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Combinatorial manifolds with complementarity

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Abstract. A simplicial complex is said to satisfy complementarity if exactly one of each complementary pair of nonempty vertex-sets constitutes a face of the complex.

We show that if a d-dimensional combinatorial manifold M with n vertices satisfies complementarity then $d = 0$, 2, 4, 8 or 16 with $n = 3d/2 + 3$ and $|M|$ is a "manifold like a projective plane". Arnoux and Marin had earlier proved the converse statement.

Keywords. Combinatorial manifolds; complementarity.

1. Introduction

Recall that a *simplicial complex* K is a collection of nonempty sets (sets of *vertices*) such that all nonempty subsets of a member of the collection are again members. A member of K with $i+1$ vertices is called an *i-face* (or simplex of dimension i). For $\sigma \in K$ $\mathrm{Lk}(\sigma)$ (= *link* of σ) $:= \{\gamma \in K; \gamma \cap \sigma = \emptyset, \gamma \cup \sigma \in K\}$. A simplicial complex may be thought of as a prescription for the construction of a topological space by pasting together geometric simplexes. The topological space thus obtained from a simplicial complex K is called a *polyhedron* and is denoted by $|K|$. Let K_1 and K_2 be two simplicial complexes. A map $f: |K_1| \to |K_2|$ is called PL if there are subdivisions K_1' and K_2' of K_1 and K_2 respectively such that $f: K_1' \to K_2'$ is simplicial. We write $|K_1| \approx |K_2|$ if $|K_1|$ and $|K_2|$ are PL homeomorphic. A simplicial complex K (respectively $|K|$) is called a *combinatorial d-manifold* (respectively PL *d-manifold*) if for every vertex v in K $Lk(v)$ is a $(d-1)$-dimensional combinatorial sphere.

In 1962, Eells and Kuiper [5] proved that a PL manifold M^d with PL Morse number $\mu(M^d) = 3$ has dimension $d = 0$, 2, 4, 8 or 16. If $d = 0$ M^d consists of three points. If $d = 2$ M^d is the real projective plane. For $d = 4$, 8 or 16, M^d is a simply connected cohomology projective plane over complex numbers, quaternions or Cayley numbers, respectively. Each of the manifolds of above type is called a *manifold like a projective plane*. This classification turned up in the 1987 paper [3] of Brehm and Kühnel on combinatorial manifolds with few vertices. Specifically, they proved that: Let M_n^d be a combinatorial d-manifold with n vertices,

(BK1) if $n < 3[d/2] + 3$ then $|M_n^d| \approx S^d$,
(BK2) if $n = 3(d/2) + 3$ and $|M_n^d| \not\approx S^d$ then $d = 2$, 4, 8 or 16 and $|M_n^d|$ must be a "manifold like a projective plane". Moreover for $d = 2$ $M_n^d = \mathbb{R}P_6^2$ and for $d = 4$ $M_n^d = \mathbb{C}P_9^2$.

It is classically known that there exists a unique (up to simplicial isomorphism) 6-vertex triangulation (denoted by $\mathbb{R}P^2_6$) of the real projective plane $\mathbb{R}P^2$. It is also known (see [2], [6] and [7]) that there exists a unique (up to simplicial isomorphism) 9-vertex triangulation (denoted by $\mathbb{C}P^2_9$) of the complex projective plane $\mathbb{C}P^2$.

Implicit in [3] is the result that $\mathbb{C}P^2_9$ satisfies complementarity. This result was made explicit by Arnoux and Marin [1] in 1991. More generally, they proved that any manifold as in (BK2) satisfies complementarity. In this article we prove the converse:

Theorem. *Let M^d_n be a combinatorial d-manifold with n vertices. If M^d_n satisfies complementarity then $d = 0$, 2, 4, 8 or 16 with $n = 3(d/2) + 3$ and $|M^d_n|$ is a "manifold like a projective plane".*

2. Preliminaries

Let K be a triangulation of the sphere S^{p-1} with n vertices. The *f-vector* of K is $f(K) := (f_0, \ldots, f_{p-1})$, where f_i is the number of i-faces in K. Thus $f_0 = n$ and $f_i \leqslant \binom{n}{i+1}$ for $1 \leqslant i \leqslant p-1$. Let $\mathbb{N}$ denote the non-negative integers, and define $H: \mathbb{N} \to \mathbb{N}$ as follows

$$H(m) = \begin{cases} 1 & \text{if } m = 0 \\ \sum_{i=0}^{p-1} f_i \binom{m-1}{i} & \text{if } m > 0. \end{cases} \tag{1}$$

Then there exists (see [8]) integers $h_0, \ldots, h_p$ such that

$$(1-x)^p \sum_{m=0}^{\infty} H(m) x^m = h_0 + h_1 x + \cdots + h_p x^p \tag{2}$$

is an identity in the formal power series ring $\mathbb{C}[[x]]$.

For $k \leqslant p < n-1$ (equating the coefficients of x^k from both sides of $(1+x)^{-(p-k+1)}(1+x)^n = (1+x)^{n+k-p-1}$) we get

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{p-j}{p-k} \binom{n}{j} = \binom{n+k-p-1}{k}. \tag{3}$$

By substituting $i - 1 = p$ and $l = p + 1 - k$ we get

$$\binom{n-l}{i-l} = \sum_{j=0}^{i-l} (-1)^{i-l-j} \binom{i-1-j}{l-1} \binom{n}{j} = \sum_{m=l}^{i} (-1)^{l-m} \binom{n}{i-m} \binom{m-1}{l-1}. \tag{4}$$

Then from (1) and (2) by using (4) we get (see [9])

$$h_i = \sum_{l=0}^{p} (-1)^{i-l} \binom{p-l}{p-i} f_{l-1}, \tag{5}$$

where we set $f_{-1} = 1$.

If $f_{j-1} = \binom{n}{j}$ for $1 \leqslant j \leqslant q \leqslant p$ then by (3) we have

$$h_i = \binom{n+i-p-1}{i} \quad \text{for } i \leqslant q. \tag{6}$$

The Dehn–Sommerville equations, which hold for any triangulation of the sphere S^{p-1}, are equivalent to the statement (see [9]):

$$h_i = h_{p-i} \quad 0 \leqslant i \leqslant p. \tag{7}$$

3. Proof of the theorem

Throughout, M is an n-vertex combinatorial d-manifold satisfying complementarity. It is trivial from the definition that, for $d=0$ M consists of three points, and since clearly there is no 1-manifold satisfies complementarity, we may take $d \geqslant 2$.

We shall repeatedly use the following obvious consequences of complementarity. Since no set of $\geqslant d+2$ vertices constitute a face, $n \leqslant 2d+3$ and every set of $\leqslant n-d-2$ vertices is a face. That is, for $i \leqslant n-d-3$, all i-faces occur in M. More generally the number of i-faces + the number of $(n-i-2)$-faces $= \binom{n}{i+1}$. As each vertex forms a 0-face, therefore $n > d+2$. Thus, $d+2 < n \leqslant 2d+3$.

Throughout this section we put $c = [d/2]$. Thus, $d = 2c-1$ or $2c$.

If F_i is the number of i-faces in M then we have:

$$\sum_{i=0}^{n-3} F_i = \begin{cases} F_0 + (F_1 + F_{2m-3}) + \cdots + (F_{m-2} + F_m) + F_{m-1} & \text{if } n = 2m, \\ F_0 + (F_1 + F_{2m-2}) + \cdots + (F_{m-1} + F_m) & \text{if } n = 2m+1 \end{cases}$$

$$= \begin{cases} \binom{2m}{1} + \binom{2m}{2} + \cdots + \binom{2m}{m-1} + \frac{1}{2}\binom{2m}{m} & \text{if } n = 2m, \\ \binom{2m+1}{1} + \binom{2m+2}{2} + \cdots + \binom{2m+1}{m} & \text{if } n = 2m+1 \end{cases}$$

$$= 2^{n-1} - 1,$$

which is an odd integer, where we set $F_i = 0$ for $i > d$. Therefore the Euler characteristic of $M = \Sigma_{i=0}^{n-3}(-1)^i F_i$ is odd.

If $n = d+3$ then (by (BK1)) M is a sphere.

If $n > d+3$ then all the i-faces occur in M for $i \leqslant n-d-3 \geqslant 1$. Therefore the link of any vertex in M is an $(n-1)$-vertex combinatorial $(d-1)$-sphere with f-vector satisfying: $f_i = \binom{n-1}{i+1}$ for $0 \leqslant i \leqslant n-d-4$. Hence by (6), the h-vector of this link satisfies $h_i = \binom{n-d-2+i}{i}$ for $0 \leqslant i \leqslant n-d-3$.

If $d = 2c$ then by (7) for $n > 3c+3$, we get $\binom{n-c-3}{c-1} = h_{c-1} = h_{c+1} = \binom{n-c-1}{c+1}$. Which gives $n = 2c+2$, contrary to our assumption in this case.

If $d = 2c-1$ then for $n \geqslant 3c+3$, we get $\binom{n-c-2}{c-1} = h_{c-1} = h_c = \binom{n-c-1}{c}$. Which gives $n = 2c+1$, a contradiction.

Thus, $n \leqslant 3c+3$ if d is even and $n < 3c+3$ if d is odd. Therefore, by (BK1) and (BK2) M is either a sphere or a "manifold like a projective plane". But as Euler characteristic of M is odd, M cannot be a sphere. This completes the proof.

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