ARITHMETICALLY COHEN-MACAULAY BUNDLES ON THREE DIMENSIONAL HYPERSURFACES

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Abstract. We prove that any rank two arithmetically Cohen-Macaulay vector bundle on a general hypersurface of degree at least six in $\mathbb{P}^4$ must be split.

1. Introduction

An arithmetically Cohen-Macaulay (ACM for short) vector bundle on a hypersurface $X \subset \mathbb{P}^n$ is a bundle E for which $\mathrm{H}^i(X, E(k)) = 0$ for $0 < i < n-1$ and for all integers k. ACM bundles of large rank, which are not split as a sum of line bundles, exist on any hypersurface X of degree > 1 (see [2]), and it is also conjectured by Buchweitz-Greuel-Schreyer (*op. cit.*) that low rank ACM bundles on smooth hypersurfaces should be split. For example, it is well known [7] that there are no non-split ACM bundles of rank two on a smooth hypersurface in $\mathbb{P}^6$. In [6], it was proved that on a general hypersurface of degree ≥ 3 in $\mathbb{P}^5$, there are no non-split ACM bundles of rank two.

In the current paper, we extend this result to general hypersurfaces of degree $d \geq 6$ in $\mathbb{P}^4$:

Main Theorem. *Fix $d \geq 6$. There is a non-empty Zariski open set of hypersurfaces of degree d in $\mathbb{P}^4$, none of which support an indecomposable ACM rank two bundle.*

The special case when $d = 6$ was proved by Chiantini and Madonna [3]. The result we prove is optimal and we refer the reader to [6] for more details.

Our result can also be translated into a statement about curves on X: on a general hypersurface X in $\mathbb{P}^4$ of degree $d \geq 6$, any arithmetically Gorenstein curve on X is a complete intersection of X with two other hypersurfaces in $\mathbb{P}^4$. Yet another translation of the result is that the defining equation of such a hypersurface cannot be expressed as the Pfaffian of a skew-symmetric matrix in a non-trivial way.

In the current paper, we will need some of the results from [6]. We will also use the relation between rank two ACM bundles on hypersurfaces and Pfaffians that was observed by Beauville in [1] and which was not needed in [6].

As usual, let $\mathrm{H}^i_*(X, E)$ denote the graded module $\oplus_{k\in\mathbb{Z}}\mathrm{H}^i(X, E(k))$. The following theorem summarizes and paraphrases some of the results from [6] that are important for the proof of the Main Theorem above.

Theorem 1 ([6] Thm 1.1(3), Cor 2.3)**.** *Let E be an indecomposable rank two ACM bundle on a smooth hypersurface X of degree d in $\mathbb{P}^4$. Then $\mathrm{H}^2_*(X, \mathcal{E}nd(E))$ is a non-zero cyclic module of finite length, with the generator living in degree $-d$. If $d \geq 5$ and X is general, then $\mathrm{H}^2(X, \mathcal{E}nd(E)) = 0$.*

2. ACM bundles and Pfaffians

We work over an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Let E be an ACM vector bundle of rank two on X. By Horrocks' criterion [5], this is equivalent to saying that E has a resolution,

$$0 \to F_1 \xrightarrow{\Phi} F_0 \xrightarrow{\sigma} E \to 0, \tag{1}$$

where the F_i's are direct sums of line bundles on $\mathbb{P}^n$. We will assume that this resolution is minimal, with $F_0 = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^n}(-a_i)$ where $a_1 \leq a_2 \leq \cdots \leq a_n$. Using [1], we may write F_1 as $F_0^\vee(e-d)$, where e is the first Chern class of E, and we may assume that Φ is a skew-symmetric $n \times n$ matrix with n even. The (i,j)-th entry ϕ_{ij} of Φ has degree $d - e - a_i - a_j$. The condition of minimality implies that there are no non-zero scalar entries in Φ and thus every degree zero entry must be zero.

We quote some facts about Pfaffians and refer the reader to [9] for more details. Let $\Phi = (\phi_{ij})$ be an $n \times n$ even-sized skew symmetric matrix and let $\mathrm{Pf}(\Phi)$ denote its *Pfaffian*. Then $\mathrm{Pf}(\Phi)^2 = \det \Phi$. Let $\Phi(i,j)$ be the matrix obtained from Φ by removing the i-th and j-th rows and columns. Let Ψ be the skew-symmetric matrix of the same size with entries $\psi_{ij} = (-1)^{i+j}\mathrm{Pf}(\Phi(i,j))$ for $0 \leq i < j \leq n$. We shall refer to $\mathrm{Pf}(\Phi(i,j))$ as the (i,j)-Pfaffian of Φ. The product $\Phi\Psi = \mathrm{Pf}(\Phi)\mathrm{I}_n$ where I_n is the identity matrix.

Example 1. Let $n = 4$ above. Then

$$\mathrm{Pf}(\Phi) = \phi_{12}\phi_{34} - \phi_{13}\phi_{24} + \phi_{14}\phi_{23}.$$

The following lemma shows the relation between skew-symmetric matrices, ACM rank 2 bundles and the equation defining the hypersurface.

Lemma 1. *Let E be a rank 2 ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree d and let $\Phi : F_1 \to F_0$ be the minimal skew-symmetric matrix associated to E. Then $X = X_\Phi$, the zero locus of $\mathrm{Pf}(\Phi)$. Conversely, let $\Phi : F_1 \to F_0$ be a minimal skew-symmetric matrix such that the hypersurface X_Φ defined by $\mathrm{Pf}(\Phi)$ is smooth of degree d. Then E_Φ, the cokernel of Φ, is a rank 2 ACM bundle on X_Φ.*

Proof. Let $f \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be the polynomial defining X. Since E is supported along X, $\det \Phi = f^n$ for some n upto a non-zero constant where Φ is as in resolution (1). Locally E is a sum of two line bundles and so the matrix Φ is locally the diagonal matrix $(f, f, 1, \cdots, 1, 1)$. Since the determinant of this diagonal matrix is f^2, we get $f = \mathrm{Pf}(\Phi)$ (upto a non-zero constant).

To see the converse: let Φ be any skew-symmetric matrix and Ψ be defined as above. Let $f = \mathrm{Pf}(\Phi)$ be the Pfaffian. Since $\Phi\Psi = f\mathrm{I}_n$, this implies that the composite $F_0(-d) \xrightarrow{f} F_0 \to E_\Phi$ is zero. Thus E_Φ is annihilated by f and so is supported on the hypersurface X_Φ defined by f. Since X_Φ is smooth, by the Auslander-Buchsbaum formula, E_Φ is a vector bundle on X_Φ. Therefore locally Φ is a diagonal matrix of the form $(f, \cdots, f, 1, \cdots, 1)$ where the number of f's in the diagonal is equal to the rank of E. Since $\det(\Phi) = f^2$, we conclude that rank of E_Φ is 2. □

Let $V \subset \mathrm{Hom}(F_1, F_0)$ be the subspace consisting of all minimal skew-symmetric homomorphisms, where F_i's are as above. The following is an easy consequence of the above lemma.

Lemma 2. *Let $\Phi_0 \in V$ be an element such that E_{Φ_0} is a rank 2 ACM bundle on a smooth hypersurface X_{Φ_0}. Then there exists a Zariski open neighbourhood U of Φ_0 such that for any $\Phi \in U$, X_Φ is a smooth hypersurface and E_Φ is a rank two ACM bundle supported on X_Φ.*

3. Special cases

The proof of the Main Theorem will require the study of some special cases, which are listed below.

Lemma 3. *Consider the following three types of curves in $\mathbb{P}^4$:*

- *a curve C which is the complete intersection of three general hypersurfaces, two of which are of degree ≤ 2.*

- *a curve D which is the locus of vanishing of the principal 4×4 sub-Pfaffians of a general 5×5 skew-symmetric matrix χ of linear forms.*
- *a curve C_r, $r \geq 0$, which is the locus of vanishing of the 2×2 minors of a general 4×2 matrix Δ with one row consisting of forms of degree $1 + r$, and the remaining three rows consisting of linear forms.*

The general hypersurface X in $\mathbb{P}^4$ of degree ≥ 6 cannot contain any curve of the the first two types. The general hypersurface X of degree $d \geq \max\{6, r+4\}$ cannot contain any curve of the third type.

Proof. The curve C is smooth if the hypersurfaces are general. If χ is general, the curve D is smooth (see [10], page 432 for example). If Δ is general, the curve C_r is smooth (see *op. cit.* page 425).

The proof of the lemma is a straightforward dimension count. By counting the dimension of the set of all pairs (Y, X) where Y is a smooth curve of the described type and X is a hypersurface of degree d containing Y, it suffices to show that this dimension is less than the dimension of the set of all hypersurfaces X of degree d in $\mathbb{P}^4$. This can be done by showing that if $\mathcal{S}$ denotes the (irreducible) subset of the Hilbert scheme of curves in $\mathbb{P}^4$ parameterizing all such smooth curves Y, then the dimension of $\mathcal{S}$ is at most $h^0(\mathcal{O}_Y(d)) - 1$.

This argument was carried out in [8] where Y is any complete intersection curve in $\mathbb{P}^4$. The case where Y equals the first type of curve C in the list above is Case 2 of [8]. Hence we will only consider the types of curves D and C_r here.

If Y is of type D in the list, the sheaf $\mathcal{I}_D$ has the following free resolution ([10], page 427):

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 5} \xrightarrow{\chi} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 5} \to \mathcal{I}_D \to 0.$$

Computing Hilbert polynomials, we see that D is a smooth elliptic quintic in $\mathbb{P}^4$, and it easily computed that that $h^0(\mathcal{N}_D) = 25$. Since $h^0(\mathcal{O}_D(d)) = 5d$, for $d \geq 6$, we get $\dim \mathcal{S} \leq h^0(\mathcal{N}_D) \leq h^0(\mathcal{O}_D(d)) - 1$.

If Y is of type C_r in the list, we may analyze the dimension of the parameter space of all such C_r's as follows. Let S be the cubic scroll in $\mathbb{P}^4$ given by the vanishing of the two by two minors of the linear 3×2 submatrix θ of the 4×2 matrix Δ. The ideal sheaf of the determinantal surface S has resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-3)^2 \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^n}(-2)^3 \to \mathcal{I}_S \to 0.$$

From this one computes the dimension of the set of such cubic scrolls to be 18, since the 30 dimensional space of all 3×2 linear matrices is acted on by automorphisms of $\mathcal{O}_{\mathbb{P}^n}(-3)^2$ and $\mathcal{O}_{\mathbb{P}^n}(-2)^3$, with scalars giving the stabilizer of the action. Furthermore, by dualizing the resolution, we get a resolution for ω_S:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 3} \xrightarrow{\theta^\vee} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2} \to \omega_S \to 0.$$

A section of $\omega_S(r+3)$ gives a lift $\mathcal{O}_{\mathbb{P}^n}(-r-3) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2}$, and we obtain a 4×2 matrix $\begin{pmatrix} \theta \\ \alpha^\vee \end{pmatrix}$ of the required type. Hence C_r is a curve on S in the linear series $|K_S + (r+3)H|$, where H is the hyperplane section on S. Intersection theory on S gives $K_S.K_S = 8$, $K_S.H = -5$ and $H.H = 3$. Using this, we may compute the dimension of the linear system of C_r on S, and we get the dimension of the set $\mathcal{S}$ of all such C_r in $\mathbb{P}^4$ to be 21 if $r = 0$ and $(3/2)r^2 + (13/2)r + 24$ otherwise.

The ideal sheaf of C_r has a free resolution given by the Eagon-Northcott complex [4]

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^n}(-r-4)^{\oplus 3} &\to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(-r-3)^{\oplus 6} \\ &\to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^n}(-r-2)^{\oplus 3} \to \mathcal{I}_{C_r} \to 0. \end{aligned}$$

Let $d \geq max\{6, r+4\}$ be chosen as in the statement of the lemma. Then $d = r+s+4$ where $s \geq 0$ ($s \geq 2$ when $r = 0$; $s \geq 1$ when $r = 1$). Using the above resolution, a calculation gives

$$h^0(\mathcal{O}_{C_r}(d)) = \frac{3}{2}r^2 + \frac{29}{2}r + 3rs + 4s + 17.$$

The required inequality $\dim \mathcal{S} < h^0(\mathcal{O}_{C_r}(d))$ is now evident. $\square$

4. Proof of Main Theorem

In this section, E will be an indecomposable ACM bundle of rank two and first Chern class e on a smooth hypersurface X of degree d in $\mathbb{P}^4$. The minimal resolution (1) gives $\sigma : F_0 \to E \to 0$, and we may describe σ as $[s_1, s_2, \cdots, s_n]$ where $s_1, s_2, \cdots, s_n$ is a set of minimal generators of the graded module $H^0_*(E)$ of global sections of E, with degrees $a_1 \leq a_2 \leq \cdots \leq a_n$.

Lemma 4. *If E is an indecomposable rank 2 ACM bundle with first Chern class e on a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$, then there is a relation in degree $3-e$ among the minimal generators of S^2E.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}nd(E) \to (S^2E)(-e) \to 0.$$

$S^2E(-e)$ has the same intermediate cohomology as $\mathcal{E}nd(E)$ since the sequence splits in characteristic zero.

Choose a minimal resolution of S^2E:

$$0 \to B \to C \to S^2E \to 0,$$

where C is a direct sum of line bundles on X and B is a bundle on X with $\mathrm{H}^1_*(X, B) = 0$.

We first show that $B^\vee(e + d - 5)$ is not regular. For this, consider the dual sequence $0 \to (S^2E)^\vee \to C^\vee \to B^\vee \to 0$.

By Serre duality and Theorem 1

$$\mathrm{H}^1(X, (S^2E)^\vee(d + e - 5)) = 0.$$

Therefore

$$\mathrm{H}^0(X, C^\vee(d + e - 5)) \to \mathrm{H}^0(X, B^\vee(d + e - 5))$$

is onto. If $B^\vee(d + e - 5)$ were regular, the same would be true for

$$\mathrm{H}^0(X, C^\vee(d + e - 5 + k)) \to \mathrm{H}^0(X, B^\vee(d + e - 5 + k)) \quad \forall\ k \geq 0.$$

However, this is false for $k = d$ since by Serre duality and Theorem 1, $\mathrm{H}^1(X, (S^2E)^\vee(2d+e-5)) \neq 0$. Thus $B^\vee(e+d-5)$ is not regular. Now

$$\begin{aligned}\mathrm{H}^1(X, B^\vee(e + d - 6)) \cong \mathrm{H}^2(X, B(1 - e)) &\cong \mathrm{H}^1(X, S^2E(1 - e)) \\ &\cong \mathrm{H}^1(X, \mathcal{E}nd(E)(1)).\end{aligned}$$

By Serre duality,

$$\mathrm{H}^1(X, \mathcal{E}nd(E)(1)) \cong \mathrm{H}^2(X, \mathcal{E}nd(E)(d - 6))$$

which by Theorem 1 equals zero for $d \geq 6$ (this is the main place where we use the hypothesis that $d \geq 6$). Furthermore, $\mathrm{H}^2(X, B^\vee(e+d-7)) = 0$ since $\mathrm{H}^1_*(X, B) = 0$. Since $B^\vee(e + d - 5))$ is not regular, we must have $\mathrm{H}^3(X, B^\vee(e + d - 8)) \neq 0$.

In conclusion, $\mathrm{H}^0(X, B(3-e)) \neq 0$. In other words, there is a relation in degree $3 - e$ among the minimal generators of S^2E. $\square$

Lemma 5. *Let E be as above. Then $1 \leq a_1 + a_2 + e \leq a_1 + a_3 + e \leq 2$.*

Proof. The resolution (1) for E gives an exact sequence of vector bundles on X: $0 \to G \to \overline{F}_0 \xrightarrow{\sigma} E \to 0$, where $\overline{F}_0 = F_0 \otimes \mathcal{O}_X$ and G is the kernel. This yields a long exact sequence,

$$0 \to \wedge^2 G \to \overline{F}_0 \otimes G \to S^2\overline{F}_0 \to S^2E \to 0.$$

From the arguments after Lemma 2.1 of [6] (using formula (5)), it follows that $\mathrm{H}^2_*(\wedge^2 G) = 0$. Hence the map $S^2\overline{F}_0 \to S^2E$ is surjective on global sections. The image of this map picks out the sections s_is_j of degree $a_i + a_j$ in S^2E. Observe that the lowest degree minimal sections

s_1, s_2 of E induce an inclusion of sheaves $\mathcal{O}_X(-a_1) \oplus \mathcal{O}_X(-a_2) \overset{[s_1,s_2]}{\hookrightarrow} E$ whose cokernel is supported on a surface in the linear system $|\mathcal{O}_X(a_1 + a_2 + e)|$ on X (a nonempty surface when E is indecomposable). Hence $1 \leq a_1 + a_2 + e$. There is an induced inclusion

$$S^2[\mathcal{O}_X(-a_1) \oplus \mathcal{O}_X(-a_2)] \hookrightarrow S^2E.$$

Therefore the three sections of S^2E given by s_1^2, s_1s_2, s_2^2 cannot have any relations amongst them. Since these are also three sections of S^2E of the lowest degrees, they can be taken as part of a minimal system of generators for S^2E. It follows that the relation in degree $3-e$ among the minimal generators of S^2E obtained in the previous lemma must include minimal generators other than s_1^2, s_1s_2, s_2^2. Since the other minimal generators have degree at least $a_1 + a_3$, and since we are considering a relation amongst minimal generators, we get the inequality $a_1 + a_3 \leq 2 - e$. $\square$

Lemma 6. *For any choice of* $1 \leq i < j \leq n$*, the* (i,j)*-Pfaffian of* Φ*, is non-zero. Consequently, its degree (which is* $a_i + a_j + e$*) is at least* $(n-2)/2$*.*

Proof. On X, E has an infinite resolution

$$\cdots \to \overline{F}_0^\vee(e-2d) \xrightarrow{\overline{\Phi}} \overline{F}_0(-d) \xrightarrow{\overline{\Psi}} \overline{F}_0^\vee(e-d) \xrightarrow{\overline{\Phi}} \overline{F}_0 \to E \to 0.$$

We also have

$$\begin{array}{ccccccc} \overline{F}_0^\vee(e-2d) & \xrightarrow{\overline{\Phi}} & \overline{F}_0(-d) & \xrightarrow{\sigma} & E(-d) & & \\ & & & & \downarrow \overset{\alpha}{\cong} & & \\ & & & & E^\vee(e-d) & \overset{\sigma^\vee}{\hookrightarrow} & \overline{F}_0^\vee(e-d) & \xrightarrow{\overline{\Phi}^\vee} & \overline{F}_0. \end{array}$$

Let $\overline{\Theta} = \sigma^\vee \alpha \sigma$. Since $\sigma = (s_1, \cdots, s_n)$, we may express the (i,j)-th entry of $\overline{\Theta}$ as $\theta_{ij} = s_i^\vee s_j$ (suppressing the canonical isomorphism α). $\Phi^\vee = -\Phi$ and $\alpha\sigma : \overline{F}_0(-d) \to E^\vee(e-d)$ is surjective on global sections. Hence we have a commuting diagram

$$\begin{array}{ccccccccccc} \overline{F}_0^\vee(e-2d) & \xrightarrow{\overline{\Phi}} & \overline{F}_0(-d) & \xrightarrow{\overline{\Psi}} & \overline{F}_0^\vee(e-d) & \xrightarrow{\overline{\Phi}} & \overline{F}_0 & \to & E & \to & 0 \\ & & \downarrow \overset{B}{\cong} & & \downarrow \overset{-I}{\cong} & & || & & || & & \\ \overline{F}_0^\vee(e-2d) & \xrightarrow{\overline{\Phi}} & \overline{F}_0(-d) & \xrightarrow{\overline{\Theta}} & \overline{F}_0^\vee(e-d) & \xrightarrow{-\overline{\Phi}} & \overline{F}_0 & \to & E & \to & 0 \end{array}$$

It is easy to see that B is an isomorphism. As a result, every column of B has a non-zero scalar entry.

Now suppose that $\overline{\psi}_{ij} = 0$ for some i, j so that $\sum_k s_i^\vee s_k b_{kj} = 0$. Let Y_i be the curve given by the vanishing of the minimal section s_i with the exact sequence

$$0 \to \mathcal{O}_X(-a_i) \xrightarrow{s_i} E \xrightarrow{s_i^\vee} I_{Y_i/X}(a_i + e) \to 0.$$

Hence $s_i^\vee s_i = 0$ and $s_i^\vee s_k$ for $k \neq i$ give minimal generators for $I_{Y_i/X}$. It follows that no b_{kj} can be a non-zero scalar for $k \neq i$. Hence b_{ij} has to be a non-zero scalar and the only one in the j-th column. However, $\overline{\psi}_{jj} = 0$. So by the same argument, b_{jj} is the only non-zero scalar. To avoid contradiction, $\overline{\psi}_{ij}$ and hence $\psi_{ij} \neq 0$ for $i \neq j$. □

We now complete the proof of the Main Theorem. As in the previous lemmas, assume that X is general of degree $d \geq 6$, with E an indecomposable rank two ACM bundle on X. We will show that the inequalities of Lemma 5 lead us to the special cases of Lemma 3, giving a contradiction.

Let $\mu = a_1 + a_2 + e$. By Lemma 5, $1 \leq \mu \leq 2$.

Case $\mu = 1$**.** In this case, in order for the $(1,2)$-Pfaffian of Φ to be linear, by Lemma 6, n must equal 4. In the 4×4 matrix Φ, the $(1,2)$-Pfaffian is the entry ϕ_{34} which we are claiming is linear. Likewise the $(1,3)$-Pfaffian is the entry ϕ_{24} which by Lemma 5 has degree $a_1+a_3+e \leq 2$. By Lemma 2, we may assume that $\phi_{14}, \phi_{24}, \phi_{34}$ define a smooth complete intersection curve and X contains this curve by example 1. By Lemma 3, X cannot be general.

Case $\mu = 2$**.** In this case $a_2 = a_3$. By Lemma 6, n must be 4 or 6. The case $n = 4$ is ruled out again by the arguments of the above paragraph since Φ has two entries of degree 2 in its last column. We will therefore assume that $n = 6$. The matrix

$$\Phi = \begin{pmatrix} 0 & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} \\ * & 0 & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} \\ * & * & 0 & \phi_{34} & \phi_{35} & \phi_{36} \\ * & * & * & 0 & \phi_{45} & \phi_{46} \\ * & * & * & * & 0 & \phi_{56} \\ * & * & * & * & * & 0 \end{pmatrix}$$

is skew-symmetric and by our choice of ordering of the a_i's, the degrees of the upper triangular entries are non-increasing as we move to the right or down.

As remarked before, the degree of ϕ_{ij} is $d - e - a_i - a_j$. The $(1,2)$-Pfaffian (which is a non-zero quadric when $\mu = 2$) is given by the expression (see example 1)

$$\text{Pf}(\Phi(1,2)) = \phi_{34}\phi_{56} - \phi_{35}\phi_{46} + \phi_{36}\phi_{45}. \tag{2}$$

We shall consider the following two sub-cases, one where ϕ_{56} has positive degree (and hence can be chosen non-zero by Lemma 2) and the other where it has non-positive degree (and hence is forced to be zero):

$\underline{d-e-a_5-a_6>0}$.

Since $\phi_{34}\cdot\phi_{56}$ is one term in the $(1,2)$-Pfaffian of Φ, and since degree ϕ_{34} is at least degree ϕ_{56}, they are both forced to be linear. Therefore ϕ_{34}, ϕ_{35}, ϕ_{36} have the same degree (=1) and so $a_4=a_5=a_6$. Likewise, $a_3=a_4=a_5$. Therefore $a_2=a_3=a_4=a_5=a_6$. Hence Φ has a principal 5×5 submatrix χ (obtained by deleting the first row and column in Φ) which is a skew symmetric matrix of linear terms, while its first row and first column have entries of degree $1+r, r\geq 0$.

By Lemma 2 we may assume that the ideal of the 4×4 Pfaffians of χ defines a smooth curve C. X is then a degree $d=3+r$ hypersurface containing C. By Lemma 3, X cannot be general when $d\geq 6$.

$\underline{d-e-a_5-a_6\leq 0}$.

In this case, the entry $\phi_{56}=0$. Suppose ϕ_{46} is also zero. Then both ϕ_{36} and ϕ_{45} must be linear and non-zero since the $(1,2)$-Pfaffian of Φ (see equation 2) is a non-zero quadric. Since $a_2=a_3$, ϕ_{26} is also linear. Thus using Lemma 2, X contains the complete intersection curve given by the vanishing of ϕ_{16} and the two linear forms ϕ_{36},ϕ_{26}. By Lemma 3, X cannot be general.

So we may assume that $\phi_{46}\neq 0$. Since ϕ_{35} is also non-zero, both must be linear. Hence $a_3+a_5=a_4+a_6$, and so $a_3=a_4$ and $a_5=a_6$.

After twisting E by a line bundle, we may assume that $a_2=a_3=a_4=0\leq a_5=a_6=b$. The linearity of the entry ϕ_{46} gives $d-e-b=1$. The condition $d-e-a_5-a_6\leq 0$ yields $1\leq b$. Taking first Chern classes in resolution (1) gives $e=2-a_1$.

Let $r=-a_1,\ s=b-1$. Then $r,\ s\geq 0$, and $d=r+s+4$. If we inspect the matrix Φ, the non-zero rows in columns 5 and 6 give a 4×2 matrix Δ with top row of degree $1+r$ and the other entries all linear. By Lemma 2, we may assume that the 2×2 minors of this 4×2 matrix define a smooth curve C_r as described in Lemma 3. Since X contains this curve, X cannot be general when $d\geq 6$.

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