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Construction of rank two vector bundles on $\mathbf{P}^4$ in positive characteristic

N. Mohan Kumar

Department of Mathematics, Washington University, St. Louis, MO 63130, USA (Kumar@math.wustl.edu)

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1. Introduction

Vector bundles on Projective spaces have been the subject of many papers and many results in this direction are known. For a somewhat dated account, the reader may see [OSS82]. One of the most interesting problems in this area is the study of small rank bundles on Projective spaces. For example, a conjecture by Hartshorne [Har74] states that there are no small rank vector bundles on Projective spaces, other than direct sum of line bundles. The solution to this tantalising problem still seems remote, though very many results are known. Let me restrict my attention to rank 2 bundles for the moment. Many interesting bundles of rank 2 are known over Projective spaces of dimension 2 and 3. But over $\mathbf{P}^4$, essentially the only interesting one known is the well known Horrocks-Mumford bundle [HM73]. There are also some interesting ones in characteristic 2, discovered by Tango [Tan76] and Horrocks [Hor80].

In this paper, we shall deal with this problem and prove a criterion relating bundles on $\mathbf{P}^{n+1}$ to bundles on $\mathbf{P}^n$. This condition on certain bundles over $\mathbf{P}^n$ is necessary and sufficient for the existence of bundles on $\mathbf{P}^{n+1}$. Though this criterion is not very pleasant, it allows you to restrict your attention to bundles just on $\mathbf{P}^n$ to construct bundles on $\mathbf{P}^{n+1}$. Using this criterion (which has nothing to do with the characteristic of the field), we construct many rank two bundles on $\mathbf{P}^4$ over a field of positive characteristic. (For the Chern classes of these bundles, see Sect. 3.1). Unfortunately we have not been able to extend our construction to complex numbers, though I feel it should be possible. The construction follows closely what we did in [Kum91].

2 General remarks

Let me start with a word about notation. We will have to deal with maps, $\phi: M \to M \otimes L$ often in this article, where M is a sheaf and L, a line bundle. Then it makes sense to talk about $\phi \otimes \mathrm{Id}: M \otimes L' \to M \otimes L \otimes L'$ for any line bundle L'. We will denote this map also for brevity by ϕ. Also it makes sense to talk about $\phi^i: M \to M \otimes L^i$, by composing ϕ. So we will talk about ϕ as an *endomorphism*, though strictly speaking, it is not. It also make sense to say when such a map is *nilpotent*, by saying that $\phi^n = 0$ for some n.

We make the following transparent observation:

Remark. If $\pi: X \to Y$ is a finite map then the category of sheaves on Y which are $\pi_* \mathcal{O}_X$ modules is the same as the category of sheaves of the form $\pi_* \mathscr{F}$ where $\mathscr{F}$ is a sheaf on X (with the appropriate homomorphisms).

A typical case where we plan to apply this is when X is the m^{th} order thickening of $\mathbf{P}^n \subset \mathbf{P}^{n+1}$ and $Y = \mathbf{P}^n$ with π the projection from a point in $\mathbf{P}^{n+1}$ away from this hyperplane. We had applied this in [Kum91] in a similar but slightly different context.

So let X be the m^{th} order thickening of $\mathbf{P}^n \subset \mathbf{P}^{n+1}$ and $Y = \mathbf{P}^n$ and $\pi: X \to Y$ the projection from a point away from this hyperplane.

1. Then $\pi_* \mathcal{O}_X = \oplus_{i=0}^{m-1} \mathcal{O}_Y(-i)$. Thus we see that giving a sheaf on X is equivalent to giving a sheaf $\mathscr{F}$ on Y and an endomorphism $\phi: \mathscr{F} \to \mathscr{F}(1)$ with $\phi^m = 0$.
2. Let (E_1, ϕ_1) and (E_2, ϕ_2) be two such sheaves on Y. Giving a map $\psi: E_1 \to E_2$ which commutes with the ϕ_i's is equivalent to giving a map between the corresponding sheaves on X.
3. We want to apply this to the special case when E_1 arises as the restriction of a direct sum of line bundles say F on $\mathbf{P}^{n+1}$. We will write for clarity G, for F restricted to $\mathbf{P}^n$. Then we see that

$$E_1 = \pi_*(F\mid_X) = \oplus_{i=0}^{m-1} G(-i)$$

as before. ϕ_1 can be identified with the map which just shifts the blocks. That is to say the ϕ_1 takes $G(-i)$ to the corresponding $G(-i)$ in $E_1(1)$ as identity. (Of course $G(-m+1)$ goes to zero).
4. Thus giving a map from F as above to the sheaf corresponding to (E, ϕ) is just giving a map $\theta: G \to E$. Because then we get for any i a map $G(-i) \to E$ by taking the induced map $G(-i) \to E(-i)$ and then composing it with ϕ^i.

Now let E be any vector bundle on $\mathbf{P}^{n+1}$ and let $Y = \mathbf{P}^n$ be any hyperplane. Then by Quillen–Suslin Theorem [*e.g.*, see[Qui76]], E restricted to the complement of this hyperplane is free. Thus, if we denote by $F = \mathcal{O}^r_{\mathbf{P}^{n+1}}$ where $r =$ rank E, then we have an exact sequence,

$$0 \to E(-i) \to F \to \mathscr{F} \to 0$$

for some integer i and $\mathscr{F}$ is a coherent sheaf on some X as above (m^{th} order thickening of the hyperplane for some m). Since we will be primarily interested in deciding when such an E is a direct sum of line bundles, we may as well rename $E(-i)$ by E. Using π as above we get a coherent sheaf $\pi_*\mathscr{F} = M$ on Y. But $\mathscr{F}$ has homological dimension one and therefore by Auslander-Buchsbaum Theorem [*e.g.*, see [Mat70]], M is a vector bundle. As above, we also have a nilpotent map $\phi: M \to M(1)$. Letting $G = \mathcal{O}_Y^r$ we also have a map $\psi: G \to M$ since the distinguished r sections of $\mathscr{F}$ give r sections of M. Further by the surjectivity of the above map from $F \to \mathscr{F}$, we see that

$$\phi(M(-1)) + \psi(G) = M$$

What we want to state is the converse of this remark.

Lemma 1. *Let $Y = \mathbf{P}^n$, G be a rank r bundle on Y which is a direct sum of line bundles. Assume M is a vector bundle on Y with a nilpotent map $\phi: M \to M(1)$ and a map $\psi: G \to M$ such that $\phi(M(-1)) + \psi(G) = M$. Then there exists a vector bundle E of rank r on $\mathbf{P}^{n+1}$ and an exact sequence,*

$$0 \to E \to F \to \mathscr{F} \to 0 \tag{1}$$

where F is direct sum of r line bundles on $\mathbf{P}^{n+1}$ with $F \mid_{\mathbf{P}^n} = G$ and $\pi_*\mathscr{F} = M$.

Proof. Proof is obvious using the remarks above. □

Remark. The above lemma can be also thought of as a criterion for extending vector bundles from $\mathbf{P}^n$ to $\mathbf{P}^{n+1}$. In other words, given a vector bundle E of rank r on $\mathbf{P}^n$, it can be extended to $\mathbf{P}^{n+1}$ as a vector bundle if and only if there exists a vector bundle M over $\mathbf{P}^n$ with a nilpotent endomorphism $\phi: M \to M(1)$, a map $\psi: G \to M$ where G is a direct sum of r line bundles and an exact sequence,

$$0 \to E \to M(-1) \oplus G \xrightarrow{(\phi,\psi)} M \to 0$$

Lemma 2. *Let the notation be as in the above lemma and assume $n \geq 2$. If M is not a direct sum of line bundles then E is not a direct sum of line bundles. Conversely, if $r \leq n$, and M is a direct sum of line bundles, then so is E.*

Proof. Assume M is not a direct sum of line bundles. Then by Horrock's criterion. [*e.g.*, see[OSS82]], $H^i(M(l)) \neq 0$ for some i, with $0 < i < n$ and some $l \in \mathbf{Z}$. $H^j(F(l)) = 0$ $\forall j, 0 < j \leq n$, since F is a direct sum of line bundles. Therefore from our exact sequence 1, $H^i(\mathscr{F}(l)) = H^{i+1}(E(l))$. Also $H^i(\mathscr{F}(l)) = H^i(M(l))$ since π is a finite map from supp $\mathscr{F} \to Y$. Thus $H^{i+1}(E(l)) \neq 0$ and since $0 < i+1 < n+1$, we see that E is not a direct sum of line bundles.

Conversely, assume that M is a direct sum of line bundles. Exactly as before, we get $H^i(E(l)) = 0 \ \ \forall l, 1 < i \leq n$. By duality, $H^i(E^*(l)) = 0 \ \ \forall l$, $0 < i < n$. Thus by the Syzygy theorem [EG81] rank of $E^* = r > n$ or E^* is a direct sum of line bundles. □

3 The construction

In this section, we will outline the construction of bundles M on $\mathbf{P}^3$ as described above. More generally, let $d \in \mathbf{Z}$ be any integer. We will construct a bundle M on $\mathbf{P}^3$ with a nilpotent endomorphism $\phi: M \to M(d)$ and a map $\psi: G \to M$ where G is the direct sum of two line bundles such that $\phi(M(-d)) + \psi(G) = M$ and M is not a direct sum of line bundles, over a field of positive characteristic.

Remark. The only interesting cases are $d = -1, 0, 1$. If we have M's for these values, then by taking the pull back of these by finite maps $\mathbf{P}^3 \to \mathbf{P}^3$, we can construct bundles for all d. The case $d = 0$ was treated in [Kum91].

Let $p > 0$ be the characteristic of our algebraically closed field. Choose positive numbers N, k, l so that $N - k, N - l$ both positive, $4pkl > d^2$ and $p(k + l) = (p - 1)N + d$. Let x, y, z, t be the homogeneous co-ordinates of $\mathbf{P}^3$. Let $A = x^k z^{N-k} + y^l t^{N-l}$. Let C_i be the curve defined by the vanishing of x^{pk}, y^{pl} and A^i, for $1 \leq i \leq p$. Let C be the curve (line) defined by $x = y = 0$.

1. C_i's are local complete intersection curves for $1 \leq i \leq p$ and C_p is a complete intersection of x^{pk}, y^{pl}.

Proof. C_p is a complete intersection is clear, since A^p is in the ideal generated by x^{pk} and y^{pl}. To check the rest, we need only look at points where either $z \neq 0$ or $t \neq 0$. If $z \neq 0$ one sees immediately that $x^{pk} \in (y^{pl}, A^i)$. □

2. $\omega_{C_i} \cong \mathcal{O}_{C_i}((i - 1)N + d - 4)$ for $1 \leq i \leq p$ where ω as usual denote the dualising sheaf.

Proof. This is done by descending induction on i. For $i = p$, since C_p is a complete intersection of x^{pk}, y^{pl}, this is obvious. So assume result proved for all $p \geq i > 1$. Then we have an exact seqence,

$$0 \to \mathcal{O}_{C_{i-1}}(-N) \to \mathcal{O}_{C_i} \to \mathcal{O}_{C_1} \to 0$$

which we dualise to get,

$$0 \to \omega_{C_1} \to \omega_{C_i} \to \omega_{C_{i-1}}(N) \to 0$$

Since we already know from 1) that the last term is a line bundle on C_{i-1} and then the proof is clear. □

3. Thus by Serre Construction [*e.g.*, see[OSS82]], if we denote by L_i the line bundle $\mathcal{O}_{\mathbf{P}^3}(-(i - 1)N - d)$ for all i, then we have exact sequences,

$$0 \to L_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} \mathscr{I}_{C_i} \to 0,$$

where M_i are rank two vector bundles on $\mathbf{P}^3$ for $1 \le i \le p$. In fact we can arrange these extensions to fit into the following commutative diagrams,

$$\begin{array}{ccccccccc} 0 & \to & L_i & \xrightarrow{\alpha_i} & M_i & \xrightarrow{\beta_i} & \mathscr{I}_{C_i} & \to & 0 \\ & & \downarrow \cdot A & & \downarrow \eta_i & & \downarrow & & \\ 0 & \to & L_{i-1} & \xrightarrow{\alpha_{i-1}} & M_{i-1} & \xrightarrow{\beta_{i-1}} & \mathscr{I}_{C_{i-1}} & \to & 0 \end{array}$$

where $L_i \to L_{i-1}$ is multiplication by A and $\mathscr{I}_{C_i} \to \mathscr{I}_{C_{i-1}}$ is the natural inclusion of ideals.

Proof. This is just Serre construction. Assume we have constructed the exact sequences up to $i-1$ with the commutative diagrams, the first one is just by the usual Serre construction. By taking the natural map $L_i \to L_{i-1}$ given by multiplication by A, we get a map,

$$H^0(\mathscr{O}_{C_i}) = \mathrm{Ext}^1(\mathscr{I}_{C_i}, L_i) \to \mathrm{Ext}^1(\mathscr{I}_{C_i}, L_{i-1}) = H^0(\mathscr{O}_{C_i}(N))$$

which is just multiplication by A. We also have a natural map, induced from the inclusion, $\mathscr{I}_{C_i} \subset \mathscr{I}_{C_{i-1}}$,

$$H^0(\mathscr{O}_{C_{i-1}}) = \mathrm{Ext}^1(\mathscr{I}_{C_{i-1}}, L_{i-1}) \to \mathrm{Ext}^1(\mathscr{I}_{C_i}, L_{i-1}) = H^0(\mathscr{O}_{C_i}(N))$$

In this case also, it is clear that the element $1 \in H^0(\mathscr{O}_{C_{i-1}})$ goes to A $\in H^0(\mathscr{O}_{C_i}(N))$, which is also the image of $1 \in H^0(\mathscr{O}_{C_i})$ by multiplication by A. But these 1's give extensions as desired and the commutative diagram as desired. $\square$

4. There exists a nilpotent endomorphism $\phi\colon M_1 \to M_1(d)$ given as follows: Notice that $L_1 = \mathscr{O}_{\mathbf{P}^3}(-d)$. So we can identify $\mathscr{I}_{C_1} \subset L_1(d)$ and then define $\phi = \alpha_1\beta_1$.

This is obvious.

5. $M_i/\eta_{i+1}(M_{i+1})$ is annihilated by A, $1 \le i < p$. Thus the natural map $M_i \otimes \mathscr{O}_{\mathbf{P}^3}(-N) \to M_i$ got by multiplication by A factors through η_{i+1}.

Proof. Notice that outside $\{A = 0\}$, since multiplication by A and natural inclusions of ideal sheaves are isomorphisms, η_{i+1} is also an isomorphism. So we need to verify the claim at points on $A = 0$. For such a point, which is not on C, $\mathscr{I}_{C_{i+1}} \hookrightarrow \mathscr{I}_{C_i}$ is an isomorphism. So the cokernel of η_{i+1} is the same as the cokernel of $\cdot A$, so claim is proved for such points. Now let $p \in C$. Then near p, $\mathscr{I}_{C_i} = (u, A^i)$, where $u = x^{pk}$ or y^{pl} at p. Also $\mathscr{I}_{C_{i+1}} = (u, A^{i+1})$. Pick a basis for M_i and M_{i+1}, which go to u, A^i, A^{i+1}. Then η_{i+1} is represented by a matrix of the form (v_1, v_2), where $v_i \in M_i$ and

$$v_1 = (1,0) + \lambda\alpha_i(1), \quad v_2 = (0,A) + \mu\alpha_i(1),$$

where $\lambda, \mu \in \mathcal{O}_p$. But the fact that $\eta_{i+1} \circ \alpha_{i+1} = \alpha_i \circ A$ implies immediately that the cokernel of η_{i+1} is in fact isomorphic to $\mathcal{O}_p/A\mathcal{O}_p$. (In fact, this argument shows that $M_i/\eta_{i+1}(M_{i+1})$ is a line bundle on the hypersurface $A = 0$. Moreover, one can even write down exactly this line bundle, though we will not use that fact.) Thus the map $M_i \otimes \mathcal{O}_Y(-N) \to M_i$, got by multiplication by A, factors through η_{i+1}. □

6. We have maps $g_i : L_{i+1} \to M_i(-d)$ for $1 \le i \le p$ by lifting A^i. *i.e.*, the composite $\beta_i \circ g_i$ is just given by the element $A^i \in \mathcal{I}_{C_i}$. We can arrange these maps so that $\eta_i \circ g_i = g_{i-1} \circ A$ and $\phi \circ g_1 = \alpha_1 \circ A$.

Proof. This follows essentially from the fact that $H^1(L_i^{-1} \otimes L_{i-1}(-d)) = 0$. We will construct the g's by induction. By the stated vanishing, we have $g_1 : L_2 \to M_1(-d)$ by lifting $A \in \mathcal{I}_{C_1}$. Clearly $\phi \circ g_1 = \alpha_1 \circ A$. So assume we have constructed g_{i-1} with the required property. So we have $g_{i-1} \circ A$: $L_{i+1} \to M_{i-1}(-d)$. This is just the composite,

$$L_{i+1} = L_i \otimes \mathcal{O}_Y(-N) \xrightarrow{g_{i-1}\otimes 1} M_{i-1}(-d) \otimes \mathcal{O}_Y(-N) \xrightarrow{A} M_{i-1}(-d).$$

Now by the previous claim, we see that the last map factors through η_i. So we get a map $g_i : L_{i+1} \to M_i(-d)$ such that $\eta_i g_i = g_{i-1} \circ A$. To compute $\beta_i g_i$ we may compose it with the natural inclusion of ideals and then it is just

$$\beta_{i-1}\eta_i g_i = \beta_{i-1} g_{i-1} \circ A = A^{i-1} \circ A = A^i.$$

□

Let

$$\mathcal{L} = L_p \oplus L_{p-1}(-d) \oplus \cdots \oplus L_2(-d(p-2))$$

and

$$\mathcal{M} = M_p \oplus M_{p-1}(-d) \oplus \cdots \oplus M_1(-d(p-1))$$

We have a map $f : \mathcal{L} \to \mathcal{M}$ given by sending $(x_p, x_{p-1}, \ldots, x_2) \in \mathcal{L}$ to

$$(-\alpha_p(x_p), -\alpha_{p-1}(x_{p-1}) + g_{p-1}(x_p), \ldots, -\alpha_2(x_2) + g_2(x_3), g_1(x_2)) \in \mathcal{M}$$

Let the cokernel be called M.

7. M is a rank $p+1$ vector bundle on Y.

Proof. We must show that f is injective at every point. So let $P \in C$. Then α_i's are all zero. So if $f(x_p, \ldots, x_2) = 0$ at P, then $g_i(x_i) = 0$. But since near P, A^{i-1} is part of a minimal set of generator of $\mathcal{I}_{C_{i-1}}$ and thus $g_i(x_i) = 0$ implies $x_i = 0$ at this point for all i. Now let $P \notin C$. Then α_i's are injective at this point. If $f(x_p, \ldots, x_2) = 0$ at P, we will use descending induction to prove that all the x_i's are zero. Clearly $\alpha_p(x_p) = 0$ implies $x_p = 0$. Assume we have proved $x_p = \ldots = x_k = 0$. Then by looking at the definition of f, we see that $\alpha_{k-1}(x_{k-1}) = 0$ and thus $x_{k-1} = 0$. □

We have an endomorphism $\theta \colon \mathscr{M} \to \mathscr{M}(d)$ given by,

$$(x_p, x_{p-1}, \ldots, x_1) \mapsto (0, \eta_p(x_p), \ldots, \eta_3(x_3), \eta_2(x_2) + \phi(x_1))$$

Since $\phi^2 = 0$, one can easily see that $\theta^{p+1} = 0$.

8. θ descends to a nilpotent endomorphism, $\varphi \colon M \to M(d)$

Proof. We should show that $\operatorname{Im} \theta \circ f \subset \operatorname{Im} f$.

$$\begin{aligned} &\theta \circ f(x_p, \ldots, x_2) \\ &= \theta(-\alpha_p(x_p), -\alpha_{p-1}(x_{p-1}) + g_{p-1}(x_p), \ldots, -\alpha_2(x_2) + g_2(x_3), g_1(x_2)) \\ &= f(0, Ax_p, \ldots, Ax_3) \end{aligned}$$

□

We have the natural map $\psi \colon M_p \to M$. Notice that since C_p is a complete intersection, M_p is the direct sum of two line bundles. Finally we have,

9. $$\psi(M_p) + \varphi(M(-d)) = M$$

Proof. For this it clearly suffices to prove that

$$\mathscr{M}' = \operatorname{Im} f(\mathscr{L}) + \operatorname{Im} \theta(\mathscr{M}(-d)) + M_p = \mathscr{M}.$$

So let $b = (b_p, \ldots, b_1) \in \mathscr{M}$.

First let us look at a point $P \notin C$. We will show inductively that there exists a $c_i \in \mathscr{M}'$ such that for all $j \geq i$, the j^{th} coordinate of $b - c_i$ is zero. We may clearly take $c_p = (b_p, 0, \ldots, 0)$. So by induction we may assume that $b_j = 0$ for $j > i$. Since $P \notin C$, we see that $\alpha_i(L_i) + \eta_{i+1}(M_{i+1}) = M_i$ at P. So we may write $b_i = \alpha_i(s) + \eta_{i+1}(t)$. Let us first look at the case when $i \geq 2$. Consider

$$c_i = f(0, \ldots, 0, -s, 0, \ldots, 0) + \theta(0, \ldots, 0, t, \ldots, 0) \in \mathscr{M}'$$

By our definition of f, θ, we can easily see that $b - c_i$ has all coordinates upto i zero. Next look at the case when $i = 1$. Again since $P \notin C$, we see that $\phi(M_1(-d)) + \eta_2(M_2) = M_1$. Thus we can write $b_1 = \phi(s) + \eta_2(t)$. Let $c_1 = \theta(0, \ldots, 0, t, s) \in \mathscr{M}'$ and we are done.

Now let us look at points $P \in C$. Now we will show inductively that there exists $c_i \in \mathscr{M}'$ such that $b - c_i$ has j^{th} coordinate zero for all $j \leq i$. For $i = 1$, we have $g_1(L_2(d)) + \eta_2(M_2) = M_1$ at $P \in C$. So $b_1 = g_1(s) + \eta_2(t)$. Take $c_1 = f(0, \ldots, 0, s) + \theta(0, \ldots, 0, t, 0) \in \mathscr{M}'$. So assume that $b_j = 0$ for $j < i$. Again let us first look at the case when $i < p$. Again we have $g_i(L_{i+1}(1)) + \eta_{i+1}(M_{i+1}) = M_i$ at $P \in C$. Therefore we may write $b_i = g_i(s) + \eta_{i+1}(t)$. Consider $c_i = f(0, \ldots, s, \ldots, 0) + \theta(0, \ldots, t, \ldots, 0) \in \mathscr{M}'$. One easily checks that $b - c_i$ has all coordinates zero upto the i^{th} by using our definition of f, θ. Finally assume $i = p$. But $(b_p, 0, \ldots, 0)$ clearly belongs to $\mathscr{M}'$ and thus we are done. □

10. M_1 is not a direct sum of line bundles and hence neither is M.

Proof. If M_1 is a direct sum of line bundles, we get $\mathscr{I}_{C_1}$ is a complete intersection, say of f, g of degrees a, b. Then we see by our Koszul exact sequence, $a + b = d$. Also degree of $C_1 = ab$ by Bezout's theorem. Since C_1 is supported along the line C, we may compute its degree by computing the length of $\mathscr{O}_{C_1}$ at the generic point of C. Easy to see that this is pkl. Thus $pkl = ab \leq d^2/4$. This contradicts our choice of N, k, l.

Thus $\mathscr{M}$ is not a direct sum of line bundles. Since $\mathscr{L}$ is a direct sum of line bundles, this implies that M is also not a direct sum of line bundles. □

By taking $d = 1$ in the above construction, we get a rank 2 bundle E on $\mathbf{P}^4$ by Lemma 1. Since M is not a direct sum of line bundles, by Lemma 2, E is not a direct sum of line bundles.

3.1 Computation of Chern classes

Our vector bundle E on $\mathbf{P}^4$ is given by the exact sequence,

$$0 \to E \to \mathscr{O}(-pk) \oplus \mathscr{O}(-pl) \to \mathscr{F} \to 0$$

where $M = \pi_* \mathscr{F}$ is the vector bundle on $\mathbf{P}^3$ as we have constructed. Notice that we are looking at the case $d = 1$. So to compute the Chern classes of E, we may as well restrict to a general linear space of dimension 2, since rank of E is 2. On this $\mathbf{P}^2$ we will compute the class of E in K_0. We have our distinguished $\mathbf{P}^3 \subset \mathbf{P}^4$ and the curve $C \subset \mathbf{P}^3$. So by choosing our linear space generally, we may assume that it does not meet this curve. Then we have $\mathbf{P}^1 \subset \mathbf{P}^2$, after intersecting with this linear space. We will denote by the same letters restrictions of all our vector bundles. Since $\mathscr{I}_{C_i} = \mathscr{O}_{\mathbf{P}^1}$ now, we see that $[M_i] = [\mathscr{O}] + [L_i]$ in $K_0(\mathbf{P}^1)$. Thus,

$$\begin{aligned}[M] &= [\mathscr{O}] + [\mathscr{O}(-1)] + \ldots + [\mathscr{O}(-(p-1))] + [L_1(-(p-1))] \\ &= p[\mathscr{O}] + \left[\mathscr{O}\left(-\frac{p(p+1)}{2}\right)\right].\end{aligned}$$

Thus on $\mathbf{P}^2$, we see that,

$$[\mathscr{F}] = p[\mathscr{O}] - p[\mathscr{O}(-1)] + \left[\mathscr{O}\left(-\frac{p(p+1)}{2}\right)\right] - \left[\mathscr{O}\left(-1 - \frac{p(p+1)}{2}\right)\right].$$

So we get $[E]$ to be,

$$[\mathcal{O}(-pk)]+[\mathcal{O}(-pl)]-p[\mathcal{O}]+p[\mathcal{O}(-1)]-$$
$$\left[\mathcal{O}\left(-\frac{p(p+1)}{2}\right)\right]+\left[\mathcal{O}\left(-1-\frac{p(p+1)}{2}\right)\right].$$

Now an easy computation will show that,

$$c_1(E)=-1-p(k+l+1)$$
$$c_2(E)=p(p+1)(k+l)+p^2kl$$

For instance, taking $p=2$ and $k=l=1$, we get,

$$c_1(E)=-7,\quad c_2(E)=16.$$

Though this bundle has the same chern classes as the Horrocks-Mumford bundle, this is not the Horrocks-Mumford bundle. Notice that from the above exact sequence, we have,

$$0\to E\to\mathcal{O}(-2)\oplus\mathcal{O}(-2)=F\to\mathscr{F}\to 0$$

and $M=\pi_*\mathscr{F}$ is of rank 3. Thus, $\phi^3=0$ on M. This implies, $F(-3)\subset E$ and thus $E(5)$ has non-zero sections. If E were the Horrocks-Mumford bundle, then $E(6)$ is the first twist when it is supposed to have sections.

Let me analyse the stability of these bundles we have constructed. Twisting the above sequence by $m=[-c_1(E)/2]$, and calling this twisted bundle again by E, and similarly for $\mathscr{F}$, for notational simplicity, we have,

$$0\to E\to\mathcal{O}(l')\oplus\mathcal{O}(k')=F\to\mathscr{F}\to 0$$

where $l'=-pk+m, k'=-pl+m$ and $c_1(E)=0,-1$. Thus E is stable if and only if E has no sections. As before $F(-(p+1))\subset E$, since $M=\pi_*\mathscr{F}$ is a rank $p+1$ bundle. Also without loss of generality assume that $k\geq l$. Thus if $k\geq l+2$, one checks that $F(-(p+1))$ has a section and thus E is not stable. This follows from the fact that if $k\geq l+2$, then $k'\geq p+1$. So the only possibilities are $k=l,l+1$. Notice that if p is odd, then $k\neq l$ by our restrictions on k,l. In this case the only possibility is $k=l+1$. A somewhat tedious analysis in these cases will reveal that these will indeed yield stable bundles. In particular, the above bundle with similar chern classes as Horrocks-Mumford bundle is stable.

By choosing appropriate k,l, one can construct vectorbundles with $c_1^2>4c_2$ for example, in any characteristic, $p>0$. For instance, let $s\geq 1$ be any integer and $k=1,l=ps-s$. Then $N=ps+1$ and the corresponding rank two vector bundle has

$$c_1^2-4c_2=\alpha s^2+\beta s+\gamma$$

where α, β, γ depend only on p and $\alpha = p^2(p-1)^2 > 0$. Thus by choosing $s \gg 0$, we can make the vector bundle to be of the required type.

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