Nonlinear Dynamics **20:** 247–265, 1999.

Non-Linear Vibration of a Travelling Beam Having an Intermediate Guide

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(Received: 31 December 1998; accepted: 2 June 1999)

Abstract. The free and forced responses of a non-linear travelling beam having an intermediate guide have been analysed. The guide is modelled as a purely elastic constraint with no inertia. While a suitably located guide increases the natural frequencies, the friction present in the guide-beam interface tends to destabilize the system. The presence of the guide reduces the vibration level by avoiding resonance conditions. The effect of the non-linear term is very sensitive to the location of the guide if the guide stiffness is small. It is suggested that the guide is placed near the antinode of the predominantly excited mode.

Keywords: Non-linear complex normal modes, divergence instability, eigenvalue-inclusion principle.

1. Introduction

The problem of vibration in many axially moving continuous systems, e.g., saw-bands, moving threadlines, has been considered over the last four decades [1]. For vibration analysis of these systems, modelled as travelling strings or beams, the well-known technique of the separation of variable cannot be used due to the presence of the gyroscopic term in the equation of motion. A complex normal-mode method has been developed for such gyroscopic systems [2, 3]. Both free and forced, linear responses of a travelling beam or a string have been reported. But at a high axial speed, especially near the critical speed [4] when the divergence instability occurs, the effects of non-linearities cannot be neglected [5]. Consequently, the non-linear-free vibration of a travelling beam has also been studied [6]. The near-resonance responses of a harmonically and/or parametrically excited travelling beam have also been obtained using the non-linear complex normal modes [7, 8].

In several systems like a capstan, reading-writing devices in a magnetic tape, etc., the travelling member is allowed to pass through intermediate guides. The friction present in the interface of the guide and the travelling member significantly alters the dynamics [9]. It has been shown [10] that guides with hydrodynamic action can reduce the vibration of a travelling beam. Although there exist active-control strategies [11] to reduce the vibration level, the inclusion of a guide as a passive controller can be very easily implemented.

In this paper, the non-linear vibration of a travelling beam passing through an intermediate guide is presented. The non-linear complex normal modes for the beam have been derived and are subsequently used to obtain the near-resonance response to a harmonic excitation. Both the stiffness of and the friction in the guide have been taken into consideration. It has been shown that the frictional force has a destabilizing effect. To reduce the frictional force between the guide and the beam, rollers can be used to maintain contact between them. It has also been shown that the position of the guide plays a crucial role in suppressing the vibration.

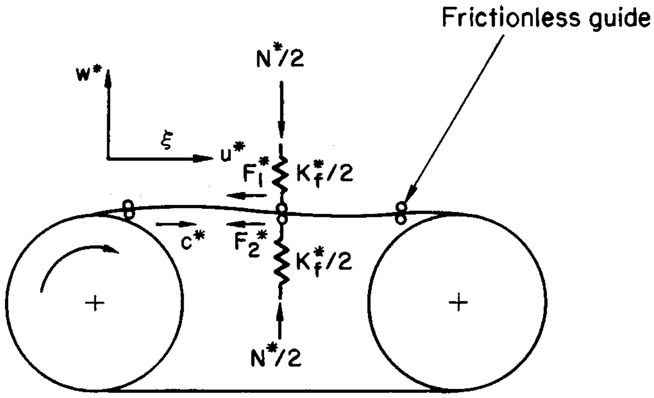

Figure 1. Schematic diagram of a travelling beam with an intermediate guide.

2. Theoretical Analysis

2.1. Equation of Motion

Consider a slender beam moving axially between two frictionless guides, as shown in Figure 1. The axial speed, c^*, is maintained by means of two rigidly mounted pulleys. An intermediate guide, consisting of two free rollers, with finite compliance is held in contact with the beam by means of an external force $N^*/2$, provided by the precompression of the compliant member. In addition, as indicated in the figure, the stiffness of the guides is assumed to be same in both directions.

Under the usual assumption of small longitudinal vibration (in comparison to the transverse vibration, i.e., assuming $u^* = O(w^{*^2})$), the non-linear equations of motion for the coupled vibration in the axial and transverse directions are [5]

$$\rho A\left[\frac{\partial^2 u^*}{\partial t^2} + 2c^*\frac{\partial^2 u^*}{\partial\xi\partial t} + c^{*^2}\frac{\partial^2 u^*}{\partial\xi^2}\right] - EA\frac{\partial^2 u^*}{\partial\xi^2}$$
$$= (EA - T_0^*)\frac{\partial w^*}{\partial\xi}\frac{\partial^2 w^*}{\partial\xi^2} - (F_1^* + F_2^*)\delta(\xi - d^*) \tag{1}$$

and

$$\rho A\left[\frac{\partial^2 w^*}{\partial t^2} + 2c^*\frac{\partial^2 w^*}{\partial\xi\partial t} + c^{*^2}\frac{\partial^2 w^*}{\partial\xi^2}\right] - T_0^*\frac{\partial^2 w^*}{\partial\xi^2} + EI_z\frac{\partial^4 w^*}{\partial\xi^4}$$
$$= (EA - T_0^*)\frac{\partial}{\partial\xi}\left[\frac{\partial u^*}{\partial\xi}\frac{\partial w^*}{\partial\xi} + \frac{1}{2}\left(\frac{\partial w^*}{\partial\xi}\right)^3\right] - K_f^* w^*\delta(\xi - d^*), \tag{2}$$

where

$$F_1^* = \mu\left[\frac{N^*}{2} + \frac{K_f^*}{2}w^*(d^*, t)\right], \tag{3}$$

$$F_1^* = \mu\left[\frac{N^*}{2} - \frac{K_f^*}{2}w^*(d^*, t)\right], \tag{4}$$

and $\delta(x)$ is the Dirac δ-function. The other symbols are explained in Appendix I.

The boundary conditions are obtained by neglecting the small curvature outside the frictionless guides. It is well known [12-16] that the dynamics of the two spans of a band-wheel system are not independent and it is the end curvature which is responsible for the coupling. However, the coupling becomes negligible for the high initial tension or large pulley radius. The end-curvature and hence the coupling becomes small enough in the presence of the frictionless guides at the end. Consequently, the boundary conditions can be written as

$$u^*(0,t) = u^*(l,t) = 0, \tag{5}$$

$$w^*(0,t) = w^*(l,t) = 0, \tag{6}$$

and

$$\frac{\partial^2 w^*(0,t)}{\partial \xi^2} = \frac{\partial^2 w^*(l,t)}{\partial \xi^2} = 0. \tag{7}$$

Using the following non-dimensional parameters,

$$\begin{aligned}
u &= u^*/l, \quad w = w^*/(l\gamma^2), \quad x = \xi/l, \quad \tau = (E/\rho)^{1/2}\gamma t/l,\\
c &= c^*(E/\rho)^{-1/2}/\gamma, \quad r^2 = I_z/A, \quad \gamma = r/l, \quad T_0 = T_0^*/(EA\gamma^2),\\
F_i &= F_i^*/(EA\gamma^2), \quad i = 1, 2,\\
N &= N^*/(EA\gamma^2), \quad K_f = K_f^* l/(EA), \quad d = d^*/l.
\end{aligned}$$

Equations (1–7) can be written, respectively, as

$$\begin{aligned}
&\left[\frac{\partial^2 u}{\partial \tau^2} + 2c\frac{\partial^2 u}{\partial x \partial \tau} + c^2\frac{\partial^2 u}{\partial x^2}\right] - \frac{1}{\gamma^2}\frac{\partial^2 u}{\partial x^2}\\
&\qquad = \gamma^2(1-\gamma^2 T_0)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - (F_1 + F_2)\delta(x-d),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\frac{\partial^2 w}{\partial \tau^2} + 2c\frac{\partial^2 w}{\partial x \partial \tau} + (c^2 - T_0)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial x^4}\\
&\qquad = (1-\gamma^2 T_0)\frac{\partial}{\partial x}\left[\frac{1}{\gamma^2}\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\gamma^2}{2}(\frac{\partial w}{\partial x})^3\right] - K_f w\delta(x-d),
\end{aligned} \tag{9}$$

$$F_1 = \mu\left[\frac{N}{2} + \frac{K_f}{2}w(d,\tau)\right], \tag{10}$$

$$F_2 = \mu\left[\frac{N}{2} - \frac{K_f}{2}w(d,\tau)\right], \tag{11}$$

$$u(0,\tau) = u(1,\tau) = 0, \tag{12}$$

$$w(0,\tau) = w(1,\tau) = 0, \tag{13}$$

and

$$\frac{\partial^2 w(0,\tau)}{\partial x^2} = \frac{\partial^2 w(1,\tau)}{\partial x^2} = 0. \tag{14}$$

For a small value of γ (i.e., $\gamma \ll 1$), the longitudinal inertia in Equation (8) can be neglected and the equation becomes

$$-\frac{1}{\gamma^2}\frac{\partial^2 u}{\partial x^2} = \gamma^2 \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - (F_1 + F_2)\delta(x-d),$$

which, when integrated twice with respect to x, results in

$$u(x,\tau) = -\frac{\gamma^4}{2}\int_0^x \left(\frac{\partial w}{\partial x_1}\right)^2 \mathrm{d}x_1 + \gamma^2(F_1+F_2)\int_0^x H(x_1-d)\mathrm{d}x_1 + xf_1(\tau) + f_2(\tau), \quad (15)$$

where

$$\begin{aligned} H(x-d) &= 0; \quad x < d, \\ &= 1; \quad x \geq d. \end{aligned}$$

The unknown constants of integration $f_1(\tau)$ and $f_2(\tau)$ can be obtained by using Equation (12), as

$$f_2(\tau) = 0, \quad (16)$$

and

$$f_1(\tau) = \frac{\gamma^4}{2}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x - \gamma^2(F_1+F_2)(1-d). \quad (17)$$

Now combining Equations (10), (11), (15–17), and (9), the equation of motion for the transverse vibration can be written in the following simplified form:

$$\begin{aligned} &\frac{\partial^2 w}{\partial \tau^2} + 2c\frac{\partial^2 w}{\partial x \partial \tau} + [c^2 - \{T_0 - \mu N(1-d) + \mu N H(x-d)\}] \\ &\quad \times \frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial x^4} - \left(\mu N \frac{\partial w}{\partial x} - K_f w\right)\delta(x-d) \\ &\quad = \varepsilon\left[\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2}, \end{aligned} \quad (18)$$

where ε $(= \gamma^2/2)$ is a small parameter, i.e., $\varepsilon \ll 1$. This equation can also be written in the familiar state-space form [3] as

$$\mathbf{A}\frac{\partial \mathbf{W}}{\partial \tau} + \mathbf{B}\mathbf{W} = \varepsilon\mathbf{N}, \quad (19)$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}, \quad \mathbf{W} = \begin{Bmatrix} \frac{\partial w}{\partial \tau} \\ w \end{Bmatrix}, \\ K &\equiv (c^2 - T_0)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4} - \mu N H(x-d)\frac{\partial^2}{\partial x^2} - \delta(x-d)\left(\mu N\frac{\partial}{\partial x} - K_f\right), \\ G &\equiv 2c\frac{\partial}{\partial x}, \end{aligned}$$

I as the identity operator and

$$\mathbf{N} = \left\{ \left(\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x \right) \frac{\partial^2 w}{\partial x^2}, 0 \right\}^T . \tag{20}$$

It is not difficult to verify that K is a self-adjoint and positive definite operator. This fact is essential to obtain the orthogonality relations between complex normal modes which are presented in the next section.

In order to facilitate computation, the non-analytic functions involved in Equation (18) can be eliminated by writing the equation separately for two domains, viz., $0 < x_1 \leq d$ and $d < x_2 < 1$ as

$$\begin{aligned} &\frac{\partial^2 w_j}{\partial \tau^2} + 2c \frac{\partial^2 w_j}{\partial x_j \partial \tau} + [c^2 - T_j] \frac{\partial^2 w_j}{\partial x_j^2} + \frac{\partial^4 w_j}{\partial x_j^4} \\ &\qquad = \varepsilon \left[\int_0^d (\frac{\partial w_1}{\partial x_1})^2 \, \mathrm{d}x_1 + \int_d^1 (\frac{\partial w_2}{\partial x_2})^2 \, \mathrm{d}x_2 \right] \frac{\partial^2 w_j}{\partial x_j^2}; \quad j = 1, 2, \end{aligned} \tag{21}$$

where w_j is the transverse displacement in the jth domain and

$$\begin{aligned} T_1 &= T_0 - \mu N(1 - d), \\ T_2 &= T_0 + \mu N d \end{aligned}$$

or

$$\Delta T = T_2 - T_1 = \mu N.$$

The matching conditions at $x = d$ can now be written as

$$\left. \begin{aligned} &w_1(d, \tau) = w_2(d, \tau), \\ &\frac{\partial w_1}{\partial x_1}(d, \tau) = \frac{\partial w_2}{\partial x_2}(d, \tau), \\ &\frac{\partial^2 w_1}{\partial x_1^2}(d, \tau) = \frac{\partial^2 w_2}{\partial x_2^2}(d, \tau), \\ &\frac{\partial^3 w_1}{\partial x_1^3}(d, \tau) - \frac{\partial^3 w_2}{\partial x_2^3}(d, \tau) + \Delta T \frac{\partial w_1}{\partial x_1}(d, \tau) - K_f w_1(d, \tau) = 0. \end{aligned} \right\} \tag{22}$$

The non-linear response of the beam can be studied by solving Equations (21) and (22). Since the non-linear term can be taken as a small perturbation to the linear equation of motion, the response of a linear beam will be discussed first and then the response of the non-linear beam.

2.2. Free and Forced Responses of the Linear System

In this section, the free and forced responses of a linear travelling beam (i.e., with $\varepsilon = 0$ in Equation (19)) are discussed. It is well known that no stationary mode shape exists for a travelling beam, but the harmonic oscillation is still possible at some frequencies, known as the 'natural frequencies'. Considering the complex normal modes [3], the natural frequencies and the complex mode shapes can be obtained. The response of the beam $w(x, \tau)$, at any one of the natural frequencies ω^l can be written as

$$\mathbf{W}(x, \tau) = \frac{a}{2}\Phi(x)\,\mathrm{e}^{i\omega^l \tau} + \frac{\overline{a}}{2}\overline{\Phi}(x)\,\mathrm{e}^{-i\omega^l \tau}, \tag{23}$$

with $\Phi(x) = \begin{Bmatrix} i\omega^l \phi \\ \phi \end{Bmatrix}$, where $\phi(x)$ is the complex normal mode shape. In Equation (23), the overbar denotes the complex conjugate. Substituting Equation (23) into Equation (19) (with $\varepsilon = 0$) and equating the coefficients of $\mathrm{e}^{i\omega^l \tau}$ and $\mathrm{e}^{-i\omega^l \tau}$ separately from both sides, one gets

$$i\omega^l \mathbf{A}\Phi + \mathbf{B}\Phi = \mathbf{0} \tag{24}$$

and its complex conjugate, respectively. It is to be pointed out that the partial derivatives appearing in **A** and **B** (see Equation (19)) are replaced by total derivatives. Equation (24) is now solved together with the boundary conditions to get different sets of values of ω^l and ϕ. The values of the nth set are called the nth natural frequency and mode shape, and will be denoted by ω_n^l and ϕ_n, respectively.

It is to be noted that the above solution is to be obtained numerically. Instead of solving Equation (24), which contains several non-analytic functions, we break it in two domains, as explained in Section 2.1. Thus Equation (24) reduces to

$$-(\omega^l)^2 \phi_j + 2ic\omega^l \frac{\mathrm{d}\phi_j}{\mathrm{d}x_j} + (c^2 - T_j)\frac{\mathrm{d}^2\phi_j}{\mathrm{d}x_j^2} + \frac{\mathrm{d}^4\phi_j}{\mathrm{d}x_j^4} = 0, \tag{25}$$

with $j = 1, 2$. The matching conditions at $x = d$ are still given by Equation (22) with the partial derivatives replaced by the total derivatives and w replaced by ϕ. Assuming the solution in the form

$$\phi_j(x_j) = \sum_{k=1}^{4} \alpha_{jk}\,\mathrm{e}^{p_{jk} x_j}, \qquad j = 1, 2$$

and applying the boundary conditions, one obtains

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{12} \end{Bmatrix} = [K_1] \begin{Bmatrix} \alpha_{13} \\ \alpha_{14} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \alpha_{21} \\ \alpha_{22} \end{Bmatrix} = [K_2] \begin{Bmatrix} \alpha_{23} \\ \alpha_{24} \end{Bmatrix}.$$

The four unknowns, α_{13}, α_{14}, α_{23} and α_{24}, are now substituted in the matching conditions to obtain a relation like

$$[K_0] \begin{Bmatrix} \alpha_{13} \\ \alpha_{14} \\ \alpha_{23} \\ \alpha_{24} \end{Bmatrix} = 0. \tag{26}$$

The existence of a non-trivial solution implies

$$\det[K_0] = 0. \tag{27}$$

The values of ω^l and p_{jk}'s ($j, k = 1, 2, 3, 4$) are obtaind numerically by solving Equations (25) and (27). It is to be noted that the matrix $[K_0]$ is a complex one. Hence, both the real and imaginary parts of the determinant must vanish simultaneously.

Considering the symmetric and antisymmetric nature of the matrices **A** and **B**, respectively, the following orthogonality relations amongst the complex modes ϕ_n's and their conjugates are easily obtained in terms of the following complex inner products:

$$\int_0^1 \Phi_m^T \mathbf{A} \Phi_n \, \mathrm{d}x = 0 \quad \text{for all } m \text{ and } n, \tag{28}$$

and

$$\int_0^1 \overline{\Phi}_m^T \mathbf{A} \Phi_n \, \mathrm{d}x = 0 \quad \text{for all } m \neq n. \tag{29}$$

The above normal modes are used to derive the response of such a beam when an external force $f(x, \tau)$ is applied. In the following, the steady-state response of the beam to a harmonic excitation, i.e., $f(x, \tau) = f(x) \cos \Omega\tau$, is derived. The equation of motion is

$$\mathbf{A} \frac{\partial \mathbf{W}}{\partial \tau} + \mathbf{B}\mathbf{W} = \mathbf{f}, \tag{30}$$

with

$$\mathbf{f} = \{f(x, \tau),\ 0\}^T. \tag{31}$$

The steady-state response $w(x, \tau)$ is assumed to be

$$\mathbf{W}(x, \tau) = \frac{1}{2} \mathbf{P}(x)\, \mathrm{e}^{i\Omega\tau} + \text{c.c.}, \tag{32}$$

where c.c. denotes the complex conjugate of the previous term. Using the modal expansion theorem, $\mathbf{P}(x)$ can be expanded as

$$\mathbf{P}(x) = \sum_{n=1}^{\infty} (p_n \Phi_n + q_n \overline{\Phi}_n). \tag{33}$$

After substituting Equations (32) and (33) into Equation (30), the coefficients of $\mathrm{e}^{i\Omega\tau}$ from both sides are equated. Then the orthogonality relations (28) and (29) are used to obtain p_n and q_n as

$$p_n = \tilde{p}_n\, \mathrm{e}^{i\theta_{pn}} = \frac{2 \int_0^1 \overline{\Phi}_n^T \mathbf{f}_1 \, \mathrm{d}x}{i(\Omega - \omega_n^l) \int_0^1 \overline{\Phi}_n^T \mathbf{A} \Phi_n \, \mathrm{d}x} \tag{34}$$

and

$$q_n = \tilde{q}_n\, \mathrm{e}^{i\theta_{qn}} = \frac{2 \int_0^1 \Phi_n^T \mathbf{f}_1 \, \mathrm{d}x}{i(\Omega + \omega_n^l) \int_0^1 \overline{\Phi}_n^T \mathbf{A} \Phi_n \, \mathrm{d}x}, \tag{35}$$

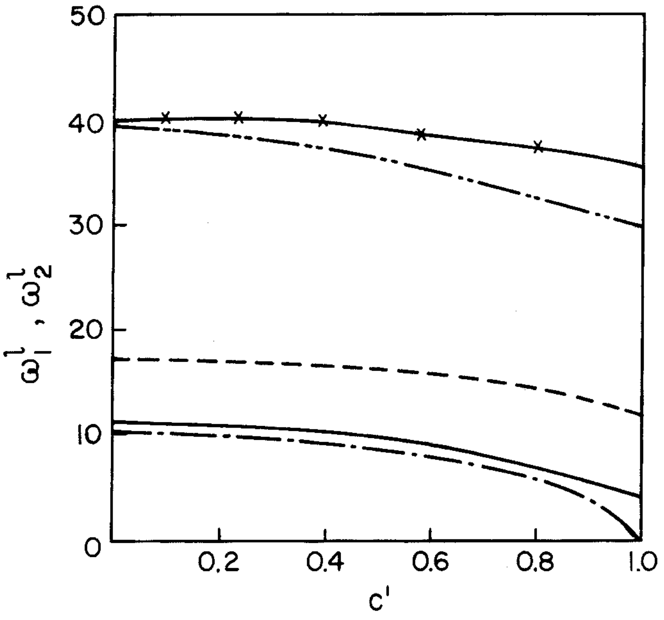

Figure 2. Variation of ω_1^l and ω_2^l with c'. $d = 0.5$, $\Delta T = 0$, —·—: ω_1^l with no guide; —: ω_2^l, $K_f = 10$, - - - : ω_1^l, $K_f = 100$, — - - —: ω_1^l, $K_f = 1000$, —×—: ω_2^l, $K_f = 0, 10, 100, 1000$.

with $\mathbf{f}_1 = \{f(x)/2,\ 0\}^T$. It is readily seen that at or near resonance, the magnitude of p_n will be very high. Hence, the aim of controlling the vibration is to maintain p_n at a low value.

2.2.1. *Numerical Results and Discussion*

Numerical results obtained from the linear analysis (presented in the previous section) are now discussed. All the results, unless otherwise mentioned, belong to a travelling beam having a tension $T_2 = 1$. The important physical parameters which control the performance of the system are

1. the speed of the beam, expressed as a parameter $c' = c/(C_{cr})_1$, where $(C_{cr})_1 = \sqrt{\pi^2 + T_2}$,
2. the location of the guide, d,
3. the stiffness of the guide, K_f; and
4. the coefficient of friction between the beam and the guide, μ.

It is well known that an increase in the axial speed reduces the natural frequency of vibration of a beam without any intermediate guide [4]. The same effect is also observed, even in the presence of a guide. In Figure 2, the first and second natural frequencies are plotted against the axial speed for various values of K_f. For the chosen guide location, i.e., $d = 0.5$, the second natural frequency, ω_2^l, as expected, is insensitive to any change in the stiffness K_f. However, the first natural frequency ω_1^l and, consequently, the first critical speed considerably increases with increasing K_f. Thus, for large values of K_f (i.e., when the guide behaves like a rigid support) the first natural frequency of the guided beam tends towards the second natural frequency of the unguided beam.

Figures 3a and 3b show the variation of the natural frequencies with the guide location. It is observed that the first natural frequency ω_1^l is maximum with $d = 0.5$, whereas the value of ω_2^l attains a maximum when $d = 0.25$ and $d = 0.75$. For a stationary simply-supported beam, the point $x = 0.5$ corresponds to the antinode of the first mode and the node of the

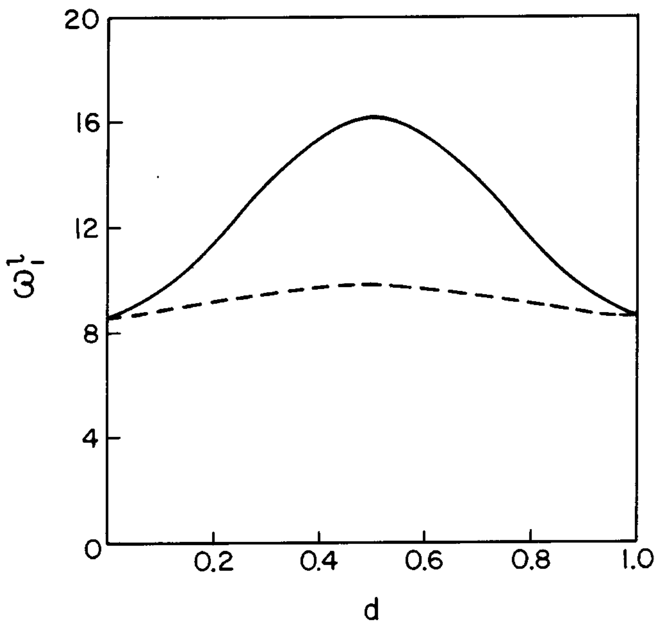

Figure 3a. Variation of ω_1^l with the guide-location. $c' = 0.5$, $\Delta T = 0$, - - -: $K_f = 10$, —: $K_f = 100$.

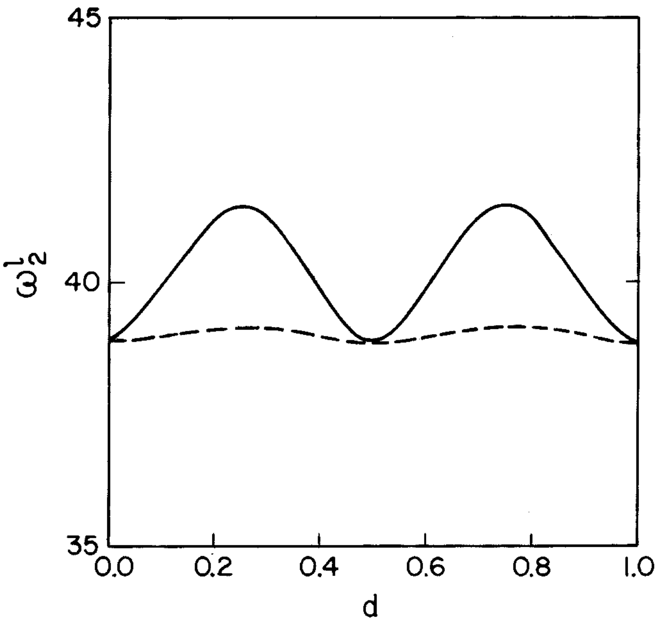

Figure 3b. Variation of ω_2^l with the guide-location. $c' = 0.5$, $\Delta T = 0$, - - -: $K_f = 10$, —: $K_f = 100$.

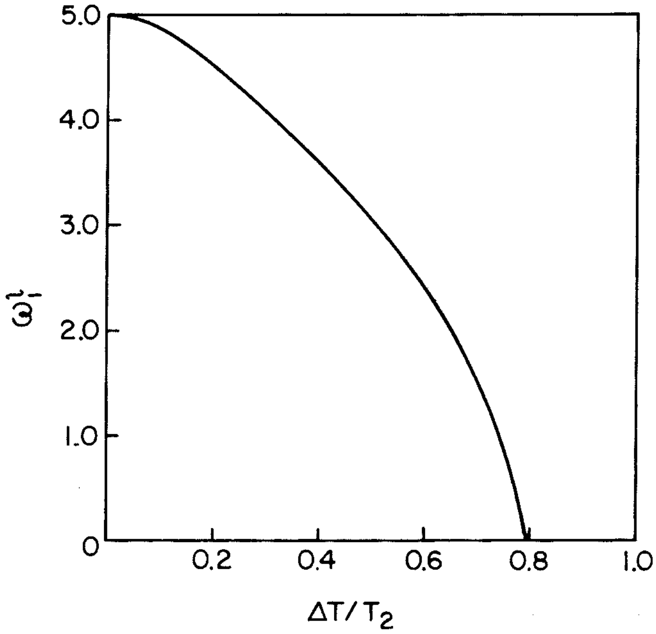

Figure 4. Variation of ω_1^l with ΔT. $c' = 0.95$, $T_2 = 10.0$, $d = 0.5$, $K_f = 10$.

second mode. Although the concepts of nodes and antinodes cannot be used for a travelling beam, the response envelope, however, becomes maximum or minimum at $x = 0.5$ when the beam vibrates at its first or second natural frequency, respectively [7]. Similarly, for the second natural frequency, the response envelope reaches a maximum value at $x = 0.25$ and $x = 0.75$. Thus, from this example, it is confirmed, much satisfying our expectation, that the guide makes the system stiff if it is placed where the response envelope reaches its maximum. Attention may be drawn to one of the general results for a constrained flexible system, known as the ‘eigenvalue inclusion principle’. For a gyroscopic system, it has been shown [17] that for a beam or string having a stiffness constraint, the nth natural frequency of a constrained system (say $\Omega_n^{(c)}$) satisfies the following inequalities

$$\omega_n^{(nc)} \leq \Omega_n^{(c)} \leq \omega_{n+1}^{(nc)},$$

where $\omega_n^{(nc)}$ is the nth natural frequency of the unconstrained system. Numerical results obtained for the guided travelling beam are in conformity with the above result.

Figure 4 shows the variation of ω_1^l with the coefficient of the guide friction, μ. It is seen that the friction reduces the stiffness of the system. From Equation (21), it can be concluded that the friction introduces a compressive load in the first span ($0 < x < d$). This compressive load may become so high that the span may undergo divergence instability, as shown in Figure 4. Thus, in addition to damaging the beam surface, the presence of friction may cause instability in the system.

No ‘frequency loci veering’ has been observed for a guided travelling beam. This supports the observation reported in [13]. The non-cyclic disordered beam, discussed in this reference, corresponds to the present model with $K_f \to \infty$ and $d = 0.5$. However, the phenomena of ‘frequency loci veering’ and ‘mode localization’ have been observed for a guided travelling string [9].

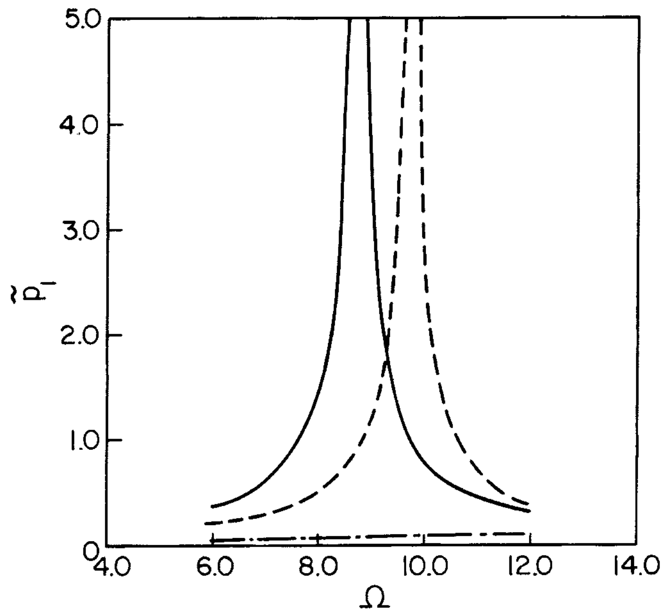

Figure 5. Amplitude of the frequency response of the first linear mode. $F_0 = 10$, $\Delta T = 0$, $c' = 0.5$, $d = 0.5$; —: no guide ($K_f = 0$), - - -: $K_f = 10$, —·—: $K_f = 100$.

The effect of the intermediate guide on the steady-state harmonic response is shown in Figure 5. Only the magnitude of p_1, when the beam is excited by a point harmonic load $f(x) = F_0\delta(x - x_0)$ at $x_0 = 1/3$ with the excitation frequency $\Omega \approx \omega_1^l$, is plotted. By changing the natural frequency, the guide helps to avoid the resonance at a particular speed. As expected, the near-resonance response decreases with the stiffness of the guide.

Figures 6a and 6b show the effects of the guide-location on the steady-state responses. As seen from the figures, the guide, when suitably placed, can attenuate the response level of a resonantly excited beam. As the application of the guide shifts the natural frequency, the resonance condition can be avoided. If the guide is flexible, then its optimal location corresponds to the position where the response envelope attains a maximum and depends on the mode which is resonantly excited. In other words, to minimize the vibration, the guide is placed at a location where the response of the uncontrolled system is maximum. However, as the stiffness of the guide increases, the location of the guide looses its importance as long as it does not coincide with any of the nodal points of the response envelope. The shift of the natural frequency, after placing the stiff guide anywhere within two successive nodal points, is so large that the response amplitude becomes practically insensitive to the guide-location.

2.3. Effects of the Non-Linearity

In this section the effects of the non-linearity on the free and near-resonance forced responses are presented. It may be mentioned that the relationship between different linear natural frequencies depends on the various system parameters. Some typical relationships may give rise to internal and combinatorial resonances which are not considered in this work. Thus, only such combinations of parameters are taken where all the natural frequencies are distinct and no special relation between them exists. For such a system, the free and near-resonance forced harmonic responses are analysed.

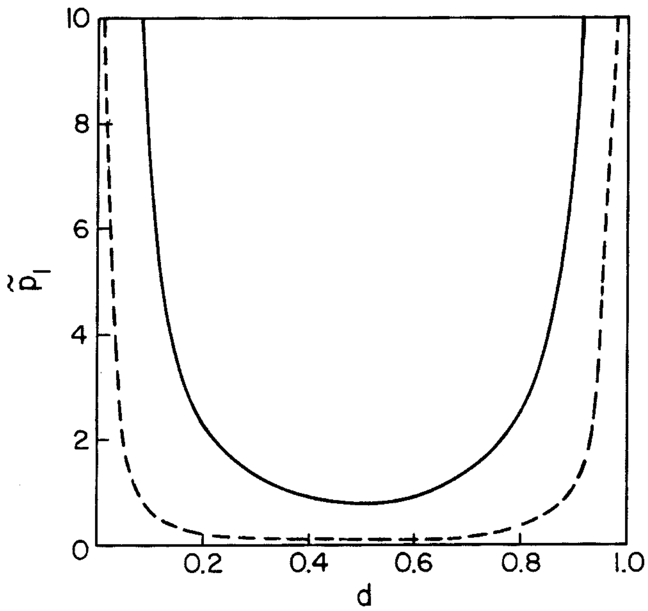

Figure 6a. Effect of the guide-location on $\tilde{p}_1$. $c' = 0.5$, $\Delta T = 0$, $\Omega = 8.7$, $F_0 = 10$; —: $K_f = 10$, - - -: $K_f = 100$.

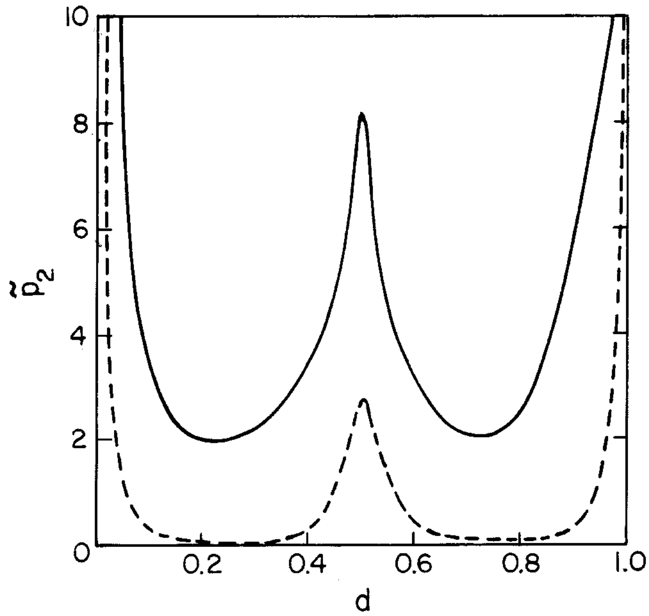

Figure 6b. Effect of the guide-location on $\tilde{p}_2$. $c' = 0.5$, $\Delta T = 0$, $\Omega = 39.0$, $F_0 = 10$; —: $K_f = 10$, - - -: $K_f = 100$.

In the absence of any internal resonance, the harmonic response of a free travelling beam has already been presented in reference [7]. A concept of complex non-linear normal mode has been developed. For the free vibration in the nth non-linear normal mode, the response is assumed as

$$\mathbf{W}(x,\tau) = \frac{a}{2}\Psi_n(x)\,\mathrm{e}^{i\omega_n\tau} + \frac{\overline{a}}{2}\overline{\Psi}_n(x)\,\mathrm{e}^{-i\omega_n\tau}, \tag{36}$$

where $\Psi_n = \begin{Bmatrix} i\omega_n\psi_n \\ \psi_n \end{Bmatrix}$, with ψ_n and ω_n as the nth non-linear mode shape and the corresponding frequency, respectively. Both these quantities, however, are amplitude-dependent. Assuming small non-linearity, ω_n and Ψ_n are expanded, respectively, as

$$\omega_n = \omega_n^l + \varepsilon\beta_1^{(n)} + \cdots, \tag{37}$$

$$\Psi_n = \Phi_n + \varepsilon\Delta_1 + \cdots, \tag{38}$$

where $\Delta_1^{(n)}$ does not contain Φ_n. Now Equation (35) is substituted into Equation (19). Thereafter, balancing the harmonics (i.e., equating the coefficients of $\mathrm{e}^{i\omega_n\tau}$ and $\mathrm{e}^{-i\omega_n\tau}$ separately), the following equation and its complex conjugate are obtained:

$$i\omega_n\frac{a}{2}\mathbf{A}\Psi_n + \frac{a}{2}\mathbf{B}\Psi_n = \varepsilon\mathbf{N}_\psi, \tag{39}$$

where

$$\mathbf{N}_\psi = \left\{\frac{a^2\overline{a}}{8}\left[\left(\int_0^1\left(\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right)^2\mathrm{d}x\right)\frac{\mathrm{d}^2\overline{\psi}_n}{\mathrm{d}x^2} + 2\left(\int_0^1\frac{\mathrm{d}\psi_n}{\mathrm{d}x}\frac{\mathrm{d}\overline{\psi}_n}{\mathrm{d}x}\,\mathrm{d}x\right)\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2}\right], 0\right\}^T. \tag{40}$$

Using expansions (37) and (38) together with the orthogonality relations (26–29), a perturbation technique yields

$$\beta_1^{(n)} = -\frac{\omega_n^l}{4}a\overline{a}\frac{\lambda(\overline{\phi}_n,\phi_n)}{t_n}, \tag{41}$$

$$\Delta_1 = a\overline{a}\left[\sum_{m\neq n} g_m\Phi_m + \sum_{m=1}^{\infty} h_m\overline{\Phi}_m\right], \tag{42}$$

where

$$g_m = -\frac{1}{4}\frac{\omega_m^l}{\omega_n^l - \omega_m^l}\frac{\lambda(\overline{\phi}_m,\,\phi_n)}{t_n}, \quad m \neq n,$$

$$h_m = \frac{1}{4}\frac{\omega_m^l}{\omega_n^l + \omega_m^l}\frac{\lambda(\phi_m\,,\phi_n)}{t_n}, \quad m = 1, 2, 3, \ldots,$$

$$t_n = \int_0^1 \overline{\Phi}_n^T\mathbf{A}\Phi_n\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots,$$

and

$$\lambda(\phi_m, \phi_n) = 2\left(\int_0^1 \phi_m \frac{\mathrm{d}^2\phi_n}{\mathrm{d}^2 x}\,\mathrm{d}x\right)\left(\int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\frac{\mathrm{d}\overline{\phi}_n}{\mathrm{d}x}\,\mathrm{d}x\right) + \left(\int_0^1 \phi_m \frac{\mathrm{d}^2\overline{\phi}_n}{\mathrm{d}x^2}\,\mathrm{d}x\right)\left(\int_0^1 \left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right).$$

The above-mentioned non-linear normal modes can be used to obtain the near-resonance harmonic response of the beam when excited by an external force $f(x)\cos\Omega\tau$, i.e., with the equation of motion

$$\mathbf{A}\frac{\partial \mathbf{W}}{\partial \tau} + \mathbf{BW} - \varepsilon\mathbf{N} = \mathbf{f}, \tag{43}$$

where $\mathbf{f} = \{f(x)\cos\Omega\tau, 0\}^T$.

The principal harmonic response of the beam can be assumed as

$$\mathbf{W}(x,\tau) = \frac{1}{2}(\mathbf{\Lambda}(x)\,\mathrm{e}^{i\Omega\tau} + \overline{\mathbf{\Lambda}}(x)\,\mathrm{e}^{-i\Omega\tau}). \tag{44}$$

For the near-resonance excitation $\Omega \approx \omega_n^l$, the response is approximated as [7]

$$\mathbf{\Lambda} = a_n\Psi_n + \sum_{m\neq n}^{\infty} a'_m\Psi_m + \sum_{m=1}^{\infty} b'_m\overline{\Psi}_m + o(\varepsilon^2), \tag{45}$$

where $a'_m = \varepsilon a_m$, $b'_m = \varepsilon b_m$. Now Equations (44) and (45) are substituted into Equation (43) and the coefficients of $\mathrm{e}^{i\Omega\tau}$ are equated from both sides. Then using the orthogonality relations (28) and (29), the following equations [7] are obtained, after neglecting terms $o(\varepsilon^2)$:

$$a_n = \frac{2\int_0^1 \overline{\Psi}_n^T \mathbf{f}_1\,\mathrm{d}x}{i(\Omega - \omega_n)\int_0^1 \overline{\Psi}_n^T \mathbf{A}\Psi_n\,\mathrm{d}x}, \tag{46}$$

$$\varepsilon a_m = a'_m = \frac{2\int_0^1 \overline{\Phi}_m^T \mathbf{f}_1\,\mathrm{d}x}{i(\Omega - \omega_m^l)\int_0^1 \overline{\Phi}_m^T \mathbf{A}\Phi_m\,\mathrm{d}x}; \quad m \neq n$$

and

$$\varepsilon b_m = b'_m = \frac{2\int_0^1 \Phi_m^T \mathbf{f}_1\,\mathrm{d}x}{i(\Omega + \omega_m^l)\int_0^1 \overline{\Phi}_m^T \mathbf{A}\Phi_m\,\mathrm{d}x}; \quad m = 1, 2\ldots,$$

where $\mathbf{f}_1 = \{f(x)/2, 0\}^T$. The complex cubic equation (46) can be solved numerically. Assuming $a_n = \tilde{a}_n\,\mathrm{e}^{i\theta_n}$, Equation (46), for a point load $f(x) = F_0\delta(x - x_0)$, takes the form

$$A^6 + \left[\frac{2\omega_n^l(1-r_1)}{S} - \nu\left((M_1^R)^2 + (M_1^I)^2\right)\right]A^4 + \left[\frac{(\omega_n^l)^2(1-r_1)^2}{S^2} - 2\nu(M_1^R Q_1^R + M_1^I Q_1^I)\right]A^2 - \nu[(Q_1^R)^2 + (Q_1^I)^2] = 0, \tag{47}$$

where

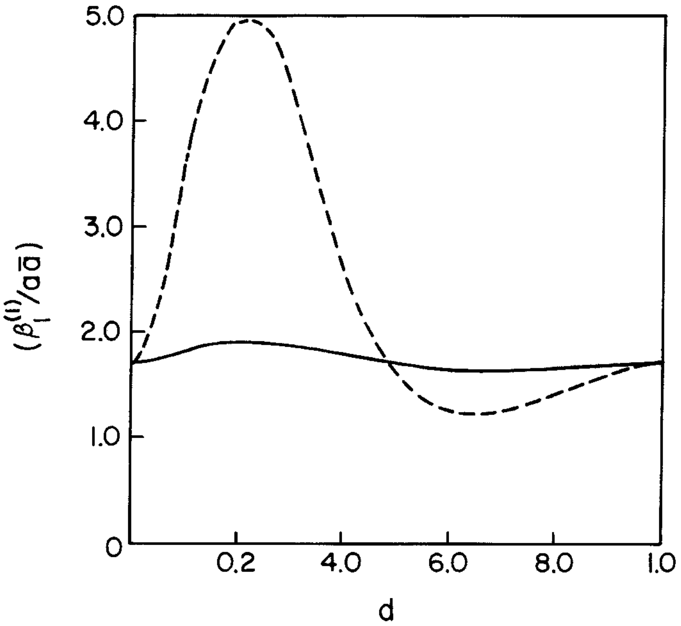

Figure 7. Effect of the guide-location on $\beta_1^{(1)}/a\overline{a}$. $c' = 0.5$, $\Delta T = 0$; —: $K_f = 10$, - - -: $K_f = 100$.

$$Q_1 = -\frac{\omega_n^l}{F_0}\int_0^1 \overline{\phi}_n f\,\mathrm{d}x = Q_1^R + iQ_1^I,$$

$$M_1 = \frac{1}{F_0}\left\{-\sum_{m\neq n}\overline{g}_m\omega_m^l\int_0^1\overline{\phi}_m f\,\mathrm{d}x\right\} + \frac{1}{F_0}\left\{\sum_{m=1}^{\infty}\overline{h}_m\omega_m^l\int_0^1\phi_m f\,\mathrm{d}x\right\} = M_1^R + iM_1^I,$$

$$S = \frac{\beta_1^{(n)}}{\tilde{a}_n^2},\quad r_1 = \frac{\Omega}{\omega_n^l},\quad \nu = \left(\frac{F_0}{t_n}\right)^2\frac{\varepsilon}{S^2},\quad A = \sqrt{\varepsilon}\tilde{a}_n.$$

It may be mentioned that depending upon the system parameters, either one or three roots may exist. As shown in [7], the intermediate root is always unstable and is not observable in reality. It was also shown in [7] that the steady-state response of the beam, up to the order $o(1)$, is

$$w(x,\tau) = \tilde{a}_n\phi_1^*(x)\cos(\Omega\tau + \theta_n + \rho_n),$$

where $\tan\rho_n = -\phi_1^I/\phi_1^R$ and $\phi_1^* = \sqrt{(\phi_1^R)^2 + (\phi_1^I)^2}$.

2.3.1. *Numerical Results and Discussion*

In this section, numerical results showing the effect of non-linearity on the free and forced responses of a travelling beam are presented. The tension of the beam, T_2, is again taken as unity, i.e., $T_2 = 1$.

Owing to the presence of the non-linear term, the natural frequency shows a hardening characteristics, since the term $\beta_1^{(n)}/a\overline{a}$ (see Equation (37)) is positive. In Figure 7, the variation of $\beta_1^{(1)}/a\overline{a}$ with the guide location, d, is shown. As seen from the figure, the non-linear effects

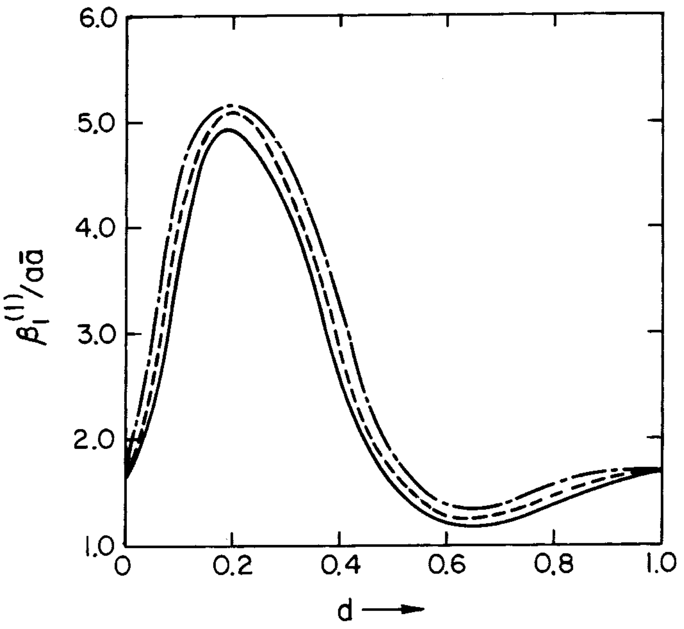

Figure 8. Effect of frictional force on $\beta_1^{(1)}/a\bar{a}$. $c' = 0.5$, $d = 0.5$, $K_f = 100$, $T_2 = 1.0$; —: $\Delta T/T_2 = 0$, - - -: $\Delta T/T_2 = 0.1$, — - —: $\Delta T/T_2 = 0.3$.

depend strongly on the location and the stiffness of the guide. For the values of the parameters considered in this paper, the term $\beta_1^{(n)}/a\bar{a}$, however, changes little with increasing frictional force. This is shown in Figure 8.

As seen from Figure 9, the linear theory, when compared with the non-linear theory for such a hard system, overestimates the steady-state response when the excitation frequency is equal to the natural frequency i.e., $\Omega = \omega_n^l$ and underestimates the response when $\Omega > \omega_n^l$. Therefore, special attention should be given to the response at $\Omega > \omega_n^l$. Figure 10 shows the variation of the maximum amplitude (since there exists a possibility of getting multiple amplitudes depending upon the initial conditions), i.e., the maximum value of A_1, obtained by solving Equation (47) with $n = 1$. Three excitation frequencies, $\Omega = 8.7$, 12 and 15, are considered. These frequencies are such that the first mode is primarily excited. The significance of the choice of the guide location, so far as the maximum amplitude is concerned, can be clearly seen in Figure 10. It is seen that for $\Omega = 8.7$ (i.e., $\Omega < \omega_1^l = 10.87$), the non-linear response is less compared to the linear response. If a guide is placed, the linear natural frequency increases (see Section 2.2.1) and the excitation frequency becomes much less compared to the natural frequency, i.e., the frequency at which the resonance occurs. For a suitable guide-location, the difference between the excitation and the natural frequencies becomes so large that the effect of non-linearity is hardly perceptable. When $\Omega > \omega_1^l$ (for example, $\Omega = 12$ or 15), the response amplitude predicted by the linear theory is much smaller than that obtained by the non-linear analysis. If a guide is now placed, the difference between Ω and ω_1^l decreases and, as per the characteristics of a hard system, the response amplitude again decreases. It is to be pointed out that even if the linear theory may not suggest the requirement of a guide by underestimating the response, a guide may be required because of the presence of the non-linear term.

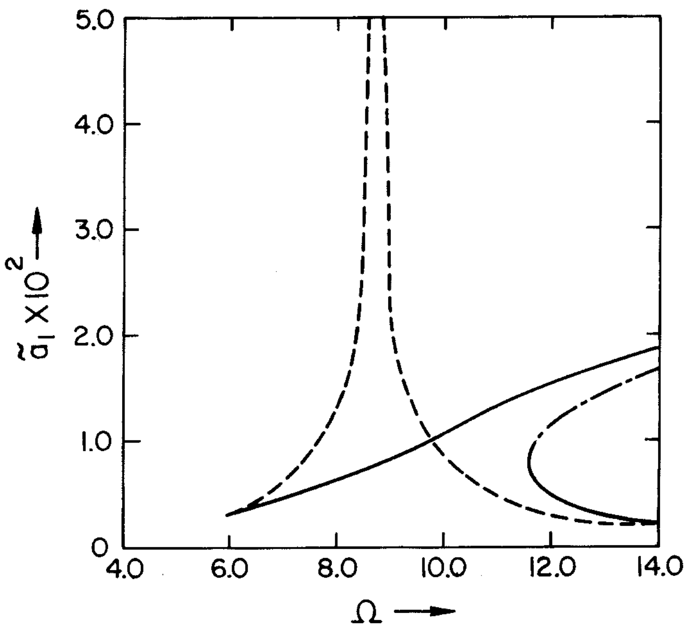

Figure 9. Near-resonance response of the travelling beam with an intermediate guide. $c' = 0.5$, $d = 0.5$, $\Delta T = 0$, $K_f = 100$, $F_0 = 1000$, $\varepsilon = 0.0001$; —: stable solution (non-linear theory), — - —: unstable solution (non-linear theory), - - -: linear theory.

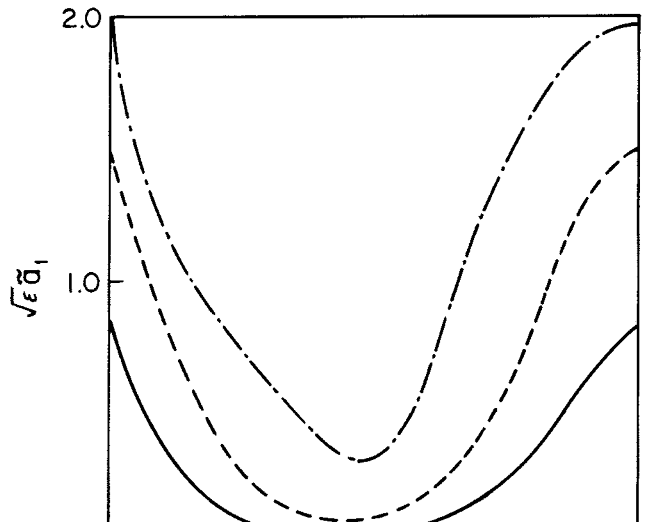

Figure 10. Effect of the guide-location on the maximum value of the roots of Equation (47). $c' = 0.5$, $\Delta T = 0$, $K_f = 100$, $F_0\sqrt{\varepsilon} = 10$; —: $\Omega = 8.7$, - - -: $\Omega = 12$, —·—: $\Omega = 15$.

3. Conclusions

The free and forced responses of a guided travelling beam, including the non-linear effects, have been studied in this paper. The guide is modelled as an elastic constraint having frictional contact. While the stiffness of the guide has a stabilizing effect so far as the divergence instability is concerned, the friction between the guide and the beam adds to the instability. Further, if the guide is placed at a proper location, the steady-state amplitude of vibration under a harmonic excitation can be reduced significantly. Thus, the guide can serve the purpose of a passive controller. The choice of a suitable guide location plays a very important role in controlling the vibration. The effect of non-linear terms for proper selection of the guide-location has also been clearly brought out.

Appendix I: List of Symbols

w^*	=	transverse displacement of the beam
u^*	=	longitudinal displacement of the beam
c^*	=	uniform axial speed
T_0^*	=	initial tension in the beam
d^*	=	location of the guide
$k_f^*/2$	=	stiffness of the guide
$N^*/2$	=	precompression of the guide
ρ	=	density of the beam material
E	=	Young's modulus of the beam material
A	=	area of cross-section of the beam
l	=	length of the beam
I_z	=	second moment of area of cross-section about the neutral axis
r	=	radius of gyration of the beam cross-section = $\sqrt{I_z/A}$
γ	=	slenderness ratio, $r/l \ll 1$
ε	=	$\gamma^2/2$
ξ	=	longitudinal distance of a point on the beam from left support
t	=	time
x	=	non-dimensional distance
τ	=	non-dimensional time
w	=	non-dimensional transverse displacement
u	=	non-dimensional longitudinal displacement
c	=	non-dimensional axial speed
T_0	=	non-dimensional tension
T_j	=	tension in the jth span, where $j = 1, 2$
ΔT	=	difference between the tensions in different spans $(= T_2 - T_1)$
c'	=	$c/\sqrt{\pi^2 + T_2}$
d	=	non-dimensional location of the guide
$k_f/2$	=	non-dimensional stiffness of the guide
$N/2$	=	non-dimensional precompression of the guide
ϕ_n	=	nth linear non-stationary complex normal mode $= \phi_n^R + i\phi_n^I$
i	=	$\sqrt{-1}$
ω_n^l	=	linear natural frequency of nth linear mode

ψ_n	=	nth non-linear complex normal mode $= \psi_n^R + i\psi_n^I$
ω_n	=	frequency corresponding to nth non-linear normal mode
f^*	=	transverse force per unit length
f	=	non-dimensional transverse force
Ω	=	non-dimensional frequency of excitation
ϕ_n^*	=	$\sqrt{(\phi_n^R)^2 + (\phi_n^I)^2}$.
a_n	=	participation of the nth linear mode $= \tilde{a}_n\, e^{i\theta_n}$.

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