# Geometry and string theory

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THERE is something almost mystical about the connection between geometry and the laws of nature. An encounter at the seashore with the perfection of a logarithmic spiral in a seashell can be a source of wonder. A wonder which is multiplied when the self-same spirals appear in telescopic images of distant galaxies.

That Nature is, at various levels, amenable to a geometrical understanding has been a rich theme in science. Though little more than a belief, it has consistently yielded beautiful insights. But it is perhaps fair to say that at no time in the past has geometry played as profound a role as it currently does in string theory's endeavours to understand quantum gravity. This should not be too surprising. Since the time when Einstein taught us that gravity is geometry, we have learnt to think of space–time not as a passive arena for events, but rather as a dynamical entity. Hence any theory of quantum gravity such as string theory must describe the quantum fluctuations of geometry.

Much of the elusiveness of a consistent quantum theory of gravity has to do with this notion of describing quantum fluctuations of geometry. Recall that in the quantum mechanics of particles, instead of classical trajectories from point *A* to point *B*, one explores all possible paths from *A* to *B*, each weighted by some amplitude. Amplitudes for paths far from the classical one are suppressed if their action is large compared to Planck's constant ($h$) (note 1). In this regime, a nearly classical ('semi-classical') description can be applied. However, when the action is comparable to Planck's constant, then trajectories far from the classical contribute equally. Such quantum trajectories are by and large far from being smooth – they are more like the paths in Brownian motion (see Figure 1).

The situation is very analogous with gravity. Einstein's equations of gravity determine the classical geometry of space–time, just as Newton's laws fix particle trajectories. This is a good approximation for macroscopic systems of the kind we are accustomed to. However, again quantum fluctuations are important when the action is of order $h$. For gravity, this is true at distance scales set by Newton's constant $G_N$, namely $l_p = (hG_N/c^3)^{1/2} \approx 10^{-33}$ cm (called the Planck length scale) or equivalently at characteristic energies which are enormously large $E_p = (hc/l_p) \approx 10^{28}$ eV (the Planck energy scale). At these scales, which would only have been encountered in the very hot, very early universe, one must come to grips with what fluctuations in geometry mean. As for trajectories, one expects contributions from geometries which are wild and Brownian, so to say. In other words, we no longer have the reassurance of a fixed (classical) background space–time and moreover the majority of the many space–times that enter into a quantum process are far from the smooth (or near-smooth) manifolds conventionally studied by mathematicians. How are we to describe these weird geometries? Are there geometrical structures that can replace our conventional notions of space–time? Or perhaps geometry emerges only as an approximate notion at distances large compared to the Planck length.

Though we are still far from having all the answers, string theory, in its attempt to address these issues, has given us some inklings as to what to expect, while also expanding our notion of geometry. Geometry has also played a central role in string theory's attempts to understand other questions of particle physics like the number of generations, the origin of gauge symmetries, the hierarchy between the scale of the standard model and that of gravity. At its most ambitious, it could perhaps even explain the number of space–time dimensions we live in. And, almost as an aside, it has had fruitful interaction with mathematics leading to a renewal of the traditional links between the two subjects.

In this article I will try to convey something about this varied interplay between geometry and string theory. We will walk through several layers of this interplay, proceeding from the relatively well-understood towards the more novel contours of our picture of quantum gravity that is still emerging. More operatively, we

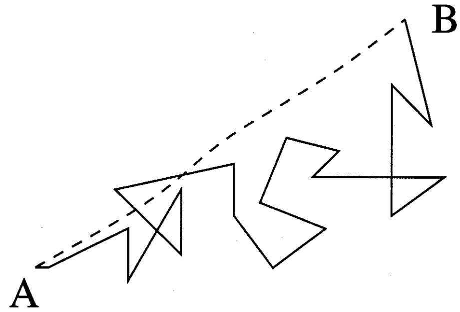


**Figure 1.** Typical trajectory contributing to a quantum amplitude. (The dotted line is the smooth classical path).

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will start with discussing how classical strings see a background space–time very differently from the way point particles do. Then we will be ready to elaborate on the basic connection between strings and gravity. We will also see that string theory can combine gravity with the other forces in a very geometric way. We will also briefly look at an example of the profound influence of string theory on mathematics. Having thus far discussed various aspects of string theory at the classical level involving no fluctuations, we step onto the less-understood quantum terrain. Here we mostly concentrate on the various pictures of quantum gravity that seem to be emerging. Though these ideas have not yet been integrated together into one coherent story, they are provocative hints of the ultimate story.

## How strings see space–time

Strings are extended objects. This is the crucial difference with the notion of fundamental particles as point-like, localized entities. A localized object 'sees' space–time in a much more limited way compared to an extended one. To give a simple illustration, we know that a cylindrical tube is topologically different from a plane sheet. The cylinder has paths on it (which wind around the circular direction) that cannot be continuously shrunk to zero size. A small ball via its very local probing cannot distinguish this property of the cylinder from the plane. But a rubber band can. The rubber band can wrap the circular direction of the cylinder and we know that such a wrapped band is qualitatively different from an unwrapped one. While we can easily lift the unwrapped one off the cylinder, we cannot do so with the wrapped one without breaking the band (note 2). In a similar way, closed strings, which can be thought of as rubber bands, see more of the topology of space–time than do point particles. Thus by assuming that the familiar particles are excitations of a more fundamental extended object, string theory immediately introduces new ways of probing space–time.

The extended nature of strings brings some additional surprises. Our usual concept of size goes topsy-turvy when we use strings as probes. There is a characteristic string size (denoted by $l_s$) (note 3) such that closed strings really cannot see a space–time of size smaller than $l_s$. The idea is that when you try to probe features smaller than size $l_s$ with a closed string, then the uncertainty principle demands that the string have a lot of momentum and thus energy. But pumping energy into a string only makes it grow larger.

As a consequence of these features one finds, for example, that closed string propagation on a circle of size $R < l_s$ is the same as that on one of radius $(l_s^2/R) > R$. More specifically, closed strings on a circle have ('momentum') modes whose energy, like those of a particle on a circle, is in integer multiples of $1/R$. But unlike particles, there are ('winding') modes where the string wraps an integer number of times around the circle. The energy of such states is proportional to the length of the wrapped string which is thus an integer times $R$ (since the string has uniform tension or energy per unit length). It is the presence of these winding modes that allow one to have an equivalent description in terms of strings on a circle of radius $l_s^2/R$. The winding modes on our original circle have exactly the energy to be the momentum modes in the new description. This interchange of momentum and winding modes is a consequence of the stringy nature of our probe. The fact that string theory on a circle of radius smaller than the string scale $l_s$ is equivalent to one on a circle larger than $l_s$ is the first hint that the geometry seen by strings has fundamentally novel features at small distances (note 4).

This equivalence, called T-duality, is the first of many mysterious equivalences or *dualities* that string theories exhibit. There is a powerful generalization of *T*-duality when strings propagate on more complicated space–times. This is called mirror symmetry and we will come to it presently. But at this point we should stress that all that we have said thus far follows simply from the extended nature of strings. There is no dynamical effect of gravity playing any role.

## How strings make space–time

It is therefore time to take a look at how string theory can describe the dynamics of space–time itself.

A closed relativistic string in flat space–time, just like a rubber band, has many oscillation modes in which to vibrate. Among these oscillations is one which turns out to be massless. The surprise is that this massless state carries the quantum numbers and has the symmetry of the quantum that mediates gravitational interaction, the graviton. Since the leading interactions of this particle are also those of the graviton, the conclusion is inescapable (note 5) that a quantum theory of strings contains gravity.

Though we started in flat space, we expect that if we excite enough of these oscillation modes or graviton quanta, then the resulting configuration would curve space–time (because of its energy) in precisely a way that satisfies Einstein's equations. This connection is, in fact, one of the beautiful results of string theory. The consistency of the $(1+1)$ dimensional 'world volume' (note 6) theory of the string requires that it be a so-called conformal field theory. This in turn requires the 'target' space–time on which the string propagates to satisfy Einstein's equations. In other words, a sensible string theory implies Einstein's theory of gravity! Actually, one also finds tiny corrections (note 7) to Einstein's equations which are proportional to the string scale $l_s$. These corrections are a consequence of the presence of an infinite tower of massive excitations, in addition to the graviton, which are the higher oscillator modes of the closed string. It is these oscillator states that help to cure many of the sicknesses that plague the conventional quantization of gravity. In particular, at high energy, the contributions of quantum loops is no longer divergent. We will come back to the quantum behaviour of string theory.

## Strings and the dimension of space–time

Another remarkable consequence of the requirement of the consistency of the world volume theory of the string is that the dimension of the target space–time is also constrained to be ten (at least in the best defined supersymmetric versions of the theory – the superstring theories).

Now this, at first sight, poses a very big problem if string theory is to describe nature. But actually, the fact that strings propagate only in ten-dimensional space–time may be a virtue in explaining the origin of forces other than gravity. The idea of obtaining forces like electromagnetism in four dimensions, from gravity in higher dimensions, is an idea going back to Kaluza and Klein in the 1930s. It geometrizes nongravitational forces by postulating that there are additional curled-up ('compactified') spatial dimensions. The graviton in the higher dimensional space has oscillation modes ('polarizations') along the extra dimensions as well. These additional polarizations manifest themselves in our four-dimensional space–time as gauge fields like the photon (note 8). The size of these additional dimensions must be quite small since we see a four-dimensional world around us. It is conventionally believed that the extent of these dimensions must be comparable to the Planck scale (note 9).

In the case of the superstring the idea is that the six additional dimensions are of size $l_s$ and hence are too small to be directly observable. However, the topology of this internal space would determine the particle content in our world. One of the simplest choices for the geometry of this space, fixed by requiring consistency with Einstein's equations together with some symmetry requirements, is that it be of a very special kind which mathematicians call a Calabi–Yau (CY) manifold (note 10). These manifolds come in a variety of topologies. In particular, in one of the simplest scenarios involving the superstring known as the heterotic string, the number of families ('generations') of chiral quarks and leptons (there are three in our world) is given by a topological number (the Euler character) which roughly measures the number of holes of different dimensions in the manifold. Moreover, the gauge groups of the strong and weak interactions can also be obtained by appropriate choice of the internal manifold. Thus in this scenario, much of the physics we see around us would be totally determined by geometry. The central role played by the somewhat abstruse geometries of CY manifolds has motivated extensive study of string theory on these spaces which has also had a mathematical impact.

However, this is simply one of many possible scenarios whereby the real world with its intricate pattern of particles and interactions could be realized in string theory. In recent years, others have been proposed which rely on solitonic objects in string theory – *D*-branes – which we will encounter soon. All of them involve geometrizing, to varying extents, many of the observed features of the standard model. More input, perhaps from experiments, will be necessary to choose between these various scenarios.

## A pause

So far we have touched on novel aspects of string theory which are to do with the extended nature of strings and its relation to gravity. However everything that has been said thus far has been in the context of what is called classical string theory and is something that is relatively well understood. We have seen that the extended nature of strings even at the classical level leads to strings seeing space–time differently. We also mentioned how classical gravity, namely Einstein's equations, emerges from string theory. Kaluza–Klein compactifications of the classical theory can broadly reproduce the observed spectrum of light particles in nature. But we have not yet talked about effects which are genuinely quantum gravitational. Quantum string theory is a subject in which we have made some headway in recent years, but is still little understood. Nevertheless the little we have understood thus far is enormously encouraging in terms of the rich and consistent structure that is indicated. As we will see geometry plays ever more novel roles in this context. But before we proceed to describe that, let us take a digression into some of the fruits of interaction between classical string theory and mathematics.

## Classical strings and mathematics

Mathematics in the middle of the 20th century developed along lines which were somewhat removed from motivations or applications in physics. However, in the seventies and eighties, modern ideas in mathematics, specifically in topology and geometry, were seen to be crucial in the understanding of various nonperturbative aspects of quantum field theory. By now, the topological nature of various solitons like vortices and monopoles inform any discussion of the topic. The understanding of the role played by various characteristic classes and the Atiyah–Singer index theorem in the anomalous symmetries of field theory helped in generalizations to various anomalies associated with gravity. This in turn was crucial in establishing the consistency of string theories.

However, much of geometry has developed from local concepts like points and smoothness in small neighbourhoods. The global structure is sought to be captured in simple topological notions. It has come to be realized that even classical strings require a generalization of these notions of geometry and topology to reflect what is seen by extended (string-like) objects. One concrete example of this is in mirror symmetry. This is a generalization to Calabi–Yau manifolds of the *T*-duality that we mentioned in the context of a circle. String theory compactified (in the Kaluza–Klein sense of the word) on a CY manifold *M* is equivalent to that on another topologically distinct CY manifold *W*, where *W* bears quite a precise mathematical relation to *M*. More accurately, IIA string theory on *M* is the same as IIB string theory on *W* (note 11). Thus from the mathematical point of view, it reveals a totally unexpected relation among pairs of Calabi–Yau manifolds. Unexpected, because our intuitions about geometry are those of the smooth geometry of points and not of extended objects.

To roughly understand the relation between the manifolds *M* and *W*, it is helpful to think first in terms of a two-dimensional space in the shape of a cycle tube – the torus. The torus can be characterized by two parameters, say $(t, u)$. One $(t)$ describes the size (or area) of the torus. The other $(u)$ captures the shape of the torus, roughly how thin the tube is relative to its overall diameter (Figure 2). Now the surprise is that string theory on a torus with some fixed parameters $(t_0, u_0)$ is equivalent to strings on the distinct torus with parameters $(u_0, t_0)$. In other words, there is a 'mirror' torus whose shape and size parameters are interchanged with respect to the original one. And there is an equivalence of string theories on the mirror and the original.

For the more complicated case of CY manifolds, the idea is essentially the same. The mirror *W* to *M* is one in which the size and shape parameters have been interchanged. An important difference is that for CY manifolds, the number of size and shape parameters can be more than one and in fact need not even be equal in number. Thus the mirror *W* generically has a different topology from *M* (note 12).

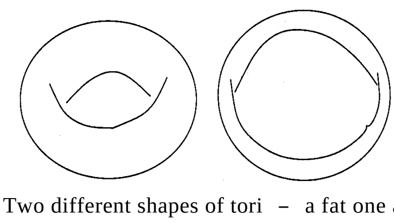

**Figure 2.** Two different shapes of tori – a fat one and a thin one.

Mirror symmetry has some very nontrivial physical and mathematical consequences. This is because certain interesting computations in IIA string theory on *M* are very difficult even classically (note 13). In particular, there are contributions which are intrinsically stringy, coming, for instance, from the nontrivial wrapping of the two-dimensional string world sheet on various topological spheres inside the Calabi–Yau (note 14). The difficulty lies in the mathematical question of counting how many different (holomorphic) maps exist from the two-dimensional sphere $(S^2)$ into *M*. But mirror symmetry transforms this computation in IIA to one in IIB string theory on the mirror *W*. It turns out that the corresponding computation in IIB is special in that it does not have any intrinsically stringy contribution. By following through the map of the variables from IIA to IIB, one can obtain the nontrivial answer in IIA with all its worldsheet instanton contributions. And thus also provide an answer to the mathematical question of counting of maps mentioned above. Impressive agreement with partial results known earlier to mathematicians bears out the physical arguments used in arriving at the answer. It has led to extensive investigations of mirror symmetry by mathematicians. In part, the surprise lies in the manner in which string theory managed to extract answers about extended objects (maps of 2*d* surfaces into a CY manifold) through a seemingly unrelated point particle like computation. It hints at a broader conception of geometry which is reflected in ideas like 'quantum cohomology', the appropriate generalization of ordinary cohomology, that has cropped up in the study of mirror symmetry (note 15).

This is merely one instance where string theory has stimulated research in mathematics by revealing hitherto unsuspected connections. The future is bound to deepen this contact between the two disciplines.

## *D*-branes in string theory

It is the quantum aspects of string theories that are potentially the most striking and to which we will now turn to. When quantum effects are small, there is a definite prescription in string theory for their inclusion. One can write a Taylor expansion in powers of a dimensionless 'string coupling constant' $(g_s)$ which captures the strength of quantum effects. The coefficent of $g_s^{2n}$ for some process is given in terms of a computation on a 2*d* worldsheet with *n* holes (essentially like *n* cycle tubes attached in a chain). This expansion in powers of $g_s$ is what is called a perturbative expansion and breaks down at finite coupling (note 16). Essentially, the perturbative expansion misses some important physics which does not show up in the power series (note 17). Such 'nonperturbative' effects have played an important role in understanding the dynamics of the field theories of the strong and electroweak interactions. In particular, phenomena like the confinement of quarks are nonperturbative in nature.

In field theories, these nonperturbative effects are typically associated with field configurations that are topologically nontrivial. These objects such as vortices or monopoles are finite energy topological defects (or solitons) in space–time very much like the defects and dislocations in crystals. It turns out that string theories have analogous defects which can have infinite extent in some directions. Such a string soliton extended in $p$ spatial directions is called a $Dp$ brane and has a finite tension or mass per unit volume. Thus, for example, a $D0$ brane is a point-like defect while a $D3$ brane fills three spatial dimensions. Note that a $D1$ brane is a string-like excitation distinct from the so-called fundamental strings in the theory. These are nonperturbative excitations of string theory since their tension is proportional to $1/g_s$.

The existence of $D$-branes is a consequence of $T$-duality on open strings. If we start with a geometry (as seen by closed strings) of a small circle of radius $R$, then $T$-duality tells you that there is an equivalent description in terms of a large circle of radius $l_s^2/R$. In particular, a zero size circle is transformed into an infintely extended dimension. However, this is true only for closed strings as we mentioned in note 4. Even when we describe things in terms of closed strings in an infinitely extended dimension, the open strings continue to see the original zero size circle. And therefore it is as if the open strings are stuck to a (hyper) plane of one less dimension (the position of the plane is arbitrary). To put it in other words, in the $T$-dual description, the open strings propagate in effectively one lower dimension than the closed strings. Further $T$-dualities can result in the open strings living only in $p$ spatial dimensions. And this $p$ dimensional hyperplane on which open strings can end can be taken as the definition of a $Dp$-brane.

There can also be multiple such hyperplanes and open strings can stretch between them, i.e. have their endpoints on two different branes. This has the consequence that the two endpoints of an open string can be labelled by which brane they end on. This structure is exactly like that of a matrix ($M_{ij}$) with the two indices ($i, j$) labelling the two endpoints. In fact, the degrees of freedom that live on the $D$-brane are matrix valued and this is the origin of the so-called nonabelian symmetries that the theory possesses. In particular, nonabelian gauge theories similar to those encountered in the description of the strong and the electroweak force arise very naturally in this way. This has been the origin of speculations that our real $3+1$ dimensional world might be a brane or a defect in a higher dimensional world. The nongravitational forces could arise as above and only gravity would (weakly) probe the presence of the extra dimensions.

Coming back to the matrix valued fields on the $D$-brane, one of the surprises is that the coordinates representing the position of the branes in the transverse space are also promoted to matrices. For a collection of $N$ $D$-branes, the eigenvalues of the $N \times N$ matrices representing the transverse directions, are the conventional coordinates of the individual branes. However, there are additional (off-diagonal) degrees of freedom in the matrix description and these are quite crucial to the properties of $D$-branes. This matrix description of positions suggests a nonabelian generalization of ordinary geometry. A matrix generalization of the general covariance underlying Einstein's equations is probably called for. These ideas are yet to be fleshed out and the future will probably see innovations in geometry along these directions.

## The connectedness of all string vacua

### *The conifold transition*

$D$-branes have allowed us to understand several nonperturbative aspects of string theory. At small values of the coupling $g_s$, these effects are usually tiny but one semiclassical situation in which such effects can become dramatically important is when one considers string compactifications on a singular CY manifold. Here singular refers to a degeneration of the manifold in such a way that some classical geometric quantities (like the curvature) go to infinity. A simple example to keep in mind is a two-dimensional ice cream cone which is singular at its tip. In geometric terms this is due to the circle (or $S^1$) of the cone shrinking to zero size at the apex. Similarly, what happens in string theory is that a three-dimensional sphere ($S^3$) inside the six-dimensional CY can shrink to zero size in such a way that the space looks like a higher dimensional version of our cone. Classical string theory on such a space, called the conifold, is also singular in that calculable quantities become infinite. So strings, at least classically, do not cure the singularity of the geometry.

It has now been understood that the singularity of classical string theory is a result of not including in the theory a light particle that one obtains when one wraps a $D3$ brane around the $S^3$. Normally one would be justified in ignoring the effects of such a particle when $g_s$ is small since it would be very massive and give small contributions. But when the $S^3$ is of zero size, one has an additional massless particle in the theory whose effects can be ignored only at the cost of getting infinite answers.

Inclusion of the wrapped brane enables one to actually 'go through' the singularity smoothly to another (topologically distinct) CY manifold. Now, the topology of a manifold is something that is normally preserved in any smooth process – one cannot change the topology without cutting or tearing. But what we see here is that quantum nonperturbative effects in string theory have smoothened out the singularity that connects the two distinct CY manifolds and thus managed to mediate topology change! Topology changes in a mild form were present at the classical level. But the change is much more dramatic here at the quantum level (note 18). This topology change is a signature of the nonsmooth nature of quantum fluctuations of geometry that we mentioned in the introduction. In fact, it has been conjectured that all the topologically different compactifications on CY manifolds to four dimensions are continuously connected by a chain of such conifold transitions.

## String dualities and *M*-theory

The connectedness of various CY compactifications is part of a large mesh of quantum string 'dualities' that seems to relate different string backgrounds. The general conjecture is of the form: string theory $A$ on a manifold $M_1$ at large values of coupling $g_{s(A)}$ behaves like weakly interacting string theory $B(g_{s(B)} \sim \frac{1}{g_{s(A)}})$ on a different manifold $M_2$. Here $A$ and $B$ refer to one of the five different kinds of supersymmetric string theories that are known in ten dimensions (note 19). These equivalences between different theories typically relate nonperturbative objects (like *D*-branes) in one theory to perturbative ones (like the oscillator states of the fundamental string) in another. In fact, the detailed properties of *D*-branes, such as their matrix degrees of freedom, is crucial for the dualities to work. We refer to the articles by Sen in the references for a more detailed exposition.

Since $M_1$ and $M_2$ are totally distinct ($M_1$ might be a torus, for instance, while $M_2$ might be a more complicated geometrical object like a CY manifold), this makes classical geometry a very malleable concept. When quantum effects are very large, the theory behaves as if it were another string theory classically probing another geometry. Thus quantum effects are effectively incorporated, at least when large, into a change of geometry!

The culmination of this geometrization which provides the basis of the connectedness of all the known supersymmetric string theories, was in the discovery of *M*-theory. Unlike all the 10$d$ supersymmetric string theories, it lives in eleven dimensions. The size of the eleventh dimension turns out to be proportional to the strength of string interactions. The theory in flat eleven-dimensional space–time thus corresponds to infinitely coupled strings and has no small parameter in which to make a perturbative expansion. It is a fundamentally interacting quantum theory whose complete microscopic definition still eludes us. However, what we do know is that this unknown theory on different compactifications gives rise to all the known string theories. From the viewpoint of M-theory, many of the quantum dualities between string theories, mentioned in the previous paragraphs becomes a simple consequence of some geometrical symmetry. For example, *M*-theory on a vanishingly small two-dimensional torus ($S^1 \times S^1$) is the same as the Type IIB string theory in ten dimensions. The simple geometric symmetry in M-theory of exchanging the two circles of the torus translates into a very mysterious quantum strong-weak duality in IIB string theory where the coupling $g_s$ is exchanged for its inverse. This is a rather remarkable geometrization of quantum effects and points to the importance of understanding *M*-theory.

The general lesson is that we have a fundamental description of quantum gravity in *M*-theory. Geometry in a form we are familiar with emerges in limits of this theory. Some of these limits are very quantum domains but which nevertheless seem to have a geometric description. This is quite a surprise from the viewpoint of quantum gravity.

Dualities have understandably been very influential in shaping the way we think about string theory. But they have also brought tangible gains in terms of applications. One of the immediate spinoffs of the new nonperturbative understanding of string theory was in the description (in terms of *D*-branes) of certain kinds of black holes in string theory. The gains on this ground are described in the articles by S. Das and L. Susskind in the bibliography. We merely remark that it is now hopeful that string theory will be able to address the issue of singularities and associated puzzles in black hole physics.

## The gauge theory/geometry correspondence

One of the most exciting notions that has sprung from the study of dualities and, more importantly, the properties of *D*-branes has been that of a duality between open and closed string descriptions. This has given us, for the first time, a well-defined prescription for quantum gravity in a class of negatively curved backgrounds.

The idea goes back to the description of the modes on the *D*-brane in terms of open strings. At the same time we mentioned that these defects have a finite tension or energy per unit volume. This implies that they curve space–time. We can therefore have two descriptions of the region in the vicinity of a *D*-brane. One in terms of the curved geometry of space (a closed string description) and the other in terms of modes of an open string. Since these descriptions overlap, the conclusion is then inescapable that these are two completely different descriptions of one and the same system. This becomes even more striking when one takes a limit in which on the open string side only the lowest oscillation modes, namely that of massless gauge fields (and some partner fermions and scalars) survive. This simple field theory limit is then, by this reasoning, an equivalent description of the negatively curved space in the immediate vicinity of the *D*-brane. The space–time in the vicinity of a *Dp* brane is effectively a $p+2$ dimensional one in that the other directions are compact. In certain cases, such as the $p=3$ case, the noncompact part of this space–time has constant negative curvature and is what is known as Anti deSitter (AdS) space, in this case $AdS_5$.

We refer the reader to the article by S. Minwalla in the bibliography for a more detailed account of this correspondence. For our purposes it suffices to note that in the strongest form, this conjecture implies that the quantum theory of gravity on AdS space is defined by an ordinary nongravitational quantum field theory. A crucial point is that the field theory lives only on the $(p+1)$ dimensional world volume of the *D*-brane and is supposed to describe the quantum gravity of the effectively $(p+2)$ dimensional space–time (in the vicinity of the *D*-brane). In fact, it has been understood that the field theory really lives, in a precise sense, on the boundary of the space–time (see Figure 3). This is a realization of an idea called holography. The idea basically is that any theory of quantum gravity really possesses only the degrees of freedom of a local theory (such as a field theory) of one lower dimension.

Holography is quite a remarkable proposal. It says that the geometry of $d+1$ dimensional space–time is encoded in terms of a $d$ dimensional local theory. The local nature of Einstein's equations (or its quantum version) is completely hidden in a holographic formulation. In the specific case we have been discussing, the lower dimensional local theory involves nonabelian interactions, matrix valued fields in the simplest cases. We thus learn that there is some equivalent notion of quantum geometry that is described in terms of matrix degrees of freedom. In particular, even the classical geometry of Einstein's equations is supposed to be equivalently described by the dynamics of matrix fields whose size is infinite (note 20). Understanding this better is likely to provide the answers to many of the questions raised in the introduction.

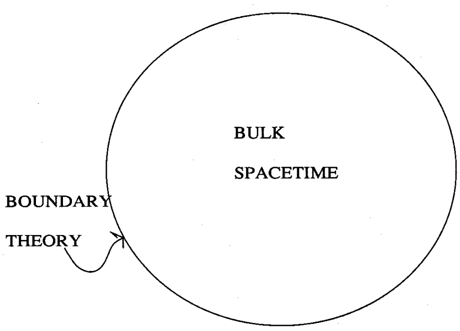


**Figure 3.** A schematic depiction of holography between gravity in the bulk of space–time (the disc) and a theory on the boundary (the circle).

One of the aspects of this correspondence is that information about the extra dimension of the bulk space–time is encoded in the energy scales of the boundary field theory. For instance, events in the bulk, far from the boundary are low energy processes in the boundary theory and similarly, events close to the boundary are reflected in high energy (or short distance) processes in the field theory. Thus local events in space–time can be mapped on to very nonlocal processes in the boundary description and vice versa. This is one ingredient that helps to capture the subtleties of quantum fluctuations of geometry in terms of a simple field theory description.

The intriguing nature of the open-closed string correspondence has provoked much study. Several geometric backgrounds and their holographic duals have been investigated. In some cases which involve the conifold geometry, M-theory seems to provide a new insight into the equivalence between open and closed strings. This is somewhat in the spirit of our discussion of the M-theoretic origin of the self-duality of Type IIB string theory. It is as if even the gauge theory/gravity correspondence can be geometrized.

There has also been recently much study of the geometry underlying open strings in the hope of understanding closed strings as somehow made up of open strings. This has brought in concepts from noncommutative geometry which seems to be very natural in the description of *D*-branes. All these new inputs might help us to understand this correspondence between open and closed strings better.

## Conclusions

The connection between geometry and string theory has many facets as we have seen. Our probings are not yet at an end. As the reader must have realized, quantum string theory is only slowly uncovering its new notions of geometry. We have not yet completely answered the questions raised in the introduction. But we have found many fascinating threads that will surely go into our picture of quantum gravity. That final picture may be nothing like what we imagine now, but whatever it is, it will change the way we view space–time.

## Notes

1. The action is a quantity which is roughly the difference between the kinetic and potential energies integrated over time. The *Principle of Least Action*, dating at least to Fermat and others in

the 17th century, states that classical trajectories are those that minimize the action.

2. We have in mind an infinite cylinder so that the band cannot be slipped off the circle.
3. This size is roughly the Compton wavelength of the string which is determined by the string's tension: $l_s = h/m_s c$, with $m_s^2 c/h$ being the string tension. This fundamental length scale of the theory is usually taken to be comparable in size to the Planck length.
4. We should add here that open strings, i.e. strings which have endpoints, see space–time in a way different from closed strings. Open strings by virtue of being open cannot rigidly wind around circles. Therefore there is no equivalent of the exchange of winding and momentum and hence no equivalence of large and small circles. This has important consequences which we will encounter later.
5. It is an old result in field theory that a massless spin-2 particle with the linearized gauge invariance of a graviton must have the interactions of Einstein's gravity.
6. Just as a particle sweeps out a 0 + 1 dimensional worldline as it moves in time, a string sweeps out a 1 + 1 dimensional worldsheet or world volume as it propagates in time.
7. More accurately the corrections are tiny at usual macroscopic length scales, but become appreciable at distances of order $10^{-33}$ cm.
8. A gauge field $A_m$ in four dimensions arises from $g_{mi}$, where $g$ is the higher dimensional metric and $i$ is an index labelling one of the compactified dimensions. Thus the simplest possible compactification from ten to four dimensions will have six photons or $U(1)$ gauge fields. More complicated compactifications can give rise to nonabelian gauge bosons.
9. This belief has been questioned of late due to the realization that experiments do not rule out extra dimensions perhaps as big as a millimetre. If indeed there are these 'large extra dimensions' then one should be able to see experimental signatures in accelerators.
10. They are so-called complex Kahler manifolds which admit metrics whose Ricci curvature $R_{i\bar{j}}$ vanishes. This ensures that the four dimensional theory preserves some supersymmetry.
11. IIA/B are two different varieties of supersymmetric string theories in ten dimensions.
12. For a CY manifold, the number of size parameters are in 1–1 correspondence with the number of two dimensional topologically nontrivial 'cycles' or submanifolds, while the shape parameters are in correspondence with a class of three-dimensional cycles. These numbers, which need not be equal, are often denoted by $h_{11}$ and $h_{21}$ respectively. These numbers capture, but do not completely characterize, the topology of the CY manifold. The Euler character which determined the number of generations is equal to $2(h_{11}–h_{21})$.
13. These are computations of the so-called Yukawa couplings (which give masses to fermions) in the four dimensional theory obtained by compactifying the IIA theory on $M$.
14. These are actually complex analytic (or in other words, holomorphic) maps from the worldsheet into the target CY space which are finite action nontrivial field configurations from the point of view of the 2d world volume theory and hence known as worldsheet instantons.
15. Ordinary or classical cohomology accounts for the gross topological features of a manifold like the number of nontrivial cycles of different dimensions.
16. The analogy to keep in mind for this perturbative string expansion is with the Feynman diagram expansion.
17. For instance, effects that will go as $(–1/g_s)$.
18. For instance $h_{11}$, $h_{21}$ mentioned in footnote 12 change in the course of this transition.
19. These go by the rather abstruse names of type IIA, type IIB, type I, heterotic SO(32) and heterotic $E_8 \times E_8$.
20. This has also been exploited in the reverse direction to study the so-called large $N$ limit of gauge theories using Einstein's equations.

The references below are more or less in increasing levels of sophistication and are just meant to give the reader a chance to explore more details of topics mentioned here.


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