in n$ such that $A-N$ is decomposable. If A is decomposable (mod $\mathscr{W}$) then A is decomposable (mod n) since $\mathscr{W} \subseteq n$. If $\mathscr{B}$ is countably generated then every non-empty set in $\mathscr{B}$ is decomposable (mod $\mathscr{W}$) hence also decomposable (mod n).

The main result of this paper is the following:

2.14 Theorem. *If X is incompressible and every set in $\mathscr{B}$ is decomposable (mod $\mathscr{W}$) then there exists a function m on $\mathscr{B} \times X$ such that*

(i) $m(A,x) \geqslant 0$, $m(X,x) = 1$ *and* $m(A,x)$ *is measurable in* x

(ii) $m(A,x) = m(T^{-1}A, x) = m(A, T^{-1}x) \pmod{\mathscr{H}}$

(iii) $m(A,x) \equiv 0 \pmod{\mathscr{H}} \Leftrightarrow A \in \mathscr{H}$

(iv) *whenever* $A_1, A_2, A_3, \ldots$ *are pairwise disjoint set in* $\mathscr{B}$

$$m\left(\bigcup_{i=1}^{\infty} A_i, x\right) = \sum_{i=1}^{\infty} m(A_i, x) \pmod{\mathscr{H}}.$$

(v) *If $E \in \mathscr{B}$ is an invariant set then for all $A \in \mathscr{B}$ and $x \in E$*

$$m(A \cap E, x) = m(A, x) \pmod{\mathscr{H}}.$$

A proof of this theorem will emerge as a consequence of a series of discussions which follow.

2.15 We say that B is a copy of $A \pmod{n}$ if $A \sim B \pmod{n}$. If inside A there are two or more pairwise disjoint copies of $A \pmod{n}$ then A is compressible (mod n). If X accommodates infinitely many pairwise disjoint copies of $A \pmod{n}$ then sA is compressible (mod n). We write $A \preccurlyeq B \pmod{n}$ if there is a $C \subseteq B$ such that $A \sim C \pmod{n}$. If in addition $s(B-C) = sB \pmod{n}$, then we write $A \prec B \pmod{n}$. If $A \prec A \pmod{n}$ then $sA \prec A \pmod{n}$ and $sA \sim A \pmod{n}$. A proof of this can be found in ([8], 5·7).

2.16 If A and B are sets in $\mathscr{B}$ for which one of the relations $A \sim B$, $A \preccurlyeq B$, $A \prec B$ holds then the relation also holds for $A \cap E$ and $B \cap E$ for any T-invariant set E in $\mathscr{B}$.

2.17 *Notation.* Suppose $A, B \in \mathscr{B}$ and B admits n or more pairwise disjoint copies of $A \pmod{n}$ then we can write $B = B_1 \cup B_2 \cup \ldots \cup B_n \cup R$, where $B_1, B_2, \ldots, B_n, R$ are pairwise disjoint and $A \sim B_1 \sim \cdots \sim B_n$. We express this by writing $B = n \odot A \oplus R$, $R \subseteq B$. We also write $n \odot A \subseteq B$ to express the fact that B admits n or more pairwise disjoint copies of $A \pmod{n}$.

2.18 In the rest of this section we will deal only with the σ-ideal $\mathscr{H}$ and the relations $\sim, \prec, \preccurlyeq$ will be with respect to this σ-ideal. We will therefore drop the qualification (mod $\mathscr{H}$) after these symbols.

2.19 *Lemma. Given $A, B \in \mathscr{B}$ we can write*

$$A = E_1 \cup E_2 \cup E_3, \quad sE_i \cap sE_j = \varnothing \text{ if } i \neq j$$

$$B = F_1 \cup F_2 \cup F_3, \quad sF_i \cap sF_j = \varnothing \text{ if } i \neq j$$

such that modulo $\mathscr{H}$ we have

(i) $F_1 \prec E_1 \quad E_2 \sim F_2 \quad E_3 \prec F_2$
(ii) $sE_1 \cap B = F_1, sE_2 \cap B = F_2, sF_2 \cap A = E_2, sF_3 \cap A = E_3,$

Moreover such a decomposition is unique (mod $\mathscr{H}$).

Proof. We define sets $A(n)$, $B(n)$, A_n, B_n, $n = 0, 1, 2, 3, \ldots$ by the following inductive procedure

$$\begin{array}{ll}
A(0) = A & B(0) = B \\
A_0 = A(0) \cap B(0) & B_0 = B(0) \cap A(0) \\
A(1) = A(0) - A_0 & B(1) = B(0) - B_0 \\
A_1 = A(1) \cap TB(1) & B_1 = T^{-1}A_1 = T^{-1}A(1) \cap B(1) \\
\vdots & \vdots \\
A(n+1) = A(n) - A_n & B(n+1) = B(n) - B_n \\
A_{n+1} = A(n+1) \cap T^{n+1}B(n+1) & B_{n+1} = T^{-n-1}A(n+1) \cap B(n+1) \\
\vdots & \vdots
\end{array}$$

The sets $A(n)$ are decreasing and we write A_∞ for their intersection. Similarly the sets $B(n)$ are decreasing and we write B_∞ for their intersection. From our construction we have

$$A = \bigcup_{i=1}^{\infty} A_i \cup A_\infty \quad A_i \cap A_j = \varnothing \text{ if } i \neq j$$

$$B = \bigcup_{i=1}^{\infty} B_i \cup B_\infty \quad B_i \cap B_j = \varnothing \text{ if } i \neq j$$

$$T^i B_i = A_i \quad \text{for } i < \infty$$

so that $\cup_{i=1}^{\infty} A_i \sim \cup_{i=1}^{\infty} B_i$. Our construction also shows that for $k \geqslant 1$, $T^k B_\infty$ is disjoint from A_∞ so that $\cup_{k=1}^{\infty} T^k B_\infty$ is disjoint from A_∞. In view of remarks in 2.12 $sA_\infty \cap sB_\infty \in \mathscr{W}$ hence also in $\mathscr{H}$ since $\mathscr{W} \subseteq \mathscr{H}$. We write

$$\begin{array}{ll}
E_1 = A \cap sA_\infty & F_1 = B \cap sA_\infty \\
E_3 = A \cap sB_\infty & F_3 = B \cap sB_\infty \\
E_2 = A - (E_1 \cup E_3) & F_2 = B - (F_1 \cup F_3).
\end{array}$$

Now

$$F_1 = B \cap sA_\infty \doteq \bigcup_{n=1}^{\infty} (B_i \cap sA_\infty) \cup (sA_\infty \cap B_\infty) = \bigcup_{n=1}^{\infty} (B_i \cap sA_\infty) (\text{mod } \mathscr{W})$$

$$E_1 = A \cap sA_\infty = A_\infty \cup \bigcup_{n=1}^{\infty} (A_i \cap sA_\infty).$$

Hence we see that $F_1 \prec E_1$ Similarly $E_3 \prec F_3$.

Finally

$$F_2 = B - (F_1 \cup F_3) = \bigcup_{n=1}^{\infty} B_i - s(A_\infty \cup B_\infty)$$

$$E_2 = A - (E_1 \cup E_3) = \bigcup_{n=1}^{\infty} A_i - s(A_\infty \cup B_\infty)$$

so that $E_2 \sim F_2$. This proves (i). Now (ii) follows from our construction. It remains to prove the uniqueness part. To this end suppose that $A = E_1' \cup E_2' \cup E_3'$, $B = F_1' \cup F_2' \cup F_3'$ is another decomposition of A and B as in the lemma. Then the sets $E_1 \cap E_2'$ and $F_1 \cap F_2'$ satisfy the relations $E_1 \cap E_2' \sim F_1 \cap F_2'$ and also $F_1 \cap F_2' \prec E_1 \cap E_2'$ which is a contradiction unless $E_1 \cap E_2'$, $F_1 \cap F_2'$ belong to $\mathscr{H}$. Indeed

$$F_1 \cap F_2' = sE_1 \cap B \cap F_2' = sE_1 \cap F_2' \sim sE_1 \cap E_2' = sE_1 \cap A \cap E_2' = E_1 \cap E_2'$$

$$F_1 \cap F_2' = F_1 \cap B \cap sE_2' = F_1 \cap sE_2' \prec E_1 \cap sE_2' = E_1 \cap A \cap sE_2' = E_1 \cap E_2'.$$

Similarly we can show that if $i \neq j$, $E_i \cap E_j'$, $F_i \cap F_j'$ belong to $\mathscr{H}$. Thus the decomposition of A and B satisfying the requirements of the lemma is unique (mod $\mathscr{H}$). (q.e.d.)

2.20 COROLLARY

Given $A, B \in \mathscr{B}$ we can write

$$A = E \cup F \quad sE \cap sF = \varnothing$$

$$B = G \cup H \quad sG \cap sH = \varnothing$$

such that (mod $\mathscr{H}$) (i) $G \prec E$ $F \preccurlyeq H$, (ii) $sE \cap B = G$, $sH \cap A = F$. *Moreover such a decomposition is unique (mod $\mathscr{H}$).*

Proof. We set $E = E_1$, $F = E_2 \cup E_3$, $G = F_1$, $H = F_2 \cup F_3$. Then G and H satisfy (i) and (ii). The uniqueness follows as before. (q.e.d.)

2.21 COROLLARY

If $A \in \mathscr{B} - \mathscr{H}$ is decomposable then a decomposition $A = C \cup D$, $C \cap D = \varnothing$, $sC = sD = sA$ (mod $\mathscr{H}$) can be so chosen that $C \preccurlyeq D$.

2.22 A repeated application of corollary 2.20 gives us the following important lemma.

2.23 *Lemma. Given $P, Q \in \mathscr{B}$, $P \preccurlyeq Q$ we can decompose P and Q as follows*

(i) $$P = \bigcup_{i=1}^{\infty} P_i \cup P_\infty, sP_i \cap sP_j = \varnothing \quad \textit{for } i \neq j$$

(ii) $$Q = \bigcup_{i=1}^{\infty} Q_i \cup Q_\infty, sQ_i \cap sQ_j = \varnothing \quad \textit{for } i \neq j$$

(iii) *for each* $i<\infty$, $sP_i = sQ_i$ *and* $Q_i = i\odot P_i \oplus R_i, R_i \prec P_i$

(iv) $P_\infty \in \mathscr{H}$.

Moreover such a decomposition of P *and* Q *is unique* (*mod* $\mathscr{H}$).

Proof. Since $P \preccurlyeq Q$ we can write $Q = 1\odot P \oplus R, R \subseteq Q$. We apply corollary 2.20 with $A=P$ and $B=R$ and let $P=E\cup F$, $R=G\cup H$ be the decompositions of P and R provided by the corollary. Put $P_1 = E = sE\cap P$, $Q_1 = sE\cap Q$ $P_1' = F = P - P_1$, $Q_1' = Q - Q_1$. We have

$$P = P_1 \cup P_1' \quad sP_1 \cap sP_1' = \varnothing$$

$$Q = Q_1 \cup Q_1' \quad sQ_1 \cap sQ_1' = \varnothing$$

$sP_1 = sQ_1$, $Q_1 = 1\odot P_1 \oplus R_1$, $R_1 \prec P_1$, $2\odot P_1' \subseteq Q_1'$. Suppose we have obtained $P_1,\dots,P_n$, P_n', $Q_1,\dots,Q_n$, Q_n' such that

$$\left.\begin{array}{l} P = \bigcup_{i=1}^{n} P_i \cup P_n', sP_i \cap sP_j = \varnothing, i\neq j, sP_i \cap sP_n' = \varnothing \text{ for all } i \\ Q = \bigcup_{i=1}^{n} Q_i \cup Q_n', sQ_i \cap sQ_j = \varnothing, i\neq j, sQ_i \cap sQ_n' = \varnothing \text{ for all } i \\ sQ_i = sP_i, Q_i = i\odot P_i \oplus R_i, R_i \prec P_i, (n+1)\odot P_n' \subseteq Q_n' \end{array}\right\} \quad *$$

Since $(n+1)\odot P_n' \subseteq Q_n'$ we can wirte

$$Q_n' = (n+1)\odot P_n \oplus R_n', R_n' \subseteq Q_n'.$$

We now apply corollary 2.20 with $A = P_n'$, $B = R_n'$. If $P_n' = E_n \cup F_n, R_n' = G_n \cup H_n$ be the decomposition of P_n', R_n' provided by the corollary, then we set

$$P_{n+1} = E_n = P_n' \cap sE_n \quad P_{n+1}' = P_n' - P_{n+1} = F_n$$

$$Q_{n+1} = Q_n' \cap sE_n \quad Q_{n+1}' = Q_n' - Q_{n+1}.$$

The sets $P_1, P_2, \dots, P_{n+1}, P_{n+1}', Q_1, Q_2, \dots, Q_{n+1}, Q_{n+1}'$ satisfy (*) with n replaced by $(n+1)$. We observe also that for each $n, P_{n+1}' \subseteq P_n', Q_{n+1}' \subseteq Q_n', (n+1)\odot P_n' \subseteq Q_n'$. If we set $P_\infty = \cap_{n=1}^{\infty} P_n'$ and $Q_\infty = \cap_{n=1}^{\infty} Q_n'$ then Q_∞ accommodates infinitely many pairwise disjoint copies of P_∞, whence $P_\infty \in \mathscr{H}$. We thus have

$$P = \bigcup_{n=1}^{\infty} P_i \cup P_\infty, sP_i \cap sP_j = \varnothing \quad \text{if } i \neq j$$

$$Q = \bigcup_{n=1}^{\infty} Q_i \cup Q_\infty, sQ_i \cap sQ_j = \varnothing \quad \text{if } i \neq j$$

$sP_i = sQ_i, Q_i = i\odot P_i \oplus R_i, R_i \prec P_i,$ for $i<\infty, P_\infty \in \mathscr{H}$ which is the decomposition required by the lemma. The uniqueness (mod $\mathscr{H}$) of the decomposition follows as before.

Remark. If $E\in\mathscr{B}$ is invariant and P,Q are as in the above lemma then the decomposition of $E\cap P, Q$ provided by the lemma will be

$$E\cap P=\bigcup_{i=1}^{\infty}(E\cap P_i)\cup(E\cap P_{\infty})$$

$$Q=\bigcup_{i=1}^{\infty}(E\cap Q_i)\cup\left(Q-\bigcup_{i=1}^{\infty}E\cap Q_i\right).$$

2.24 Let $A,B\in\mathscr{B}$ and let

$$A=E\cup P \quad sE\cap sP=\varnothing$$

$$B=G\cup Q \quad sG\cap sQ=\varnothing$$

$G\prec E$, $P\preccurlyeq Q$ be the decomposition of A and B provided by corollary 2.20. Let $(P_i)1\leqslant i\leqslant\infty$, $Q_i 1\leqslant i\leqslant\infty$ be the decompositions of the pair P,Q provided by lemma 2.23. If $A\in\mathscr{B}-\mathscr{H}$ then we define

$$\left[\frac{B}{A}\right](x)=\begin{cases} i & \text{if } x\in sP_i \quad i<\infty \\ 0 & \text{otherwise.}\end{cases}$$

In view of the uniqueness (mod $\mathscr{H}$) of the decompositions provided by corollary 2.20 and lemma 2.23 the function $[B/A]$ is unambiguously defined (mod $\mathscr{H}$). We further observe that

(i) $\quad C\sim B\Rightarrow\left[\frac{C}{A}\right]=\left[\frac{B}{A}\right]$ (mod $\mathscr{H}$)

(ii) $\quad C\sim A\Rightarrow\left[\frac{B}{C}\right]=\left[\frac{B}{A}\right]$ (mod $\mathscr{H}$)

(iii) $\quad$ if $A\preccurlyeq B$ then $\left[\frac{A}{C}\right]\leqslant\left[\frac{B}{C}\right]$ (mod $\mathscr{H}$) whereas if

$C\preccurlyeq D$ then $\left[\frac{A}{C}\right]\geqslant\left[\frac{A}{D}\right]$ (mod $\mathscr{H}$)

(iv) $\quad$ if E is T-invariant then for $x\in E$ $\left[\frac{B}{A}\right](x)=\left[\frac{B\cap E}{A}\right](x)$ (mod $\mathscr{H}$)

2.25 PROPOSITION

If E be the set of points x where $[B/A](x)<[C/A](x)$, then E is T-invariant and $E\cap B\preccurlyeq E\cap C$.

Proof. Clearly the set E is T-invariant. Let $i=[B/A](x)$ and $j=[C/A](x)$ and assume that $i<j$. Let I be the set of points x where $[C/A](x)=j$ and $[B/A](x)=i$. Then I is T-invariant and $B\cap I=i\odot(A\cap I)\oplus R$, $R\prec A\cap I$ whereas $C\cap I=j\odot(A\cap I)\oplus S$, $S\prec A\cap I$. Since $i<j$ and $R\prec A\cap I$, we see that $B\cap I\prec C\cap I$. (q.e.d.)

2.26 *Lemma.* (i) $[A/B][B/C]\leqslant[A/C]<([A/B]+1)([B/C]+1)$ (*mod* $\mathscr{H}$)
(ii) $[A/C]+[B/C]\leqslant[A\cup B/C]<[A/C]+[B/C]+2$ (*mod* $\mathscr{H}$) *where A and B are disjoint.*

Proof. Let $[A/B](x) = i$, $[B/C](x) = j$, $[A/C](x) = k$ and let E be the set of points where $[A/B] = i$, $[B/C] = j$, and $[A/C] = k$. We can ignore triplets (i, j, k) for which the set E is compressible since we wish to prove the result (mod $\mathscr{H}$). Assume therefore that the set E is incompressible. Now $E \cap B$ admits j pairwise disjoint copies of $E \cap C$ and $E \cap A$ admits i pairwise disjoint copies of $E \cap B$ whence $E \cap A$ admits at least ij pairwise disjoint copies of $E \cap C$. Thus $k =$ number of copies of $E \cap C$ inside $E \cap A$ is bigger than or equal to ij. This proves first half of (i). To prove the second half assume that $k \geqslant (i+1)(j+1)$. Then $E \cap A$ admits at least $(i+1)(j+1)$ pairwise disjoint copies $E \cap C$. Each collection of $(j+1)$ such copies of $(C \cap E)$ will admit a copy of $B \cap E$, since $[B/C] = j$ on E. Thus $E \cap A$ admits at least $(i+1)$ pairwise disjoint copies of $E \cap B$. Since $[A/B] = i$ on E and E is incompressible we have a contradiction. Hence $k < (i+1)(j+1)$ and (*i*) is proved.

Proof of (ii). If $x \notin sA \cap sB$ then at least one of the integers $[A/C](x)$, $[B/C](x)$ is zero and the other is equal to $[(A \cup B)/C](x)$ so that the inequality is valid for $x \notin sA \cap sB$. Assume therefore that $x \in sA \cap sB$ and let $i = [A/C](x)$ and $j = [B/C](x)$. Let E be the set of points where $[A/C] = i$ and $[B/C] = j$. We can ignore the pairs i, j where E is compressible. Now

$$E \cap A = i \odot (E \cap C) \oplus R, \quad R \prec E \cap C$$

$$E \cap B = j \odot (E \cap C) \oplus S, \quad S \prec E \cap C.$$

Since $A \cap B = \varnothing$. We have

$$(E \cap A) \cup (E \cap B) = (i+j) \odot (E \cap C) \oplus (R \cup S), \quad R \prec E \cap C, S \prec E \cap C.$$

Thus $[(A \cup B)/C]$ is at least equal to $(i+j)$ on E but it cannot exceed $(i+j+1)$ since $R \cup S$ can admit atmost one copy of $E \cap C$ inside it. Thus we have $(i+j) \leqslant [(A \cup B)/C] < i+j+2$ on E, which proves (ii). (q.e.d.)

2.27 Let X be incompressible. A sequence $(F_n)_{n=1}^{\infty}$ of sets in $\mathscr{B}$ is said to be fundamental if for all n, $sF_n = X$ and $[F_n/F_{n+1}](x) \geqslant 2$ (mod $\mathscr{H}$). We note that the requirement $[F_n/F_{n+1}](x) \geqslant 2$ (mod $\mathscr{H}$) implies that $F_{n+1} \prec F_n$ (mod $\mathscr{H}$). If X is incompressible and every set in $\mathscr{B} - \mathscr{H}$ is decomposable then there exists a fundamental sequence $(F_n)_{n=1}^{\infty}$ in $\mathscr{B}$. This follows on setting $F_1 = X$ and using successively the decomposability in the manner described in corollary 2.21.

2.28 *Lemma. Let $(F_n)_{n=1}^{\infty}$ be a fundamental sequence in $\mathscr{B}$. Then for $A \in \mathscr{B}$,* $\lim_{n \to \infty}$ *$[A/F_n](x)$ exists (mod $\mathscr{H}$). The limit is equal to zero on $X - sA$ and equal to ∞ on sA.*

Proof. Since $F_{n+1} \prec F_n$, $[A/F_n] \leqslant [A/F_{n+1}]$. Further by lemma 2.26 (i)

$$\left[\frac{A}{F_{n+k}}\right] \geqslant \left[\frac{A}{F_n}\right]\left[\frac{F_n}{F_{n+k}}\right] \geqslant 2^k \left[\frac{A}{F_n}\right] (\text{mod } \mathscr{H}).$$

Hence on the set of points x where

$$\left[\frac{A}{F_n}\right](x) \neq 0, \quad \left[\frac{A}{F_{n+k}}\right](x) \to \infty \text{ as } k \to \infty.$$

Thus either $[A/F_n](x)=0$ for all n or $[A/F_n](x)\to\infty$ as $n\to\infty$ (mod $\mathscr{H}$), and so the limit in question exists. If $A\in\mathscr{H}$, then the limit is zero (mod $\mathscr{H}$). Now if $A\in\mathscr{B}-\mathscr{H}$, then $[F_{n+1}/A]\leqslant[F_n/A]$ since $F_{n+1}\subseteq F_n$. Further

$$\left[\frac{F_1}{A}\right]\geqslant\left[\frac{F_1}{F_n}\right]\left[\frac{F_n}{A}\right]\geqslant 2^n\left[\frac{F_n}{A}\right].$$

Since $[F_1/A]$ is finite valued and $[F_n/A]$ is non-negative integer valued, we see that $[F_n/A](x)\to 0$ as $n\to\infty$.

Let E_n be the set of points x where $[F_n/A](x)$ is zero. Then $E_{n+1}\supseteq E_n$ and their union is X (mod $\mathscr{H}$). Now $A\cap E_n\to A$(mod $\mathscr{H}$) and since $[F_n/A]$ vanishes on E_n, $[A/F_n]>0$ (mod $\mathscr{H}$) on $s(A\cap E_n)$ whenever $A\cap E_n\in\mathscr{B}-\mathscr{H}$. From the first part of this lemma we see that $[A/F_n](x)\to\infty$ on $s(A\cap E_k)$. Since this holds for all k, $\lim[A/F_n](x)=\infty$ for $x\in sA$(mod $\mathscr{H}$). (q.e.d.)

For any two sets A, $B\in\mathscr{B}$ and any $C\in\mathscr{B}-\mathscr{H}$, the ratio $[A/C](x)\div[B/C](x)$ will be assigned the value zero whenever the numerator is zero (even in the case where the denominator is also zero). If the numerator is non-zero and the denominator is zero, the ratio will be assigned the value $+\infty$.

2.29 *Lemma. Let $(F_n)_{n=1}^{\infty}$ be a fundamental sequence in $\mathscr{B}$. Then for all A, $B\in\mathscr{B}$, $\lim_{n\to\infty}\{[A/F_n](x)\}/\{[B/F_n](x)\}$ exists (mod $\mathscr{H}$) and it assumes non-zero finite value on $sA\cap sB$(mod $\mathscr{H}$).*

Proof. The numerator is zero on $X-sA$ and the denominator is zero on $X-sB$. Hence on $X-sA$ the limit is zero and on $X-sB$ the limit is zero or ∞ depending on the value of the numerator. If $sA\cap sB\in\mathscr{H}$ then there is nothing left to prove since the limit is claimed to exist (mod $\mathscr{H}$). Assume therefore that $sA\cap sB\in\mathscr{B}-\mathscr{H}$. There is no loss of generality if we assume that $sA=sB$. Now by lemma 2.26 (i)

$$\left[\frac{A}{F_{i+j}}\right]\leqslant\left(\left[\frac{A}{F_i}\right]+1\right)\left(\left[\frac{F_i}{F_{i+j}}\right]+1\right)$$

$$\left[\frac{B}{F_{i+j}}\right]\geqslant\left[\frac{B}{F_i}\right]\left[\frac{F_i}{F_{i+j}}\right],\quad\text{whence}$$

$$\frac{\left[\dfrac{A}{F_{i+j}}\right]}{\left[\dfrac{B}{F_{i+j}}\right]}\leqslant\frac{\left[\dfrac{A}{F_i}\right]+1}{\left[\dfrac{B}{F_i}\right]}\cdot\frac{\left[\dfrac{F_i}{F_{i+j}}\right]+1}{\left[\dfrac{F_i}{F_{i+j}}\right]}\leqslant\frac{\left[\dfrac{A}{F_i}\right]+1}{\left[\dfrac{B}{F_i}\right]}\cdot\left(1+\frac{1}{2^j}\right)$$

we have

$$\limsup_{j\to\infty}\frac{\left[\dfrac{A}{F_{i+j}}\right](x)}{\left[\dfrac{B}{F_{i+j}}\right](x)}\leqslant\frac{\left[\dfrac{A}{F_i}\right](x)+1}{\left[\dfrac{B}{F_i}\right](x)},\quad\text{that is,}$$

$$\limsup_{n\to\infty}\frac{\left[\dfrac{A}{F_n}\right]}{\left[\dfrac{B}{F_n}\right]}\leqslant\frac{\left[\dfrac{A}{F_i}\right]+1}{\left[\dfrac{B}{F_i}\right]}.$$

Now $[B/F_i](x) \nearrow \infty$ on $sB(\text{mod}\,\mathscr{H})$ hence there is an invariant compressible set N such that for each $x \in sB - N$, there is an i for which $[B/F_i](x) \neq 0$. The lim sup on the left hand side is therefore finite for $x \in sB - N$. Now $i \to \infty$ gives, considering $[B/F_i](x) \nearrow \infty$ on $sB - N$,

$$\limsup_{n\to\infty} \frac{\left[\dfrac{A}{F_n}\right](x)}{\left[\dfrac{B}{F_n}\right](x)} \leqslant \liminf \frac{\left[\dfrac{A}{F_n}\right](x)}{\left[\dfrac{B}{F_n}\right](x)}, x \in sB - N.$$

Thus

$$\lim_{n\to\infty} \frac{\left[\dfrac{A}{F_n}\right](x)}{\left[\dfrac{B}{F_n}\right](x)} \text{ exists and is finite valued } (\text{mod}\,\mathscr{H})$$

on $sA = sB = sA \cap sB$. Interchanging the role of A and B proves that the limit is non-zero on $sA \cap sB$. (q.e.d.)

2.30 For $A \in \mathscr{B}$ write $m(A, x) = \lim\limits_{n\to\infty} \dfrac{\left[\dfrac{A}{F_n}\right](x)}{\left[\dfrac{X}{F_n}\right](x)}$

where $(F_n)_{n=1}^{\infty}$ is a fundamental sequence in $\mathscr{B}$. The function $m(A, x)$ is measurable in x and in addition has the following properties:

Lemma. (i) $A \sim B \Rightarrow m(A, x) = m(B, x)$ *(mod* $\mathscr{H}$*)*
(ii) $m(A, x) = m(A, T^{-1}x)$ *(mod* $\mathscr{H}$*)*
(iii) $m(A, x) = \lim_{n\to\infty} [A/F_n](x)/[sA/F_n](x)$ *(mod* $\mathscr{H}$*)*
(iv) $m(A, x) \equiv 0$ *(mod* $\mathscr{H}$*)* $\Leftrightarrow A \in \mathscr{H}$, $m(A, x) > 0(\text{mod}\,\mathscr{H})$ *on* sA
(v) *If* $A \cap B = \varnothing$ *then* $m(A \cup B, x) = m(A, x) + m(B, x)$, $(\text{mod}\,\mathscr{H})$
(vi) *If* E *be the set of points where* $m(A, x) < m(B, x)$, *then* $A \cap E \preccurlyeq B \cap E$.

Proof. Properties (i), (ii) and (iii) follow immediately from the definition of $m(A, x)$. Property (iv) is proved in lemma 2.29. To prove (v) we note that (from lemma 2.26 (ii))

$$\frac{\left[\dfrac{A}{F_n}\right] + \left[\dfrac{B}{F_n}\right]}{\left[\dfrac{X}{F_n}\right]} \leqslant \frac{\left[\dfrac{A \cup B}{F_n}\right]}{\left[\dfrac{X}{F_n}\right]} \leqslant \frac{\left[\dfrac{A}{F_n}\right] + \left[\dfrac{B}{F_n}\right] + 2}{\left[\dfrac{X}{F_n}\right]}.$$

Since $[X/F_n] \geqslant 2^n$ we see on letting $n \to \infty$ that $m(A, x) + m(B, x) \leqslant m(A \cup B, x) \leqslant m(A, x) + m(B, x)$ which proves (v). To prove (vi) let E_n be the set of points x where

$$\frac{\left[\dfrac{A}{F_n}\right](x)}{\left[\dfrac{X}{F_n}\right](x)} < \frac{\left[\dfrac{B}{F_n}\right](x)}{\left[\dfrac{X}{F_n}\right](x)},$$

i.e., E_n is the set of points x where $[A/F_n](x) < [B/F_n](x)$. From the proposition 2.25, $E_n \cap A \preccurlyeq E_n \cap B$. Now $E = \cup_{n=1}^{\infty} E \cap E_n = \cup_{n=1}^{\infty} G_n$, where $G_1, G_2, G_3, \ldots$ is the usual disjointification of $E \cap E_1, E \cap E_2, \ldots$, i.e., $G_n = E \cap E_n - \cup_{i=1}^{n-1}(E \cap E_i)$, $n = 1, 2, 3, \ldots$. Each G_n is T-invariant and $G_n \subseteq E_n$, whence $G_n \cap A \preccurlyeq G_n \cap B$. Since G_n's are pairwise disjoint we see that $A \cap E \preccurlyeq B \cap E$. This proves (vi). (q.e.d.)

2.31 *Lemma. If* $m(A,x) > \Sigma_{k=1}^{\infty} m(A_k, x)$ *(mod* $\mathscr{H}$*) where* $A_1, A_2, A_3, \ldots$ *are pairwise disjoint sets in* $\mathscr{B}$ *then* $\cup_{i=1}^{\infty} A_i \preccurlyeq A$.

Proof. Since $m(A,x) < m(A_1,x)$, by 2.30 (vi) there is a set $B_1 \subseteq A$ such that $B_1 \sim A_1$. By 2.30 (v) $m(A,x) = m(A - B_1, x) + m(B_1, x) > \Sigma_{k=1}^{\infty} m(A_k, x)$. Since $m(A_1, x) = m(B_1, x)$ by 2.30 (i), we have $m(A - B_1, x) > \Sigma_{k=2}^{\infty} m(A_k, x)$. We apply the same argument to obtain a copy B_2 of A_2 in $A - B_1$. Proceeding thus we see that $\cup_{i=1}^{\infty} A_i \preccurlyeq A$. (q.e.d.)

2.32 *Lemma. If* $A_1, A_2, A_3, \ldots$ *are pairwise disjoint sets in* $\mathscr{B}$ *then* $m(\cup_{k=1}^{\infty} A_k, x) = \Sigma_{k=1}^{\infty} m(A_k, x)$ *(mod* $\mathscr{H}$*).*

Proof. If the function m is not countably additive in first variable then there exist $A_1, A_2, A_3, \ldots$ in $\mathscr{B}$ pairwise disjoint such that $m(\cup_{i=1}^{\infty} A_i, x) \neq \Sigma_{i=1}^{\infty} m(A_i, x)$ (mod $\mathscr{H}$). In as much as $m(\cup_{i=1}^{\infty} A_i, x) \geqslant m(\cup_{i=1}^{\infty} A_i, x) = \Sigma_{i=1}^{\infty} m(A_i, x)$ we see that $m(\cup_{i=1}^{\infty} A_i, x) \neq \Sigma_{k=1}^{\infty} m(A_i, x)$ means that $m(\cup_{i=1}^{n} A_i, x) > \Sigma_{i=1}^{n} m(A_i, x)$ on an incompressible set in $\mathscr{B}$. There is no harm in assuming that $\cup_{i=1}^{\infty} A_i = X$, for otherwise we can simply replace A_1 by $A_1 \cup (X - \cup_{i=1}^{\infty} A_i)$. Now, since the set on which $m(\cup_{i=1}^{\infty} A_i, x) > \Sigma_{i=1}^{\infty} m(A_i, x)$ is incompressible, for some k, the set E of points x where $m(\cup_{i=1}^{\infty} A_i, x) - 1/k > \Sigma_{i=1}^{\infty} m(A_i, x)$ is incompressible. We can assume without loss of generality that $E = X$, otherwise we can treat E as our space X. Choose n such that $m(F_n) \leqslant 1/2^n < 1/2^k$. Then $m(X - F_n, x) \geqslant 1 - (1/2^n) > 1 - (1/k) = m(X, x) - (1/k)$. We have assumed that $\cup_{i=1}^{\infty} A_i = X$. Thus $m(X - F_n, x) > m(\cup_{i=1}^{\infty} A_i, x) - (1/k) > \Sigma_{i=1}^{\infty} m(A_i, x)$. By 2.31, $\cup_{i=1}^{\infty} A_i \preccurlyeq X - F_n$. Since $sF_n = s(X - F_n) = X$ we see that X is compressible which is a contradiction. (q.e.d.)

2.33 If $E \in \mathscr{B}$ is T-invariant then for all $B \in \mathscr{B}$ and for $x \in E$

$$m(B, x) = m(B \cap E, x) \ (\mathrm{mod}\ \mathscr{H}).$$

This holds because from the definition of the quantity $[B/F_n](x)$ we note that for $x \in E$,

$$\left[\frac{B}{F_n}\right](x) = \left[\frac{B \cap E}{F_n}\right](x) \ (\mathrm{mod}\ \mathscr{H}).$$

Summing up we have

2.34 Theorem. *Let* X *be incompressible (mod* $\mathscr{W}$*) and assume that there exists a fundamental sequence in* $\mathscr{B}$*. Then there exists a function* $m\colon \mathscr{B} \times X \to [0,1]$ *such that*

(i) $m(A, x)$ *is measurable in* x *for all* $A \in \mathscr{B}$, $m(X, x) = 1$
(ii) $m(A, x) = m(T^{-1}A, x) = m(A, T^{-1}x)$ *(mod* $\mathscr{H}$*)*
(iii) $m(A, x) \equiv 0$ *(mod* $\mathscr{H}$*) if and only if* $A \in \mathscr{H}$
(iv) $m(\cup_{i=1}^{\infty} A_i, x) = \Sigma_{i=1}^{\infty} m(A_i, x)$ *(mod* $\mathscr{H}$*) whenever* $A_1, A_2, A_3, \ldots$ *are pairwise disjoint sets in* $\mathscr{B}$.

(v) *If $E \in \mathscr{B}$ is T-invariant then for all $B \in \mathscr{B}$ and $x \in E$, $m(B,x) = m(B \cap E, x)$ (mod $\mathscr{H}$).*

The main theorem 2.14 follows because whenever every set in $\mathscr{B}$ is decomposable mod $\mathscr{W}$, then there exists a fundamental sequence in $\mathscr{B}$.

Remark 1. One can develop integration with respect to m in a natural fashon and prove the maximal ergodic theorem. If f is integrable with respect to m then the set

$$E = \left\{ x: \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \text{ does not converge as } n \to \infty \right\}$$

can be shown to be such that $m(E, x) \equiv 0$ (mod $\mathscr{H}$), i.e., $E \in \mathscr{H}$. This yields 'universal' form of Birkhoff's ergodic theorem.

Remark 2. Given a system $(X, \mathscr{B}; \mathscr{n}, T)$ one can form the σ-ideal $\mathscr{F}$ of those sets $B \in \mathscr{B}$ whose saturations are compressible with respect to $\mathscr{n}$. It seems that the above analysis can be carried out with respect to $\mathscr{F}$ and we can get the 'transition probability' m (with respect to $\mathscr{F}$) whenever X is incompressible (mod $\mathscr{n}$).

3. The case of standard Borel space

3.1 In this section we show that in case $(X, \mathscr{B})$ is a standard Borel space incompressible under T then there exists a probability measure on $\mathscr{B}$ invariant under T. We do this by putting together theorem 2.14 and some well known facts about measures and topologies on a standard Borel space. (see K R Parthasarathy's book on Probability Measures in Metric Spaces, Academic Press, 1967).

3.2 A result of Ramsay and Mackey states that we can assign to X a complete separable metric topology $\mathscr{T}$ such that T is a homeomorphism under this topology and the Borel σ-algebra generated by the topology coincides with $\mathscr{B}$ (see Weiss [11]). If we are given a countable collection $\tau \subseteq \mathscr{B}$, the topology can be so chosen that $\tau \subseteq \mathscr{T}$. (We will not need this latter fact.)

3.3 *Lemma. Let X be a complete metric space with metric d. Let $(F_n)_{n=1}^{\infty}$ be a decreasing sequence of closed sets such that for each n, F_n is covered by finitely many closed balls each of diameter $< 1/n$. Then the set $F = \cap_{n=1}^{\infty} F_n$ is compact. It is non-empty whenever each F_n is non-empty.*

Proof. It is clear from hypothesis that F is closed and totally bounded hence compact. If each F_n is non-empty we choose a point $x_n \in F_n$, $n = 1, 2, 3, \ldots,$. The sets G_n = closure of $\{x_n, x_{n+1}, \ldots\}$ are non-empty closed and totally bounded with $G_{n+1} \subseteq G_n \subseteq F_n$. Hence $G = \cap_{n=1}^{\infty} G_n$ is non-empty and contained in F. (q.e.d.)

3.4 An outer measure μ^* on the power set 2^X of a metric space (X, d) is called a metric outer measure if $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ whenever $d(E, F) > 0$. If μ^* is a metric outer measure then every open set (and therefore every Borel set) is μ^* measurable. Thus μ^* is a measure on the class of Borel sets of X whenever μ^* is a metric outer measure on 2^X. (see Halmos '*Measure Theory*', p. 48 exercises 8a, 8b).

3.5 We now return to our problem. Let T be a Borel automorphism without periodic points on a standard Borel $(X,\mathscr{B})$ and assume that X is incompressible with respect to T. Let m be the function on $\mathscr{B}\times X$ given by theorem 2.14.

We use Ramsay–Mackey result quoted above to assign to X a complete separable metric topology $\mathscr{T}$ (with metric d) whose Borel sets form the class $\mathscr{B}$ and under which T is a homeomorphism. Let $\mathscr{U}$ be a countable open base for $\mathscr{T}$. The collection obtained by taking finite unions and finite intersections of sets in the collection $\cup_{n=-\infty}^{\infty}\{T^n U : U\in\mathscr{U}\}$ is again a countable open base for $\mathscr{T}$ which is closed under finite unions, finite intersections, and application of T. We therefore assume that $\mathscr{U}$ itself has these properties. Let $\bar{\mathscr{U}}$ denote the closures of sets in $\mathscr{U}$ and $\mathscr{A}$ the algebra generated by $\mathscr{U}\cup\bar{\mathscr{U}}$ which is also countable. Let $\{\bar{U}_{1n},\bar{U}_{2n},\bar{U}_{3n},\ldots\}$ denote the collection of members in $\bar{\mathscr{U}}$ of diameter $<1/n, n=1,2,3,\ldots$. We know that

(i) $m(A\cup B,x)=m(A,x)+m(B,x) \pmod{\mathscr{H}}$ whenever $A\cap B=\varnothing$, $A,B\in\mathscr{A}$
(ii) for each n, $\lim_{k\to\infty} m(\cup_{i=1}^{k}\bar{U}_{in},x)=m(X,x)=1 \pmod{\mathscr{H}}$

Now (i) and (ii) above are countable number of conditions, hence there exists a T-invariant set $N\in\mathscr{H}$ such that for all $x\in X-N$ we have

(a) $m(A\cup B,x)=m(A,x)+m(B,x)$ whenever $A\cap B=\varnothing, A,B\in\mathscr{A}$
(b) for each $n, m(\cup_{i=1}^{k}\bar{U}_{in},x)\to 1$ as $k\to\infty$.

Fix an $x\in X-N$ and write $m(A,x)=m(A)$ for $A\in\mathscr{A}$. Define the outer measure m^* on the power set of X by setting for any $B\subseteq X$

$$m^*(B)=\inf\left\{\sum_{i=1}^{\infty} m(U_i): B\subseteq\bigcup_{i=1}^{\infty}U_i,\ U_i\in\mathscr{U}\text{ for all } i\right\}$$

We note that m^* is T-invariant and bounded by 1.

3.6 *Lemma.* $m^*(X)=1$.

Proof. We show that given $\varepsilon>0$ there exists a compact set $F=F_\varepsilon$ such that $m^*(F)\geqslant 1-\varepsilon$. From (*b*) above we can get for each n a k_n such that $m(\cup_{i=1}^{k_n}\bar{U}_{in})>1-(\varepsilon/2^{n+1})$. If $F_n=\cap_{j=1}^{n}\cup_{i=1}^{k_j}\bar{U}_{ij}$, then for all n, (i) $F_n\supseteq F_{n+1}$, (ii) F_n is closed, (iii) F_n is covered by finitely many closed balls of diameter $<1/n$, (iv) $F_n\in\mathscr{A}$ (v) $m(F_n)>1-\varepsilon$. The set $F=\cap_{n=1}^{\infty}F_n$ is compact by lemma 3.3. We show that $m^*(F)>1-\varepsilon$. Fix $\eta>0$ and let $U_1, U_2, U_3,\ldots$ be sets in $\mathscr{U}$ covering F such that $\Sigma_{i=1}^{\infty}m(U_i)-\eta<m^*(F)$. Since F is compact there is an integer p such that $F\subseteq\cup_{i=1}^{p}U_i\in\mathscr{U}$ (since $\mathscr{U}$ is closed under finite unions). By lemma 3.3 we conclude that for some integer $q, F_q\subseteq\cup_{i=1}^{p}U_i$. Finally $1-\varepsilon-\eta<m(F_q)-\eta<m(\cup_{i=1}^{p}U_i)-\eta<\Sigma_{i=1}^{\infty}m(U_i)-\eta<m^*(F)$. Since $\eta>0$ is arbitrary we see that $m^*(F)\geqslant 1-\varepsilon$. (q.e.d.)

3.7 *Lemma. If $K\subseteq X$ is compact then the restriction of m^* to K is a metric outer measure.*

Proof. Let $E,F\subseteq K$ be positive distance apart. Since $\bar{E}$, $\bar{F}$ are compact and positive distance apart we can find $U, V\in\mathscr{U}$, $U\cap V=\varnothing$ such that $E\subseteq U$, $F\subseteq V$. Fix $\varepsilon>0$ and let $W_1, W_2,\ldots$ be sets in $\mathscr{U}$ covering $E\cup F$ such that $\Sigma_{n=1}^{\infty}m(W_n)-\varepsilon<m^*(E\cup F)\leqslant m^*(E)+m^*(F)$. Now $W_n\cap U, n=1,2,3,\ldots$ cover E and the sets $W_n\cap V, n=1,2,3,\ldots$ cover F. Moreover these sets belong to $\mathscr{U}$.

Therefore

$$
\begin{aligned}
m^*(E)+m^*(F)-\varepsilon &\leqslant \sum_{n=1}^{\infty} m(W_n\cap U)+\sum_{n=1}^{\infty} m(W_n\cap V)-\varepsilon\\
&\leqslant \sum_{n=1}^{\infty} m(W_n)-\varepsilon\\
&\leqslant m^*(E\cup F)\leqslant m^*(E)+m^*(F).
\end{aligned}
$$

Thus if E and F are positive distance apart in K then $m^*(E\cup F)=m^*(E)+m^*(F)$, i.e., m^* is a metric outer measure on K. (q.e.d.)

By a result quoted earlier every Borel subset of K is m^* measurable and m^* is a countably additive measure on the Borel field of K. Moreover if $A\subseteq K$ is m^* measurable and if $TA\subseteq K$, then $m^*(A)=m^*(TA)$. Thus the collection of m^* measurable sets contain all compact sets and by lemma 3.3 there exists a σ-compact set $Y\subseteq X$ such that $m^*(Y)=1$. This set Y can be made to be T-invariant by replacing Y by $\cup_{n=-\infty}^{\infty} T^n Y$. We thus see that m^* is a T-invariant probability measure on Borel subsets of X, i.e. on $\mathscr{B}$.

Remark 1. If n_μ denotes the class of μ null sets of a T-invariant probability measure μ on $(X, \mathscr{B})$, then $n_\mu \supseteq \mathscr{H}$. We see now that $\mathscr{H}=\cap n_\mu$ where intersection is taken over all T-invariant probability measures μ.

Remark 2. If $X=\cup X_i$, $i\in I$ an indexing set, be a partition of X generated by countable number of T-invariant Borel sets in X, then incompressibility of X implies the incompressibility of at least one of the X_i. This follows on taking the regular conditional probability of a T-invariant probability measure on X with respect to the partition.

Remark 3. Let $\mathscr{A}=\{A_1,A_2,A_3,\ldots\}$ be a countable algebra generating $\mathscr{B}$ and consider the measurable map $X\to[0.1]^{\mathscr{N}_0}$ given by

$$x\to(m(A_1,x),m(A_2,x),\ldots).$$

This map gives a countably generated measurable partition of X into T-invariant sets. A typical element of this partition is of the form $C_x=\{y{:}m(A_i,y)=m(A_i,x)$ for all $A_i\in\mathscr{A}\}$. The incompressible members of this partition are uniquely ergodic under T and every ergodic T-invariant probability measure is supported on some member of this partition.

4. Historical background and Hopf's theorem

The question of existence of invariant integrals (equivalently finite invariant measures) was discussed by G D Birkhoff and P A Smith in their expository paper 'Structure analysis of surface transformations' *Journ. de Math. Tome* VII – Fasc IV, 1928, pp. 345–379. Many elementary and basic ideas of classical ergodic theory are set forth in this paper, although in the setting of continuous or analytic invertible maps of a surface. Let us briefly review the contents of §4 of this paper of Birkhoff and Smith.

Let T be an analytic invertible transformation of a surface S onto itself. Birkhoff and Smith show that a necessary and sufficient condition that there exists invariant integrals of a certain type on S, or part of S, is that S be not compressible into an arbitrarily small area. This result is an intuitive statement of their result which we explain below.

They begin by introducing a function $\varphi(e)$, e being a measurable set of S, defined as follows: Let e be divided into a finite number of mutually exclusive measurable sets

$$e = \bigcup_i \delta_i, \quad \delta_i \cap \delta_j = \emptyset \text{ for } i \neq j.$$

Then $\varphi(e)$ is the lower bound of the sum

$$\sum_i m(T^{n_i}\delta_i) \ (m = \text{surface area})$$

with respect to all possible methods of subdivision of e into finite number of measurable sets, and all possible choices of integers n_i. Here use is made of the fact that the property of measurability is preserved under analytic transformation.

It is clear that function φ may be identically zero, in which case, S is compressible into an arbitrarily small area. This happens, for example, when T is an analytic transformation of a sphere such that each circle parallel to the equator closes down on the north (or south) pole on indefinite iteration of T(or T^{-1}). On the other hand, for a transformation which preserves the area measure m, we have $\varphi(e) = m(e)$.

In any case, it follows immediately from the definition, that $\varphi(e) \leqslant m(e)$, and hence $\varphi(e)$ is bounded and absolutely continuous with respect to m. The importance of φ is due to the following theorem of Birkhoff and Smith.

Theorem. *φ is countably additive and invariant under T. Moreover there exists a finite, non-trivial T-invariant measure on S absolutely continuous with respect to m if and only if $\varphi(S) > 0$.*

It is interesting to note that Birkhoff and Smith are careful to qualify their invariant integrals as invariant intergrals of a certain type by which they seem to mean invariant integrals which are finite non-trivial and absolutely continuous with respect to m. The question of finding necessary and sufficient conditions for the existence of finite non-trivial invariant integrals, not necessarily absolutely continuous with respect to m, seems to have remained uninvestigated.

On the suggestion of Birkhoff, the question of the existence of a finite invariant measure was further taken up by Hopf in his paper [5] where the notion of compressibility due to Birkhoff and Smith was modified as follows: Consider a measure space $(X, \mathscr{B}, \mu)$ where we assume that $(X, \mathscr{B})$ is standard Borel and $\mu(X) = 1$. Let $T: X \to X$ be one–one onto measurable map which preserves μ-null sets. Two sets A, $B \in \mathscr{B}$ are said to be equivalent by countable decomposition (mod μ) if we can find μ-null sets $M, N \in \mathscr{B}$ such that $A - M$, $B - N$ are equivalent by countable decomposition in our sense, i.e., in the sense of definition 2.3. A set $A \in \mathscr{B}$ of positive measure is said to be compressible in the sense of Hopf if we can decompose A into pairwise disjoint sets $C, D \in \mathscr{B}$, each of positive measure and further A is equivalent to C by countable decomposition (mod μ). It is clear that if a set A is compressible in our sense then it is compressible in the sense of Hopf. But the converse need not hold. For example let $X = \mathbb{R} \cup S^1$ with the usual Borel structure and a probability measure having same null sets as Lebesgue measure on $\mathbb{R}$ and S^1. (Here S^1 = circle group). Let T act on X by translation by one on $\mathbb{R}$ and irrational rotation on S^1. Then X is compressible in the sense of Hopf with respect to the measure μ but not in our sense.

Theorem. (Hopf). *If X is incompressible in Hopf's sense then there exists a unique T-invariant probability measure on $\mathscr{B}$ having same null sets as μ.*

Hopf's proof of the theorem was considered difficult. Simplified proofs and alternative necessary and sufficient conditions for the existence of finite invariant measure equivalent to a given one for a non-singular transformation were therefore sought. The best known is the theorem of Hajian and Kakutani which says that a non-singular T on the measure space $(X,\mathscr{B},\mu)$ admits a finite equivalent T-invariant measure if and only if there is no weakly wandering set of positive measure, where a set $A \in \mathscr{B}$ is said to be weakly wandering if there exists a sequence $(n_k)_{k=1}^{\infty}$ of integers such that iterates $T^{n_k}A$ are pairwise disjoint (see [3]).

Hopf's theorem follows easily from the above quoted result of Hajian and Kakutani because the existence of a weakly wandering set of positive measure immediately implies compressibility (mod μ) in the sense of Hopf. We can use theorem 2.14 of this paper to prove Hopf's theorem as follows. If X is incompressible in the sense of Hopf (with respect to μ) then X is incompressible in the sense of definition 2.4. Also since $(X,\mathscr{B})$ is standard Borel every non-empty set in $\mathscr{B}$ is decomposable (mod $\mathscr{W}$). Thus the hypothesis of theorem 2.14 is satisfied. If m be as in theorem 2.14 we write

$$P(A) = \int_X m(A,x)\,\mathrm{d}\mu.$$

It is easy to show, using properties of m, that P is a T-invariant probability measure having same null sets as μ and that it is unique.

Since we have mentioned the theorem of Hajian and Kakutani, (*Trans. Amer Math. Soc.* **110** (1964) 136–151) it is natural to ask whether compressibility of X under T implies the existence of a weakly wandering set $W \in \mathscr{B}$ such that $sW = X$. One may assume, if necessary, that $(X,\mathscr{B})$ is a standard Borel space. A related question is whether a compressing (in the sense of definition 2.4) homeomorphism of a Polish space admits a weakly wandering non-empty open set. It is not necessarily required that the saturation of such an open set be all of X.

Another question which can be formulated is as follows: Suppose f is a non-negative measurable function on $(X,\mathscr{B})$ and T a Borel automorphism on X. Can one define a notion "X is incompressible with respect to f" which implies that whenever X is thus incompressible there exists a σ-finite measure m on $\mathscr{B}$ quasi-invariant under T such that $\mathrm{d}m_T/\mathrm{d}m = f$.

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