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Finite element modelling for evaluation of apparent resistivity over complex structures

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Abstract. The paper presents a computational algorithm designed for efficient modelling of apparent resistivity over complex geological structures, using finite element method. The algorithm can be used to study variations of apparent resistivities using any electrode configuration at any point on the earth's surface, not necessarily regular. A Schlumberger apparent resistivity sounding curve over a buried anticline, is presented here as an example and compared with the corresponding analytical curve, to demonstrate the correctness of the FEM algorithm.

The various potential derivatives required for the computation of apparent resistivities evaluated through different electrode configurations have been obtained by calculating the 'influence coefficients' using reciprocal theorems, an approach successfully applied in structural engineering. In essence, a set of self balancing nodal currents, obtained from the appropriate derivative(s) of the shape functions of the elements contributing to the point of observation, is applied as the load vector.

The resulting quantities corresponding to the potential distribution in traditional finite element method, then, turn out to be the potential derivatives at the point of observation for different positions of the current electrodes. These are known as influence coefficients.

The continuum nature of the domain beyond the region of interest has been modelled by using 'infinite elements' across which the potential is assumed to decay exponentially.

Keywords. Finite element method; resistivity sounding; anticline.

1. Introduction

The dc resistivity response of a buried structure of arbitrary shape, to a given electrode configuration, is generally obtained by using numerical methods like the finite element (FE) (Coggon 1971, 1973) or finite difference (FD) (Mufti 1976, 1978, 1980; Aiken *et al* 1973). In the usual application of these methods to dc problems, the finite region of interest is discretised and values of the potential are approximated at the nodal points for a given current distribution. But, whilst modelling for vertical electrical sounding, one has to calculate afresh, the influence coefficients for the potential derivative (first derivative in the case of Schlumberger and second derivative in the case of dipole sounding) every time the current electrodes are moved to a new position. Mufti (1976, 1978, 1980) presented an efficient FD algorithm using the reciprocity theorem to obtain the vertical sounding apparent resistivity curve for the Schlumberger configuration.

The FEM provides a generalised approach to the determination of influence field irrespective of the nature of the physical field involved. The influence coefficient technique has been successfully applied to structural engineering problems such as analysis of the effects of moving loads on bridge.

The FEM algorithm presented in this paper is a simple modification of the standard FE formulation. It makes use of the reciprocal theorem of matrices and enables one to evaluate the influence coefficients, for any derivative of the potential, at any point of the region and for any electrode configuration, in a single run.

Further, to improve the efficiency and versatility of the algorithm, an unconventional 'infinite element' has been used. Using infinite elements, one can study the problem even in an infinitely extended domain without having to truncate it with arbitrary boundaries.

2. FE formulations

In resistivity exploration, if the line of measurement is perpendicular to the strike and if current is applied to the ground by means of infinite line electrodes parallel to it, the problem can be treated as being purely a two-dimensional one. Even though resistivity surveys are normally carried out using point sources the study of the corresponding 2D problem can yield significant information (Mufti 1978) and is found quite useful in interpretation. The formulation presented here is quite general which is equally applicable to three dimensions. However, the problem dealt with here is posed in two dimensions simply to keep the cost of computations low.

The two-dimensional potential problem can be represented by the equation:

$$\frac{\partial}{\partial x}\left(\sigma_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma_y \frac{\partial \phi}{\partial y}\right) + S = A(\phi) = 0, \tag{1}$$

where ϕ is the unknown potential, σ_x and σ_y are conductivities in the x and y directions respectively and S is the current source term. Equation (1) has to be solved in the Ω domain with boundary Γ given by

$$\Gamma = \Gamma_1 + \Gamma_2,$$

where Γ_1 and Γ_2 represent component boundaries. On the entire boundary one of the following conditions has to be satisfied

$$\phi = \phi_0 \text{ on } \Gamma_1, \tag{2a}$$

$$\partial\phi/\partial n = q_0 \text{ on } \Gamma_2. \tag{2b}$$

To approximate the numerical solution of equation (1), the entire domain Ω is divided into l homogeneous finite sub-domains (finite element), Ω^e, and Galerkin method of weighted residuals is used for the FE formulation. Zienkiewicz (1977) gives a detailed description of FE technique while a comprehensive account of the weighted residual method of Galerkin is given by Finlayson (1972). A brief derivation of FE equations is however included here for the sake of completeness.

Let the unknown function ϕ be expressed in each subdomain, Ω^e, in an approximate form as

$$\phi = \Sigma_{i=1}^{n^e} N_i \phi_i = [N] \{\phi\}^e, \tag{3}$$

where n^e is the number of nodal points in the subdomain, ϕ_i's are the unknown nodal potential values and N_i's are the shape or trial functions which approximate the spatial dependence of ϕ over the element. In the present analysis, parabolic isoparametric shape functions belonging to the Serendipity family (Zienkiewicz 1977) have been used.

Substitution of the above approximate value of ϕ in equation (1) yields a residual $A(\phi)$. Galerkin's method of weighted residuals then yields the approximating equation in an integral form as follows:

$$\begin{aligned}
&\int_{\Omega^e} N_j A(\phi)\, d\Omega^e \\
&= \int_{\Omega^e} N_j \left[\sum_{i=1}^{n^e} \left\{ \frac{\partial}{\partial x}\left(\sigma_x \frac{\partial N_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma_y \frac{\partial N_i}{\partial y}\right) \right\} \phi_i + S \right] d\Omega^e = 0 \\
&= \sum_{i=1}^{n^e} \int_{\Omega^e} N_j \left[\frac{\partial}{\partial x}\left(\sigma_x \frac{\partial N_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma_y \frac{\partial N_i}{\partial y}\right) \right] \phi_i\, d\Omega^e \\
&\quad + \int_{\Omega^e} N_j S d\Omega^e = 0 \\
&\qquad\qquad\qquad\qquad j = 1, \ldots, n^e
\end{aligned} \tag{4}$$

since, for any integrable function

$$F = \Sigma F_i, \; \int \Sigma F_i d\Omega = \Sigma \int F_i\, d\Omega,$$

Assuming σ_x, σ_y to have a constant value over an element and integrating equation (4) by parts or using Green–Gauss theorem we get

$$\begin{aligned}
&\sum_{i=1}^{n^e} \left[\int_{\Omega^e} \left(\sigma_x \frac{\partial N_i}{\partial x} N_j dy + \sigma_y \frac{\partial N_i}{\partial y} N_j\, dx \right) - \int_{\Omega^e} \left(\sigma_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \right.\right. \\
&\quad \left.\left. + \sigma_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) d\,\Omega^e \right] \phi_i + \int_{\Omega^e} N_j S d\,\Omega^e = 0 \\
&\qquad\qquad\qquad\qquad j = 1, \cdots, n^e.
\end{aligned} \tag{5}$$

On the boundary of the domain the contribution due to first integral vanishes while on the interelement boundaries its contributions are cancelled during assembly of all element equations. So, we can write

$$\begin{aligned}
&\sum_{i=1}^{n^e} \int_{\Omega^e} \left(\sigma_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \sigma_y \frac{\partial N_i}{\partial y} \frac{\partial U_j}{\partial y} \right) \phi_i\, d\Omega^e - \int_{\Omega^e} N_j S d\,\Omega^e = 0 \\
&\qquad\qquad\qquad\qquad j = 1, \ldots, n^e.
\end{aligned} \tag{6}$$

Assembling the different element equations, like (6), one obtains a system of N equations written in a matrix form as follows:

$$[K]\ \{\phi\} = \{f\}, \tag{7}$$

where N is the total number of nodes in the entire domain Ω and

$$K_{ij} = \sum_e K^e_{ij} = \sum_e \int_{\Omega^e} \left(\sigma_x \frac{\partial N_p}{\partial x} \frac{\partial N_q}{\partial x} + \sigma_y \frac{\partial N_p}{\partial y} \frac{\partial N_q}{\partial y} \right) d\Omega^e, \tag{8a}$$

$$f_i = \sum_e f_i^e = \sum_e \int_{\Omega^e} N_p\, S d\, \Omega^e \tag{8b}$$

$$p = 1, \ldots, n^e\,;\ q = 1, \ldots, n^e;\ i = 1, \ldots, N;\ j = 1, \ldots, N. \tag{9}$$

Equations (7) and (8) allow the values of the potential to be evaluated by algebraic solutions under the condition that at least one element of the vector $\{\phi\}$ is specified initially. This restriction can be circumvented by designing a suitable boundary condition described later in this paper.

Once the potential values ϕ_i's are known, the potential derivatives can be evaluated from equation (3). Various potential derivatives ϕ_x, ϕ_y, etc. are, then, obtained by differentiating equation (3). Accordingly,

$$\phi_x = [\partial N/\partial x]\ \{\phi\}^e = [g_x]\ \{\phi\}^e, \tag{10a}$$

$$\phi_y = [\partial N/\partial y]\ \{\phi\}^e = [g_y]\ \{\phi\}^e, \tag{10b}$$

$$\phi_{xx} = [\partial^2 N/\partial x^2]\ \{\phi\}^e = [g_{xx}]\ \{\phi\}^e, \tag{10c}$$

and so on. Equation (10) can be rewritten in matrix form

$$\{\phi_d\} = [G]\ \{\phi\}^e, \tag{11}$$

where, $\{\phi_d\} = [\phi_x\ \phi_y\ \phi_{xx}\ \phi_{xy}\ \phi_{yy}]^T$, (12a)

$$[G] = \{[g_x]^T\ [g_y^T]\ [g_{xx}^T]\ [g_{x2}^T]\ [g_{yy}^T]\}^T. \tag{12b}$$

2.1. *Influence coefficients*

The influence coefficient technique can be used to evaluate the potential derivative at a specified node when unit current is applied at any other node. The effect of two current electrodes can then be obtained by using the superposition principle. To illustrate the technique, calculations for the first order potential derivative in the x direction, are presented below. One could similarly calculate any other derivative.

Let a unit current be fed at the ith node so that all but ith element of the load vector $\{f\}$ in (7) are zero and the ith element, is unity, *i.e.*

$$\{f\} = \{u\}_i = [00\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T. \tag{13}$$

Equations (7) and (13) then give an expression for $\{\phi\}$ as

$$\{\phi\} = [K]^{-1}\ \{u\}_i. \tag{14}$$

Premultiplying equation (14) by $[g_e]$ and using equation (10a) one obtains

$$\phi_e = [g_x]\,[K]^{-1}\,\{u\}_i.$$

Since $[K]$ is a positive definite symmetric matrix, using the reversal theorem on the transpose of a product we have,

$$\phi_e = \{u\}_i^T\,[K]^{-1}\,[g_e]^T, \tag{15}$$

or $$\phi_e = \{u\}_i^T\,\{r\} = r_i, \tag{15'}$$

where $$\{r\} = [K]^{-1}\,[g_e]^T. \tag{16}$$

Thus, when unit current is fed at the ith node the potential derivative at the point under investigation is nothing but the ith element of the vector $[r]$ which, indeed, gives the potential derivative at the specified node when unit current is fed at any other node and this would serve our purpose for computing the vertical sounding apparent resistivity curve.

The vector $[r]$ is obtained from equation (16). Rewriting equation (16) as

$$[K]\;\{r\} = [g_e]^T, \tag{17}$$

and comparing (17) with equation (7) one observes that the two are similar, so that the conventional FE algorithm, itself, yields the vector $\{r\}$ when the load vector consists of the requisite derivatives of the shape functions of elements which contain the specified node. Thus, only one additional subroutine which would generate the correct load vector, is required to get the vertical sounding results.

2.2. *Infinite element*

The efficiency of the algorithm has been improved by using infinite elements beyond the boundaries of the region of interest. Infinite elements have been successfully employed by Bettes (1977), Bettes and Zienkiewicz (1977).

The shape functions over infinite elements are based on Lagrange polynomials multiplied by exponential decay terms. The parent shape of the parabolic infinite element used in this analysis is shown in figure 1.

In this element the variation along the η-axis is represented by the conventional one-dimensional shape function while the variation along the ξ-axis is represented simply by an exponential factor $\exp(-a\xi)$, a being the decay constant which controls the rate of decay of the potential. A typical shape function is

$$N_i(\xi,\eta) = n_i(\eta)\,\exp(-a\xi), \tag{18}$$

$n_i(\eta)$ being the one-dimensional shape function.

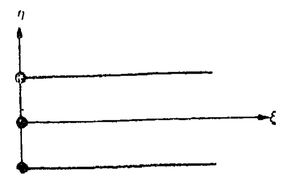

Figure 1. Parent shape of an infinite element.

Using this definition of shape functions and integrating analytically in the ξ-direction, one can obtain the elements' stiffness and incorporate that in equation (7). Thus, the effect of infinite domain is accounted for through the nodes of the element.

2.3. *Boundary conditions*

In situations where clear Dirichlet boundary condition cannot be assumed for want of symmetry or other simplifying properties, an approximate boundary condition can be designed by exploiting the fact that at very small distance from the current electrode, the electric potential can be deduced by assuming that the earth behaves as a semi-infinite medium of resistivity equal to that of the first layer.

2.4. *A modelling example*

In order to check the algorithm, an anticlinal structure of infinite resistivity buried in a homogeneous medium of resistivity 5Ω-metre (*vide* figure 2) was studied for the Schlumberger array. This model was chosen because an analytical expression for apparent resistivity over it was already available (Nek Ram 1976). The equation was solved for various values of the decay constant a. For $a = 0{\cdot}0135$ the analytical and numerical curves of apparent resistivity were found to be in close agreement as shown in figure 3.

Although, the Dirichlet boundary condition was available in this case along the line of symmetry, the solution was also obtained using the boundary condition described above. These two solutions were also found to be in quite close agreement.

3. Conclusions

The influence coefficient algorithm presented here provides an efficient means for the direct computation of an apparent resistivity curve for vertical electrical sounding over a buried structure of arbitrary shape. The technique can be used in respect of any electrode configurations desired. The use of infinite element shape functions has improved the quality of results and made it more efficient for deep electrical soundings. The algorithm can also be used to account for weathered layers. This versatile algorithm requires no significant additional memory but does require 7 sec. of additional CPU time, on an IBM—370/145 computer

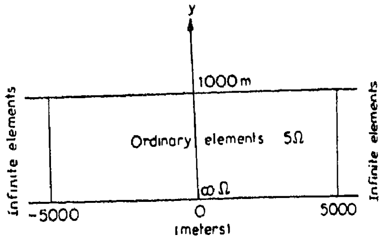

Figure 2. The anticlinal model studied.

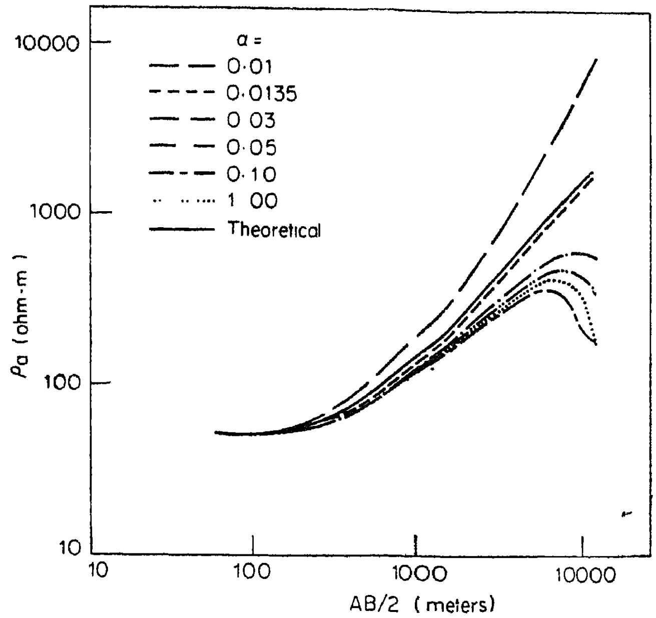

Figure 3. A plot of apparent resistivity ρ_a vs. half electrode spacing AB/2 for different values of the decay constant a.

over the 2 min and 20 sec required for the conventional FEM algorithm whose use is however limited only to resistivity profile modelling.

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