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Estimation of the waiting time distributions of earthquakes

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Abstract. Whether the earthquake occurrences follow a Poisson process model is a widely debated issue. The Poisson process model has great conceptual appeal and those who rejected it under pressure of empirical evidence have tried to restore it by trying to identify main events and suppressing foreshocks and aftershocks. The approach here is to estimate the density functions for the waiting times of the future earthquakes. For this purpose, the notion of Gram-Charlier series which is a standard method for the estimation of density functions has been extended based on the orthogonality properties of certain polynomials such as Laguerre and Legendre. It is argued that it is best to estimate density functions in the context of a particular null hypothesis. Using the results of estimation a simple test has been designed to establish that earthquakes do not occur as independent events, thus violating one of the postulates of a Poisson process model. Both methodological and utilitarian aspects are dealt with.

Keywords. Gram-Charlier series; earthquakes; Hermite polynomials; Laguerre polynomials; Poisson process; Polya process.

1. Introduction

The question whether the earthquake occurrences follow a Poisson process model has been widely addressed (Benioff 1951; Aki 1956; Shalanger 1960; Knopoff 1964; Lomnitz 1966; Ferraes 1967; Vere Jones 1970; Schlien and Toksoz 1970; Utsu 1972; Udias and Rice 1975). Under the Poisson process model, the number of earthquakes x per unit time follows a Poisson distribution

$$P(x = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad \lambda > 0, \quad k = 0, 1, 2. \tag{1}$$

where λ is the rate of the process and the inter-arrival time t_1 for earthquakes has a density function

$$p_{T_1}(t_1) = \lambda \exp(-\lambda t_1) u(t_1) \tag{2}$$

where $u(t_1)$ is a unit step function. Most frequently (1) and (2) have been fitted to the empirical data in the form of histograms using least squares criterion and bad fit taken

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as an indication that the Poisson process model is inappropriate. Rarely other density functions such as

$$p_{T_1}(t_1) = \frac{\lambda^p t_1^{p-1}}{(p-1)!} \exp(-\lambda t_1)\, u(t_1), \ \lambda > 0, \ p \text{ positive integer} \tag{3}$$

of which (2) is a special case with $p = 1$, have been fitted in the least squares sense (Udias and Rice 1975) and $p = 2$ was thought to give better fit than $p = 1$, though systematic deviation from the gamma density function of (3) with $p = 2$ has also been reported.

The approach followed in this paper is different. No particular density functions are fitted to the empirical inter-arrival times or waiting times but the density functions are estimated using Gram-Charlier type techniques.

2. Gram-Charlier series

If y is a random variable which has zero mean and unit variance (this can be achieved by standardization), its density function $p(y)$ can be written as (Whalen 1971)

$$p(y) = \varphi(y) + \sum_{n=3}^{\infty} a_n \varphi^{(n)}(y), \ -\infty < y < \infty, \tag{4}$$

where

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2), \tag{5}$$

is a standard normal density function, $\varphi^{(n)}(y)$ is its nth derivative given as

$$\varphi^{(n)}(y) = (-1)^n \varphi(y) H_n(y), \tag{6}$$

where $H_n(y)$ is a Hermite polynomial of degree n obtainable by recursive relation

$$H_{n+1}(y) = yH_n(y) - nH_{n-1}(y), \quad H_0(y) = 1, \quad H_1(y) = y. \tag{7}$$

The coefficient a_n in the Gram-Charlier expansion is given as

$$a_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} p(y)\, H_n(y)\, \mathrm{d}y, \tag{8}$$

wherein the biorthogonality relation

$$\int_{-\infty}^{\infty} H_m(y)\varphi^{(n)}(y)\,\mathrm{d}y = (-1)^n n!\, \delta_{mn} \quad n, m = 0, 1, \ldots \tag{9}$$

is used, where δ_{mn} is the Kronecker delta. Substituting for $H_n(y)$ in (8) from (7) we get

$$a_3 = -\mu_3/3!; \quad a_4 = (\mu_4 - 3)/4!;$$

$$a_5 = -(\mu_5 - 10\mu_3)/5!; \quad a_6 = (\mu_6 - 15\mu_4 + 30)/6!;$$

$$a_7 = -(\mu_7 - 21\mu_5 + 105\mu_3)/7!;$$

$$a_8 = (\mu_8 - 28\mu_6 + 210\mu_4 - 315)/8!;$$

$$a_9 = -(\mu_9 - 36\mu_7 + 378\mu_5 - 1260\mu_3)/9! \qquad (10)$$

where μ_i are standard central moments. These results are well-known (Whalen 1971) and are summarized here just because they are used in this paper as a point of departure. As standard central moments can be estimated from the empirical data, the coefficients a_n can be estimated from (10) and therefore, the density function $p(x)$ can be estimated from (4).

For reasons documented elsewhere (Kenney and Keeping 1963; Fry 1965), for truncating, terms in (4) must be grouped as 0, (3), (4, 6), (5, 7, 9), (8, 10, 12), (11, 13, 15), (14, 16, 18), every group of terms should be included or excluded collectively. The Gram-Charlier series thus truncated is called Edgeworth series.

Using microearthquake data from North-East India, (see § 5) it ... found that

$$a_3 = -0.3137; \quad a_4 = 0.1437; \quad a_5 = 0.01965;$$

$$a_6 = 0.01168; \quad a_7 = -0.005139; \quad a_9 = -0.001065 \qquad (11)$$

Using data from the Hindukush region (see § 5) it was found that

$$a_3 = -0.4567; \quad a_4 = 0.5661; \quad a_5 = -0.7936;$$

$$a_6 = 1.1851; \quad a_7 = -1.5682; \quad a_9 = -1.9469 \qquad (12)$$

Thus particularly for the large earthquakes the coefficients a_n do not decrease rapidly enough as n sweeps the Edgeworth groups. With the zeroth and the third term included the density function has a mode away from the origin. With $n = 0, 3, 4$ and 6 terms included, the density function gives nonphysical values around origin. With $n = 0, 3, 4, 6, 5, 7$ and 9 terms included the density function is monotonically decreasing. Thus the behaviour is not yet stable indicating that $n = (8, 10, 12), (11, 13, 15)$, terms must also be added. This is not very simple. Because in (10) unbiased estimates of the moments must be substituted and obtaining unbiased estimators of central moments using small sample theory (Kenney and Keeping 1963) is a fairly tedious task. Moreover, higher order central moments become less and less reliable when estimated from finite samples. Thus, it is almost certain that the Gram-Charlier series, due to its poor convergence, is not a proper method to estimate density functions in our context.

However, as Gram-Charlier series may be useful in other contexts and as unbiased estimators for central moments are not readily available beyond the fourth moment, they are listed here for easy reference

$$\hat{\mu}_2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - n_1)^2,$$

$$\hat{\mu}_3 = \frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} (x_i - n_1)^3,$$

$$\hat{\mu}_4 = \frac{N^2}{(N-1)(N^2-3N+3)} \sum_{i=1}^{N} (x_i - n_1)^4,$$

$$\hat{\mu}_5 = \frac{N^3}{(N-1)(N-2)(N^2-2N+2)} \sum_{i=1}^{N} (x_i - n_1)^5,$$

$$\hat{\mu}_6 = \frac{N^4}{(N-1)(N^4-5N^3+10N^2-10N+5)} \sum_{i=1}^{N} (x_i - n_1)^6,$$

$$\hat{\mu}_7 = \frac{N^5}{(N-1)(N-2)(N^4-4N^3+7N^2-6N+3)} \sum_{i=1}^{N} (x_i - n_1)^7,$$

$$\hat{\mu}_8 = \frac{N^6}{(N-1)(N^6-7N^5+21N^4-35N^3+35N^2-21N+7)} \sum_{i=1}^{N} (x_i - n_1)^8,$$

$$\hat{\mu}_9 = \frac{N^7}{(N-1)(N-2)(N^6-6N^5+16N^4-24N^3+22N^2-12N+4)} \sum_{i=1}^{N} (x_i - n_1)^9, \tag{13}$$

Small sample theory (Kenney and Keeping 1963) is used to derive (13), the terms of the order of N^{-1} are neglected, where N is the number of samples, $x_1, x_2, \ldots, x_N$ and n_1 is the sample mean, and the details are reported elsewhere (Malasi 1981).

3. Gram-Charlier series of other types

3.1 *The motivation*

In Gram-Charlier series of (4), the first term is the normal density function and the other terms measure deviation from it. Thus it is best suited if normal density function is the null hypothesis or prototype. In the context of estimating density function for the inter-arrival times for earthquakes the density function according to the null hypothesis of Poisson process model for earthquake occurrences is a negative exponential density function of (2) (Massey 1971; Udias and Rice 1975). It would therefore be desirable to have a series expansion for a density function in which the leading term is a negative exponential density function so that the remaining terms directly measure deviation from the null hypothesis.

Apart from the derivation using characteristic function, cumulants, etc. (Whalen 1971) the end result of the Gram-Charlier series can be interpreted as a biorthogonal expansion exploiting the biorthogonality relation of (9) between the derivatives $\varphi^{(n)}(y)$ of the normal density function and the Hermite polynomials. The waiting times for future earthquakes can take only non-negative values, whereas the Gram-Charlier series of (4) was biorthogonal expansion over the range $(-\infty, \infty)$. It would be more appropriate to use orthogonality or biorthogonality properties of some suitably chosen functions over the range $(0, \infty)$.

3.2 *Laguerre type Gram-Charlier series*

$L_n(t)$ defined as (Krishnamurthy and Sen 1976)

$$L_n(t) = (-1)^n e^t \frac{d^n}{dt^n} (t^n e^{-t}), \quad 0 \leq t < \infty. \tag{14}$$

and satisfying the recurrence relation

$$L_{n+1}(t) = (2n + 1 - t)\, L_n(t) - n^2 L_{n-1}(t);$$
$$L_0(t) = 1, \qquad L_1(t) = 1 - t \tag{15}$$

is called Laguerre polynomial of order n. Let

$$\Psi_n(t) = L_n(t)e^{-t}. \tag{16}$$

Then we have a biorthogonality property

$$\frac{(-1)^n}{(n!)^2} \int_0^\infty L_m(t)\, \Psi_n(t)\, dt = \delta_{mn}, \qquad n, m = 0, 1, 2, \ldots \tag{17}$$

using which any square integrable function $f(t)$ defined over the range $[0, \infty]$ can be expressed as

$$f(t) = \sum_{n=0}^{\infty} a_n \Psi_n(t), \qquad 0 \leq t < \infty. \tag{18}$$

where

$$a_n = (-1)^n \frac{1}{(n!)^2} \int_0^\infty f(t)\, L_n(t)\, dt \tag{19}$$

Using (15), it can be shown that

$$a_0 = 1; \quad a_1 = (1 - \nu_1); \quad a_2 = (2 - 4\nu_1 + \nu_2)/(2!)^2;$$
$$a_3 = (6 - 18\nu_1 + 9\nu_2 - \nu_3)/(3!)^2;$$
$$a_4 = (24 - 96\nu_1 + 72\nu_2 - 16\nu_3 + \nu_4)/(4!)^2;$$
$$a_5 = (120 - 600\nu_1 + 600\nu_2 - 200\nu_3 + 25\nu_4 - \nu_5)/(5!)^2;$$
$$a_6 = (720 - 4320\nu_1 + 5400\nu_2 - 2400\nu_3 + 450\nu_4 - 36\nu_5 - \nu_6)/(6!)^2; \tag{20}$$

where ν_m is the mth moment of the random variable t of which $f(t)$ is the density function.

In addition to the two motivating factors discussed in §3.1, the Laguerre type Gram-Charlier series has one more advantage. The coefficients of the Gram-Charlier series of (4) depend on the central moments as in (10), so that unbiased estimators for them have to be obtained. On the other hand, the coefficients of the Laguerre type Gram-Charlier series of (18) depend only on the moments of the random variable as in (20) and the sample moments do not have a bias in this case.

The values of the coefficients in the Laguerre type Gram-Charlier series for the inter-arrival times in the Hindukush area were found to be

$$a_0 = 1; \quad a_1 = 0.1301 \times 10^{-17}; \quad a_2 = 0.5630 \times 10^{-1};$$
$$a_3 = -0.1003 \times 10^{-1}; \quad a_4 = 0.3604 \times 10^{-2};$$
$$a_5 = 0.7429 \times 10^{-4}; \quad a_6 = -0.8283 \times 10^{-4}. \tag{21}$$

The density functions estimated by truncating the series after third, fourth, fifth and sixth terms are not very different, thus indicating good convergence of the series. The inter-arrival time data was scaled to have unit mean before fitting the density function.

3.3 *Legendre type Gram-Charlier series*

The polynomials obtained by the recurrence relation

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - nP_{n-1}(x);$$
$$P_0(x) = 1, \quad P_1(x) = x, \tag{22}$$

are called Legendre polynomials and they have a orthogonality property over the range $[-1, 1]$ that

$$\int_{-1}^{1} P_m(x) P_n(x)\, \mathrm{d}x = \frac{2}{2m+1}\, \delta_{mn}. \tag{23}$$

These polynomials cannot be applied directly to our present context, as we require functions orthogonal or biorthogonal over $[0, \infty)$. But this could be achieved by a suitable change of variable (Lee 1960). First let

$$x = 2y - 1 \tag{24}$$

so that

$$2\int_{0}^{1} P_m(2y-1) P_n(2y-1)\, \mathrm{d}y = \frac{2}{2m+1}\, \delta_{mn}. \tag{25}$$

Thus functions $P_m(2y-1)$ as functions of y are orthogonal over the range $[0, 1]$. Subsequently, let

$$y = \exp(-pt) \tag{26}$$

so that

$$\int_0^\infty pe^{-pt} P_m(2e^{-pt} - 1) P_n(2e^{-pt} - 1)\,dt = \frac{1}{2m+1}\,\delta_{mn}, \tag{27}$$

so that the functions $P_m(2e^{-pt} - 1)$ as functions of t are orthogonal in the range $[0, \infty)$ with $p\exp(-pt)$ as the weight function. Therefore, any square integrable function $f(t)$ can be expanded as

$$f(t) = \sum_{n=0}^{\infty} a_n \Psi_n(pt), \qquad 0 \leq t < \infty, \tag{28}$$

where

$$\Psi_n(pt) = P_n(2\exp(-pt) - 1)\exp(-pt), \tag{29}$$

and

$$a_n = (2n+1)p\int_0^\infty f(t) P_m(2e^{-pt} - 1)\,dt, \tag{30}$$

on the basis of (27). From (22) and (29) we have

$$\begin{aligned}
\Psi_0(pt) &= \exp(-pt),\\
\Psi_1(pt) &= -\exp(-pt) + 2\exp(-2pt),\\
\Psi_2(pt) &= \exp(-pt) - 6\exp(-2pt) + 6\exp(-3pt),\\
\Psi_3(pt) &= -\exp(-pt) + 12\exp(-2pt) - 30\exp(-3pt)\\
&\quad + 20\exp(-4pt), \text{ etc.}
\end{aligned} \tag{31}$$

If $f(t)$ is a density function, the mth moment is

$$\begin{aligned}
E(t^m) &= \sum_{n=0}^{\infty} a_n \int_0^\infty t^m P_n(2e^{-pt} - 1)\exp(-pt)\,dt,\\
&= \sum_0^\infty a_n f_{n,m}(p),
\end{aligned} \tag{32}$$

where

$$f_{n,m}(p) = \int_0^\infty t^m P_n(2e^{-pt} - 1)\exp(-pt)\,dt. \tag{33}$$

Using terms upto $n = 4$, (22), (33) and replacing mth moment $E(t^m)$, by the mth sample moment $\hat{E}(t^m)$ we get

$$p^2\hat{E}(t) = a_0 - \frac{1}{2}a_1 + \frac{1}{6}a_2 - \frac{1}{12}a_3 + \frac{1}{20}a_4,$$

$$p^3\hat{E}(t^2) = a_0 - \frac{3}{4}a_1 + \frac{17}{36}a_2 - \frac{43}{144}a_3 + \frac{247}{1200}a_4,$$

$$p^4 \hat{E}(t^3) = a_0 - \frac{7}{8} a_1 + \frac{151}{216} a_2 - \frac{937}{1728} a_3 + \frac{30689}{72000} a_4,$$

$$p^7 \hat{E}(t^6) = a_0 - \frac{63}{64} a_1 + \frac{44597}{46656} a_2 - \frac{2743363}{2985984} a_3 + \frac{303179}{345600} a_4, \tag{34}$$

where use has been made of

$$\int_0^\infty t^m \exp(-qpt)\,dt = \frac{m!}{(qp)^{m+1}}. \tag{35}$$

These equations yield a polynomial in p

$$S_6 p^7 + S_5 p^6 + S_4 p^5 + S_3 p^4 + S_2 p^3 + S_1 p^2 = 0. \tag{36}$$

Because $p = 0$ is not an acceptable choice, we have

$$S_6 p^5 + S_5 p^4 + S_4 p^3 + S_3 p^2 + S_2 p + S_1 = 0. \tag{37}$$

where

$$S_1 = 0.906854\, \hat{E}(t);\ \ S_2 = -\ 0.857505\, \hat{E}(t^2);$$
$$S_3 = 3.0437042\, \hat{E}(t^3);\ \ S_4 = -\ 4.98628859\, \hat{E}(t^4);$$
$$S_5 = 3.7042561\, \hat{E}(t^5);\ \text{and}\ S_6 = -\ 0.9948547\, \hat{E}(t^6). \tag{38}$$

Solving (37) and using a real positive root for p (34) gives a set of five equations (excluding the last) which are linear simultaneous equations in constants a_0, a_1, a_2, a_3, a_4 to be evaluated, which is a standard task. Instead or truncating the expansion in (28) after $n = 4$, terms upto $n = N$ could be retained. Then (34) will have $N + 2$ equations. The polynomial in (36) will become

$$S_{N+2} p^{N+3} + S_{N+1} p^{N+2} + \ldots + S_1 p^2.$$

Using the real positive root of this for p, (34), excluding the last equation will have $N + 1$ simultaneous linear equations in $N + 1$ constants $a_0, a_1, \ldots, a_N$. Then (28) gives an expansion for the density function.

Using data for interarrival times for earthquakes in Hindukush, we obtain

$$p = 0.12424 \times 10^1;\ \ a_0 = -\ 0.43302 \times 10^1;$$
$$a_1 = -\ 0.13783 \times 10^2;\ \ a_2 = -\ 0.22956 \times 10^2;$$
$$a_3 = -\ 0.22951 \times 10^2;\ \ a_4 = -\ 0.96675 \times 10^1. \tag{39}$$

The density function given by these coefficients after substitution in (28) was nonphysical, as suggested by the negative value of a_0 and large values of other coefficients.

3.4 *Fast convergent expansions*

The idea to be discussed has already been used in § 3.3 but here it is put in a more general context.

If functions $\varphi_n(t)$ are orthonormal, *i.e.* if

$$\int_0^\infty \varphi_n(t)\varphi_m^*(t)\,\mathrm{d}t = \delta_{nm}, \tag{40}$$

where the superscript asterisk denotes complex conjugation, it easily follows that the functions $\varphi_n(\beta t)$, where β is an arbitrary positive constant, are also orthogonal, because (40) gives

$$\int_0^\infty \varphi_n(\beta y)\varphi_m^*(\beta y)\,\mathrm{d}y = \frac{1}{\beta}\,\delta_{nm}. \tag{41}$$

Equation (40) implies that any square integrable function over $[0,\infty)$ can be expanded as

$$f(t) = \sum_{n=0}^{\infty} a_n\varphi_n(t). \tag{42}$$

Equation (41) implies that we can also have the expansion

$$f(t) = \sum_{n=0}^{\infty} b_n\varphi_n(\beta t), \qquad \beta > 0, \tag{43}$$

where β is arbitrary, but certainly the coefficients b_n will depend on the choice of β. If the expansions in (42) and (43) are not to be truncated, the choice of β is really immaterial. But if the series in (42) and (43) are to be truncated, it would seem advantageous to choose β judiciously so that the expansion has smallest possible number of terms for the same mean error. To illustrate this point, let $f(t)$ be $\varphi_p(\alpha t)$, $\alpha \neq 1$ for a particular value of p. Then in (42) we will need many terms to approximate $f(t)$. But in (43) a single term will suffice if β is chosen to be α.

This problem of choosing the scale factor β will arise for truncated orthonormal expansions over an infinite interval $(-\infty,\infty)$ also. This problem does not seem to have been discussed in literature explicitly. In the Gram-Charlier expansion of (4), standardization of the random variable amounts to choosing the scale factor to be the standard deviation.

As an example of how optimum value of β could be chosen, (19) is written as

$$\frac{(-1)^n}{(n!)^2}\,\beta\int_0^\infty L_m(\beta t)\Psi_n(\beta t)\,\mathrm{d}t = \delta_{mn}. \tag{44}$$

Then instead of (18) we could have the series expansion

$$f(t) = \sum_{n=0}^{N} a_n\Psi_n(\beta t), \tag{45}$$

where the same symbol is retained for the coefficients but their values would now depend on the choice of β. If $f(t)$ is a density function we would have

$$E(t^m) = \int_0^\infty t^m f(t)\,\mathrm{d}t,$$

$$= \int_0^\infty t^m \sum_{n=0}^{N} a_n \Psi_n(\beta t)\,\mathrm{d}t.$$

$$= \sum_{n=0}^{N} a_n \left\{ \int_0^\infty t^m L_n(\beta t) \exp(-\beta t)\,\mathrm{d}t \right\}.$$

$$= \sum_{n=0}^{N} a_n f_{n,m}(\beta). \tag{46}$$

where (16) has been used and

$$f_{n,m}(\beta) = \int_0^\infty t^m L_n(\beta t) \exp(-\beta t)\,\mathrm{d}t. \tag{47}$$

The functions $f_{n,m}(\beta)$ can be easily computed. For example, using (15)

$$f_{1,2}(\beta) = \int_0^\infty t^2 L_1(\beta t) \exp(-\beta t)\,\mathrm{d}t,$$

$$= \int_0^\infty t^2 (1 - \beta t) \exp(-\beta t)\,\mathrm{d}t,$$

$$= -4/\beta^3. \tag{48}$$

Thus all the other values of $f_{n,m}(\beta)$ could be calculated. Substituting these in (46), using $N = 4$ as an example and writing (46) for $m = 1, 2, \ldots, 6$, we get

$$\beta^2 \hat{E}(t) = a_0 - a_1,$$

$$\beta^3 \hat{E}(t^2) = 2(a_0 - 2a_1 + 2a_2),$$

$$\beta^4 \hat{E}(t^3) = 6(a_0 - 3a_1 + 6a_2 - 6a_3)$$

$$\beta^5 \hat{E}(t^4) = 24(a_0 - 4a_1 + 12a_2 - 24a_3 + 24a_4),$$

$$\beta^6 \hat{E}(t^5) = 120(a_0 - 5a_1 + 20a_2 - 60a_3 + 120a_4),$$

$$\beta^7 \hat{E}(t^6) = 720(a_0 - 6a_1 + 30a_2 - 120a_3 + 360a_4). \tag{48}$$

In general $N + 2$ such equations would be obtained. Here population moments $E(t^m)$ are replaced by sample moments $\hat{E}(t^m)$ because of the context of estimating density function from the sample moments. Equation (48) is 6 equations in 6 unknown. On solving these and ruling out the solution $\beta = 0$, we get

$$S_6\beta^5 + S_5\beta^4 + S_4\beta^3 + S_3\beta^2 + S_2\beta + S_1 = 0, \tag{49}$$

where

$$S_1 = -\hat{E}(t); \quad S_2 = 5E(t^2)/2!;$$

$$S_3 = -10\hat{E}(t^3)/3!; \quad S_4 = 10\hat{E}(t^4)/4!;$$

$$S_5 = -5\hat{E}(t^5)/5!; \quad S_6 = \hat{E}(t^6)/6! \tag{50}$$

Solving (49) for a positive real value of β and excluding the last equation in (48), we are left with 5 linear simultaneous equations in 5 unknowns $a_0, a_1, \ldots, a_4$, and these can be easily solved.

Once the coefficients a_n and β are evaluated, (45) gives the density function. In general (49) would be a polynomial of degree $N + 1$, yielding $(N + 1)$ possible choices of the scale factor β. As N increases, many different values of β would be available, making the choice of β less and less important, so that when the expansion is not truncated, any arbitrary value of β would do. This is a fortunate feature because to find out roots of a polynomial of large degree is an increasingly difficult task, whereas for a small value of N, the roots are relatively easily found and the choice of β is also more critical.

Using this method for the Hindukush data for inter-arrival times of earthquakes, we get

$$\beta = 1.1328; \qquad a_0 = 1.2425;$$
$$a_1 = -\ 0.040787; \qquad a_2 = 1.14667;$$
$$a_3 = -\ 0.0095450; \qquad a_4 = 0.014222. \tag{51}$$

Note that the value of a_0 has increased in comparison with (21). The other coefficients are still fairly small.

4. A test for independent occurrence of earthquake

It is desirable to check whether the assumption of the independent occurrence of earthquakes, which is made in the derivation of a Poisson process model (Fisz 1963) is valid. A simple test can be set for this purpose. The density function for the waiting time for the next event (*i.e.* the inter-arrival time) has already been estimated using Laguerre type Gram-Charlier series in § 3.2. If the earthquakes occur statistically independently, the density function for the waiting time for the second event would be a convolution of the density function for the waiting time for the next event with itself (Davenport and Root 1958) because the waiting time for the second event is the sum of the waiting time for the next event and the waiting time for the next event again. The density function for the waiting time for the next event is

$$p_{T_1}(t) = \sum_{n=0}^{6} a_n L_n(t) e^{-t}, \tag{52}$$

where the coefficients are given by (21). Then

$$p_{T_1}(t) * p_{T_1}(t) = \left\{\sum_{n=0}^{6} a_n L_n(t) e^{-t}\right\} * \left\{\sum_{n=0}^{6} a_n L_n(t) e^{-t}\right\} \tag{53}$$

where $*$ denotes convolution. Substituting for $L_n(t)$ from (15) and using the fact that

$$\begin{aligned}\{t^n e^{-t}\} * \{t^m e^{-t}\} &= \int_0^t u^n e^{-u} (t-u)^m e^{-(t-u)} \mathrm{d}u, \\ &= e^{-t} \int_0^t u^n (t-u)^m \mathrm{d}u, \\ &= e^{-t} \frac{(t^{n+m-1}) m!\, n!}{(n+m+1)!}\end{aligned} \tag{54}$$

we get

$$\begin{aligned}p_{T_1}(t) * p_{T_1}(t) = e^{-t}\{&1.18419t - 0.084268t^2 + 0.12952t^3 + 0.076443t^4 \\ &- 0.013856t^5 - 0.0036101t^6\}\end{aligned} \tag{55}$$

Equation (55) gives the density function for the waiting time for the second earthquake under the assumption that the earthquakes occur independently. The density function for the waiting time for the second earthquake can also be estimated empirically by the method in §3.2. For the Hindukush data it is (Goel 1982)

$$p_t(t) = e^{-t}[0.27272 + 5.17921t - 1.96555t^2 + 0.56842t^3 + 0.076003t^4 + 0.004638t^5 - 0.000103t^6]. \quad (56)$$

It is obvious that the two sets of coefficients in (55) and (56) do not match. Thus the assumption of the independent occurrence of earthquakes stands refuted.

The density function for the waiting time for the third earthquake has also been empirically estimated for the Hindukush data using the method of §3.2 and is (Goel 1982)

$$p_t(t) = e^{-t}[-0.006602 + 4.10569t - 3.09795t^2 + 0.88408t^3 - 0.11608t^4 + 0.0069548t^5 - 0.00015204t^6]. \quad (57)$$

Even this could have been used to check the hypothesis of the independent occurrence of earthquakes.

5. Data

The data regarding earthquakes in the Hindukush region are taken from the catalogue of epicentral locations of earthquakes prepared by the Indian Meteorological Department (IMD). These catalogues are prepared by IMD by using the data from USCSS, ISC, ISS and various other agencies. In all, 1535 earthquakes of magnitudes greater than 3.5, from January 1970 to December 1976, of focal depth less than 250 km in an area bounded by 69° E to 72° E longitude and 35° N to 38° N latitude, are used.

Another set of data regarding micro-earthquakes from North-Eastern region of India is also used. These data were collected by the University of Roorkee and GSI under a joint project for a period of 5.5 months from May 1979 to October 1979. There were stations at Raliang (25.47° N, 92.43° E), Borjori (26.40° N, 92.94° E), Burnihat (26.06° N, 91.89° E), and Shillong (25.57° N, 91.88° E). The Shillong station was run by the IMD. Magnitudes were between 2.2 to 5.1. Sources having distances more than about 300 km, *i.e.* those with S-P times greater than 40 sec were excluded. In all 235 events were used for the analysis.

5.1 *Conclusions*

(a) A simple test for independent occurrence of earthquakes was designed. The hypothesis that the earthquakes are independent events stands refuted.

(b) The Laguerre type Gram-Charlier series, in which the leading term is the negative exponential density function for the interarrival times for earthquakes under the Poisson process model for earthquakes, is the best method for the empirical estimation of waiting time distribution for earthquake occurrences. Some of the reasons for this are discussed in §§ 3.1 and 3.2.

(c) The scale factor β should be properly chosen for the fast convergence of orthonormal expansion over infinite and semi-infinite intervals.

(d) Though one of the assumptions of the Poisson process model for earthquakes is invalid, the estimated density function for the inter-arrival time of earthquakes is not very different from the negative exponential density function as shown by the small values of $a_1, a_2, \ldots$ in (21) and (51). The other consequences of the Poisson process model, such as other waiting time distributions, may be incorrect.

5.2 *Criticism and scope for future work*

(a) Various gamma density functions of (3) have been considered as models for the inter-arrival time of earthquakes (Udias and Rice 1975). The approach used here could be interpreted as a generalization of this notion because the Laguerre type Gram-Charlier series of § 3.2 can be viewed as a linear combination of various gamma density functions. If the gamma density function of (3) is fitted, the parameter p has been interpreted (Udias and Rice 1975) as the reciprocal of the average cluster size. Can this clue be generalized and a better index of clustering defined in terms of the coefficients of the Laguerre type Gram-Charlier series?

(b) The Laguerre type Gram-Charlier series could be interpreted as obtained by considering a linearly independent set of functions of (3), one for every positive integer value of p and obtaining a biorthonormal set of functions from them by Gram-Schmidt procedure (Barrett 1963). Similarly, one could start with a linearly independent set of functions $p\exp(-pt)$, $2p\exp(-2pt)$, $3p\exp(-3pt)$,... and obtain a biorthonormal set of functions from them by Gram-Schmidt procedure. This could be considered to be the basis of Legendre-type Gram-Charlier series which was derived in § 3.3 from a different starting point. In expanding square integrable functions, orthonormalization or biorthonormalization of a set of linearly independent functions is said to be a proper step. But in obtaining expansion for a density function, it need not be so. For example, one could have written

$$p(t) = \sum_{n=1}^{\infty} \alpha_n \frac{\lambda^n t^{n-1}}{(n-1)!} \exp(-\lambda t) u(t), \tag{58}$$

directly which amounts to writing a density function $p(t)$ as a convex combination of gamma density functions provided

$$\alpha_n \geq 0, \qquad \sum \alpha_n = 1. \tag{59}$$

Equation (18) is equivalent to (58) for square integrable bipolar functions, but nothing equivalent to (59) has been imposed therein. As a result there is a no guarantee that (18) will give a function which will satisfy $f(t) \geq 0$, so that $f(t)$ can be a valid density function. This can be clearly seen from (21), (51), (56) and (57). Similarly, one could have written

$$p(t) = \sum_{n=1}^{\infty} \alpha_n np \exp(-npt) \tag{60}$$

with (59) imposed. On the other hand Legendre type Gram-Charlier series has already yielded a blatantly nonphysical density function as reported in § 3.3. This is the consequence of using notions of orthonormality, biorthonormality, Gram-Schmidt procedure, etc. applicable for square integrable functions to the so-called class D of functions which are candidates for density functions. Suitable class D procedures should be developed. The Gram-Charlier expansion of § 2, however, cannot be interpreted as a convex combination of a family of density functions. This could be viewed as one more disadvantage of it.

(c) It has been suggested that the Laguerre type Gram-Charlier series or its class D counterpart is a particularly good choice for estimating the waiting time density functions for the future earthquakes in the context of a Poisson process model. This point should be raised to a general level. Suppose the null hypothesis is not a Poisson process model but a Polya process model (Fisz 1963; Sharma 1982; Sharma *et al* 1983). Then the various waiting time density functions are

$$p_{T_p}(t) = \frac{\nu(\nu+1)\ldots(\nu+p-1)}{(a+t)^{\nu+p}} \frac{t^{p-1}}{(p-1)!} \qquad t > 0 \quad p = 1, 2, 3, \ldots \tag{61}$$

where T_p is the waiting time for the pth earthquake, and a and ν are positive parameters. Then to use a Gram-Charlier type approach, we should obtain a set of orthonormal functions over the interval $[0, \infty)$ by Gram-Schmidt orthonormalization procedure and use this set of functions to develop a Polya type Gram-Charlier series for the estimation of waiting time density functions for future earthquakes. Or alternatively we should write

$$p(t) = \sum_{n=1}^{\infty} \alpha_n \frac{\nu(\nu+1)\ldots(\nu+n-1)}{(a+t)^{\nu+n}} \frac{t^{n-1}}{(n-1)!} \quad t > 0 \tag{62}$$

with (59) imposed.

(d) The Poisson process model of earthquakes implies that the inter-arrival times of earthquakes follow a negative exponential distribution. There are conflicting views in the literature whether negative exponential distribution for the inter-arrival times of earthquakes implies a Poisson process model for earthquakes (Massey 1971; Udias and Rice 1975; Vere Jones 1970). But the correct answer to the second question must be negative as explained below. Under the Poisson process model, the density function for the waiting time T_p for the pth earthquake can be shown to be (Goel 1982; Sharma 1982; Sharma *et al* 1983)

$$p_{T_p}(t_p) = \frac{\lambda^p t_p^{p-1}}{(p-1)!} \exp(-\lambda t_p) u(t_p), \qquad \lambda > 0, \qquad p \text{ positive integer.} \tag{63}$$

It is important to note that (63) lists various implications of a Poisson process model. It can also be seen that (63) cannot be derived from (2) alone. Thus a Poisson process model implies (2), but (2) does not necessarily imply a Poisson process model, because (63) for $p = 2, 3, \ldots$ may still be violated. It is, therefore, instructive to find out under what conditions (2) implies (63). It can be shown that (Goel 1982).

$$p_{T_m}(t) * p_{T_n}(t) = p_{T_{m+n}}(t) \tag{64}$$

where $*$ represents convolution. Thus convolution of the waiting time density functions for the nth and mth earthquakes given by (63) gives the waiting time density function for the $(n + m)$th earthquake according to (63). In particular

$$p_{T_p}(t) = p_{T_1}(t) * p_{T_1}(t) * \ldots * p_{T_1}(t), \tag{65}$$

where the term on the right side is a p-fold convolution. Convolution of the density functions of two or more random variables gives the density function of their sum provided the random variables are independent (Davenport and Root 1958). Thus (63) follows from (2) only if the events are statistically independent. Equation (2) alone does not imply that the earthquakes are independent. The conclusion in § 5(d) should be viewed in this light.

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