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Rational Double Points on a Rational Surface

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§ 0. Introduction

In [6], we tried to classify a certain class of rational singularities over a rational surface, algebraically. One condition which made the local rings of such singularities unique, was that they all had "logarithmic Kodaira dimension", $-\infty$. In this paper we study a prominent class of such local rings, the rational double points. It turns out that even in this special case, not all local rings are isomorphic in some classes. To illustrate the statement above, we give the statements of the theorems proved. Let us fix some notation:

k = an algebraically closed field of characteristic zero.

A = a normal local domain of dimension two which is the germs of functions at a point on a normal rational surface over k.

$R = k[X, Y, Z]_{(X,Y,Z)}$ where X, Y and Z are indeterminates.

$\hat{}$ denotes completion with respect to the maximal ideal.

Theorem 1. *If*

i) $\hat{A} \cong \hat{R}/(X^4 + Y^3 + Z^2)$, *then* $A \cong R/(X^4 + Y^3 + Z^2)$. (E_6)

ii) *If* $\hat{A} \cong \hat{R}/(Y^3 + X^3 Y + Z^2)$ *then* $A \cong R/(Y^3 + X^3 Y + Z^2)$. (E_7)

iii) *If* $\hat{A} \cong \hat{R}/(X^5 + Y^3 + Z^2)$, *then either* $A \cong R/(X^5 + Y^3 + Z^2)$ *or* $A \cong R/(X^4 Y + X^5 + Y^3 + Z^2)$ *and these two local rings are non-isomorphic.* (E_8)

Theorem 2. *If* $\hat{A} \cong \hat{R}/(Z^{n+1} - XY)$, *then*

i) $A \cong R/(Z^{n+1} - XY)$ *if* $n \neq 7, 8$.

ii) *If* $n=7$, $A \cong R/(Z^8 - XY)$ *or* $A \cong R/(Z^2 - (Y - X^2)(Y - X^2 - Y^2))$ *and these two local rings are non-isomorphic.*

iii) *If* $n=8$, $A \cong R/(Z^9 - XY)$ *or* $A \cong R/(Z^2 - (X + Y^3)^2 - X^3$, *and these two local rings are non-isomorphic.* (A_n)

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Theorem 3. *Let X be a normal rational surface with $p \in X$, a D_n-type singularity and $X-\{p\}$, nonsingular. Then either there exists a rational pencil with p, not a base point, whose general member is non-singular rational or $n=8$ and there exists an elliptic pencil which does not have p as a base point.*

Remark. In Theorem 3, one hopes that the local rings are unique for $n \neq 8$, but we have not been able to find a proof.

These results grew out of an attempt to prove that the Chow groups of affine rational surfaces containing at worst rational double points are zero. For instance, the above theorems tell us that Chow groups for rational affine surfaces containing exactly one rational double point and non-singular outside, are zero if it is of type A_n with $n \neq 7, 8$ or D_n with $n \neq 8$. We will prove a precise result in §4. Spencer Bloch had informed us that he has a proof for all rational double points without classifying the local rings themselves.

The proofs of these theorems, especially that of Theorem 2, are somewhat tedious. The arguments are repetitive and so we would sometimes only sketch the proofs of steps which are similar to earlier steps.

We fix some more notations and conventions. We will always be dealing with rational surfaces and hence the term "surface" without qualification would always mean so. We will write "exceptional curve" for exceptional curves of the first kind. We will say that a curve C "meets" a curve D to sometimes mean also that they meet transversally at one point. All unproved statements about rational double points can be found in [2], [3] and [8]. We may use the same letter for a divisor or its support when there is no confusion. Also "ruling" always would mean a pencil of curves with no base points whose general member is non-singular rational.

We state below some of the known results which are frequently used, for the convenience of the reader.

I. Let X be a complete non-singular rational surface and let E be an effective divisor on it. Suppose that $\kappa(E+K) = -\infty$, where K is the canonical bundle of X. Then for any $L \in \operatorname{Pic} X$, $|L+m(E+K)| = \emptyset$ for $m \gg 0$ [3].

II. Let X be a non-singular rational surface and let E be any effective divisor. Let D be a reduced irreducible curve on X. If $|D+E+K| = \emptyset$, then

$$D \approx \mathbb{P}^1 \quad \text{and} \quad (D.E) \leqq -\frac{E.(E+K)}{2}.$$

Proof. Apply Riemann-Roch inequality for $|D+K|$ and $|D+E+K|$.

In particular, if E is the fundamental cycle of a rational double point and $|D+E+K| = \emptyset$, then $(D.E) \leqq 1$, since $\dfrac{E(E+K)}{2} = -1$.

III. a) If $C \approx \mathbb{P}^1$ and $C^2 = 0$ on a non-singular rational surface X, then $l(C) = 2$ and C gives a ruling of X. If $F \in |C|$ is a reducible member, then there exists an exceptional curve of the first kind L in the support of F. Also any curve in $\operatorname{Supp} F$ which meets a section of this ruling occurs with multiplicity one in F. In particular, if L meets a section, then there exists another exceptional curve M in $\operatorname{Supp} F$. If C is a curve on X which gives a ruling and D is any other curve with $C.D=0$, then D is contained in a member of $|C|$. Also $D^2 \leqq 0$ and $D^2=0$ if and only if $C \sim D$ [6].

b) If $|C|$ is a ruling of X and $F \in |C|$, then since $|F+K|=\emptyset$, we get that every component of F is $\approx \mathbb{P}^1$. Let $E_1, \ldots, E_n$ be in $\operatorname{Supp} F$ and D any other member of $\operatorname{Supp} F$. Assume also that $\bigcup E_i$ is connected. Then $D.(\sum E_i) \leqq 1$.

Proof. By (II) it suffices to show that $\left(\sum_i^n E_i\right) \cdot \left(\sum_1^n E_i + K\right) = -2$. Induct on n. $n=1$ is clear since $E_i \approx \mathbb{P}^1$. Let E_n be a component such that $\sum_1^{n-1} E_i$ is connected. By induction $\left(\sum_1^{n-1} E_i\right)\left(\sum_1^{n-1} E_i + K\right) = -2$ and $E_n \cdot \sum_1^{n-1} E_i \leqq 1$. But since $\sum_1^n E_i$ is connected, $E_n \cdot \sum E_i = 1$. Thus

$$\left(\sum_1^n E_i\right) \cdot \left(\sum_1^n E_i + K\right) = \left(\sum_1^{n-1} E_i\right) \cdot \left(\sum_1^{n-1} E_i + K\right) + 2\left(E_n \cdot \sum_1^{n-1} E_i\right) + E_n(E_n+K)$$
$$= -2+2-2=-2.$$

c) Let $D=\sum_1^p n_i E_i$, be an effective divisor with $D.E_i = 0 \,\forall_i$ and $K.D=-2$. Then $l(D)=2$ and $|D|$ gives a ruling of X.

Proof. Since D is connected, we get $0 = E_i \cdot D = n_i(E_i^2) + \sum_{i \neq j} n_j(E_i E_j)$ and hence $E_i^2 < 0$, for every i, if $p>1$. Inducting on p, the case $p=1$ is trivial from a). If $p>1$, then since $K \cdot D = -2$, let $K.E_p < 0$. But then E_p is an exceptional curve. Let f: $X \to Y$ be the blowing down of E_p and $D' = \sum_1^{p-1} n_i f(E_i)$. Then, since $E_p \cdot D = 0$, $f^* D' = D$. Esay to see that $D'.f(E_i)=0$ $1 \leqq i \leqq p-1$. Also $K_Y \cdot D' = -2$. Now by induction we are done.

IV. Rational Double Points

The graphs in this article can be found for instance in [1]. It is also easy to compute the fundamental cycle, given the graph, from the definition in [2]. By Theorem (2.7) of [1], we get that if E = the fundamental cycle of a rational double point on a rational surface, then $|m(E+K)| = \emptyset \;\forall\, m>0$ where K is the canonical divisor. Also if E has n distinct components, then they are linearly independent in $\operatorname{Pic} X$, and do not generate $\operatorname{Pic} X$, since the corresponding intersection matrix is negative definite by [2]. So $rk \operatorname{Pic} X > n$. Also by [8], the rings in the theorems all correspond to rational double points. Moreover, it is easy to verify that non-completed rings in the theorems do correspond to rational surfaces.

V. Double Covers

The best place to look for the facts used is M. Artin's Thesis, though it may not be available. An alternate reference is Principles of Algebraic Geometry by Griffiths and Harris, though the information is somewhat scattered.

§ 1. E_6, E_7 and E_8

In this section we prove Theorem 1. We will separately analyse the three cases and prove the corresponding statements.

Case of E_6. The graph of the special fibre of the desingularisation of this singularity is:

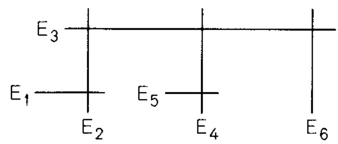

Fig. 1.

where E_i's are non-singular rational curves with $E_i^2 = -2$ and they meet as in the diagram. The scheme-theoretic inverse image of the singular normal point is $E = E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6$. So X is a non-singular projective surface containing a configuration of E_6. First we may blow down all exceptional curves not meeting E. Since X is rational and it has more than one curve with negative self intersection, X cannot be minimal. Let I be an exceptional curve on X. By our choice L must meet E. Now by I, there exists an $m > 0$, such that $|L + m(E+K)| \neq \emptyset$ and $|L + (m+1)(E+K)| = \emptyset$. Fixing this m, choose an integer r, minimal with the property that $|L + mK + rE| \neq \emptyset$ and $|L + (m+1)K + (r+1)E| = \emptyset$. Thus by minimality of r, we get $|L + mK + (r-1)E| = \emptyset$.

Now further assume that $K^2 \leqq 0$ and then $K.(L + mK + rE) = -1 + mK^2 < 0$. Therefore $|L + mK + rE|$ is not the zero divisor and hence write, $L + mK + rE = \sum n_i C_i$, C_i's curves and $n_i > 0$. Then since

$$|C_i + E + K| \subset |L + (m+1)K + (r+1)E| = \emptyset,$$

by II, we get that $P_a(C_i) = 0$ and $(C_i . E) \leqq 1\ \forall i$. If $C_i^2 \leqq -2$ for every i, then by genus formula, $K.C_i \geqq 0$, $\forall i$ and hence $K.(\sum n_i C_i) \geqq 0$ which is a contradiction. So there exists a C_i, with $C_i^2 \geqq -1$. If $C_i . E = 0$, then $C_i^2 \geqq 0$, since every exceptional curve meets E. But then by Riemann-Roch we get that $l(C_i) \geqq 2$. On the other hand, we get an exact sequence:

$$0 \to O(C_i - E) \to O(C_i) \to O_E \to 0.$$

Since $|C_i - E| \subset |\sum n_i C_i - E| = |L + mK + (r-1)E| = \emptyset$, we get $l(C_i) \leqq l(O_E) = 1$, which is a contradiction.

Thus we see that, if $C_i^2 \geqq -1$, then $C_i . E = 1$. Now blowing up C_i, if necessary, away from E, we may assume that $C_i^2 = -1$ and without loss of generality we may assume C_i meets E_1. (Note that all components except E_1 and E_5 occur with multiplicity more than one in E and E_1 and E_5 are symmetric.) Thus we have found an exceptional curve C which meets E only in E_1 and $C.E_1 = 1$. Now blow down all exceptional curves not meeting $C \cup \operatorname{Supp} E$. We now call this new surface X. Note that we have left a neighbourhood of $\operatorname{Supp} E$, intact in all these proceedings. $F = 2C + 2E_1 + 2E_2 + 2E_3 + E_4 + E_6$ is a divisor as in III c) and E_5 is a section. If F has another reducible member, then it must contain an exceptional divisor, by III a). By our assumption this exceptional curve has to meet the section E_5. But, then there must be another exceptional curve in this member by III a), which clearly cannot meet any E_i's or C. Thus we see that all other members are irreducible. But then if we blow down C, E_1, E_2, E_3 and E_6 in that order we get a minimal surface which implies $K^2 = 8 - 5 = 3 > 0$. Thus we have shown that eventually we may assume $K^2 > 0$. Again blow down all exceptional curves not meeting E and let C be any exceptional curve. If C meets E once as before we can get to the stage $K^2 = 3$. Let us assume that all exceptional curves meet E more than once. Then we get by II, $|C + E + K| \neq \emptyset$. By Riemann-Roch:

$$l(-E-K) \geqq \frac{(E+K)\cdot(E+2K)}{2} + 1 = K^2 > 0.$$

Since $C = (C+E+K) + (-E-K)$ and since $l(C) = 1$, it is immediate that either $C + E + K = 0$ or $-E - K = 0$. But since $|E + K| = \emptyset$, we get $C + E + K = 0$. But then

$$0 = K \cdot (C+E+K) = -1 + K^2 \Rightarrow K^2 = 1.$$

Also, since $E_6 . E = -1$, $E_6 . C = -E_6 . E - E_6 . K = 1$. Again, $C.E = -C^2 - C.K = 2$. But E_6 occurs with multiplicity 2 in E and therefore C can only meet E_6. Again, $F = E_2 + E_4 + 2E_3 + 2E_6 + 2C$ gives a ruling with E_1 and E_5 as sections by III c). If we blow down C, E_6, E_3 and E_4 in that order we see that the image of F is irreducible. But now $E_1^2 = -2$ and $E_5^2 = -1$ and hence this cannot be a minimal fibration. So there exists another exceptional curve D in another fibre. By our minimality D has to meet E and hence can meet only E_1 and E_5. If it met both, then since $D.C = 0$, we get,

$$0 = D.C + D.E + D.K = D.E - 1 \geqq 1,$$

which is a contradiction. So it meets either E_1 or E_5 transversally and we are reduced to the case $K^2 = 3$. Now we may also assume that the exceptional curve C meets E_1 transversally once and does not meet any other components of E. Now, if we blow down all curves in $C \cup \operatorname{Supp} E$ except E_6, we get $\mathbb{P}^2$ and image of $E_6 = l$, a line in $\mathbb{P}^2$, and image of the remaining curves is a point P on l.

By Riemann-Roch, $l(-K) \geqq K^2 + 1 = 4$. (In fact $l(-K) = 4$.) If $G \in |-K|$ is a general elliptic curve, the image of G in $\mathbb{P}^2$ is an elliptic curve flexed at P with tangent as l. Since $K^2 = 3$, any two such curves must meet at P atleast 6 times. If we assume that $P=(0,1,0)$ and l: $Z=0$, then a basis for this vector space of curves is Y^2Z-X^3, YZ^2, XZ^2, Z^3. In particular we may find non-singular elliptic curves in this family with any given value of the j-function. Thus given two such singularities we get two lines l_1 and l_2 in $\mathbb{P}^2$, points P on l_1 and Q on l_2 and elliptic curves G_1 and G_2 passing through P and Q respectively with flexes and l_1 and l_2 the respective tangents. From the above remark we may in addition assume that G_1 and G_2 have the same j. But then there exists an automorphism of $\mathbb{P}^2$ which takes l_1 to l_2, P to Q and G_1 to G_2. It is immediate that this automorphism lifts to the two surfaces containing the E_6-singularity which takes one normal point to the other. Thus we see that all E_6-singularities on rational surfaces have isomorphic local rings. This proves Theorem 1, i).

Remark. The map $X \to \mathbb{P}(H^0(-K)) = \mathbb{P}^3$ is the blowing down of the E and the image is a cubic normal surface with an E_6-singularity.

Case of E_7. The graph of E_7 is

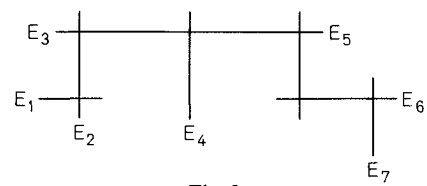

Fig. 2.

and the fundamental cycle is,

$$E=2E_1+3E_2+4E_3+2E_4+3E_5+2E_6+E_7.$$

Exactly as before, we see that, we can assume $K^2=2$ and there exists an exceptional curve C meeting only E_7, with $C.E=1$. Again, since $l(-K)\geqq 3$, there exists an elliptic curve with non-zero j-function in $|-K|$ and the argument is identical as in E_6.

Case of E_8. Here the graph is,

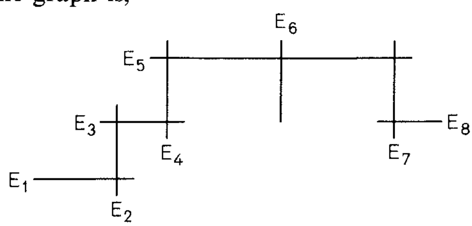

Fig. 3.

and the fundamental cycle $E=2E_1+3E_2+4E_3+5E_4+6E_5+3E_6+4E_7+2E_8$. Note that no component of E has multiplicity one and hence no other curves can meet E exactly once. As before we may reduce to the case $K^2>0$. Again we get for any exceptional curve C, either $C\cap E=\emptyset$ or $|C+E+K|\neq\emptyset$, since

$C \cap E = \emptyset$ or $C.E \geqq 2$. But since $K^2 = 10 - \text{rank of Pic} X$ and these 8 components are linearly independent in Pix X and do not generate it, $K^2 \leqq 1$ (IV).

Thus we get that $K^2 = 1$ and all exceptional curves meet E at least twice. If C is any such, $|C+E+K| \neq \emptyset$ and $|-E-K| \neq \emptyset$ implies as before $C+E+K=0$. Now one may easily verify that $C.E_1 = 1$ and $C.E_i = 0$, $i > 1$. Also by somewhat tedious computation, one may check that $1 = h^0(-K-E) = h^0(-K) - 1$, thereby proving that $h^0(-K) = 2$ and $h^1(-K) = 0$. Also one checks that $|-K|$ has exactly one base point which lies on C, away from $C \cap E$. Since general member of $|-K|$ is non-singular near the base point, we get that general member of $|-K|$ is non-singular elliptic. Now as before we may blow down these curves in the order: C, E_1, E_2, E_3, E_4, E_5, E_7 and E_8 to get as the image of E_6 a line l in $\mathbb{P}^2$ and these elliptic curves to elliptic curves in $\mathbb{P}^2$ flexed along l st the point P which is the image of $E \cup C \setminus E_6$. Also the image of $|-K|$ is generated by $3l$ and any of the image elliptic curves. Let G be a general member of this family. If $j(G) \neq 0$ we see that for almost all values of j, there exists a non-singular member in this pencil with this j. But if $j(G) = 0$, then we see that every non-singular member of this family has $j = 0$. In the case $j(G) \neq 0$, we can also see that the only members with $j = 0$ are singular. These can easily be seen by putting the equation of G in the Weierstrass form, $y^2 = 4x^3 - g_2 x - g_3$. Thus by arguments as before, we see that there are at most two local rings for E_8, one corresponding to the $j = 0$ family and the other, in which j varies. We will show that these two local rings are non isomorphic.

Claim. Let X, Y be surfaces with 'minimal' E_8-singularity configurations as above. If $\varphi: X \to Y$ is a birational map which is an isomorphism in the neighbourhoods of E_8, and takes one E_8 to another in the obvious fashion, then φ is an isomorphism.

To show this it suffices to show that φ or φ^{-1} is a morphism.

Since φ is not a morphism, there exists a rational curve C on X, $C \cap E = \emptyset$ and which after finitely many blowing ups becomes an exceptional curve. Since $C \cap E = \emptyset$, $C \sim -nK$. So $C^2 = n^2$. If $\{\alpha_i\}$ are the multiplicities of the points we are blowing up, we get

$$n^2 - \sum \alpha_i^2 = -1. \qquad (*)$$

$$\frac{n(n-1)}{2} + 1 - \sum \frac{\alpha_i(\alpha_i - 1)}{2} = 0 \qquad (**)$$

or

$$n^2 - n + 2 - \sum \alpha_i^2 + \sum \alpha_i = 0. \qquad (**)'$$

Substituting from $(*)$:

$$-n + 2 - 1 + \sum \alpha_i = 0 \quad \text{or } n = \sum \alpha_i + 1. \Rightarrow n^2 \geqq \sum \alpha_i^2,$$

contradicting $(*)$.

Now we claim that the two non-isomorphic local rings of E_8-singularities are defined in 3-space by the equations $Z^2 + y^3 + X^5 = 0$ and $Z^2 + y^3 + X^5 + X^4 y = 0$.

It suffices to show that these two rings are non-isomorphic, since both these equations give rise to E_8-singularities on rational surfaces. Note that as X

varies both equations give a family of elliptic curves on the normal surface. In the first case all non-singular members of this pencil have $j=0$ and in the second, all non-singular members of the pencil have $j\neq 0$. So the following claim proves the theorem:

Claim. On a non-singular surface with E_8-configuration as above, there exists a unique elliptic pencil which contains E as part of a member.

If the claim were false, there would exist in the minimal E_8-surface another family of curves $\{C_\lambda\}$, such that $C_\lambda \cap E = \emptyset$ for general λ. As before $C_\lambda \sim -nK$ for some $n>0$. Let $P_1, \ldots, P_r$ be the base points of this system and $a_1, \ldots, a_r$ be the multiplicity of a general C_λ, at the base points. Since we get an elliptic pencil when we blow up the base points, we get:

$$n^2 - \sum a_i^2 = 0$$

$$\frac{n(n-1)}{2} + 1 - \sum \frac{a_i(a_i-1)}{2} = 1.$$

$$\therefore n = \sum_1^r a_i.$$

$$n^2 = \sum_1^r a_i^2 \Rightarrow r=1 \quad \text{and} \quad a_1 = n.$$

But if we have a pencil of curves and a general member C has multiplicity n at a base point, then $C^2 > n$ unless $n=1$. So, $n=1$ and we see that this is the unique pencil which we have already constructed.

This proves Theorem 1.

§ 2. A_n-singularities

In this section we analyse the A_n-singularities and prove Theorem 2.

The graph of an A_n-singularity is,

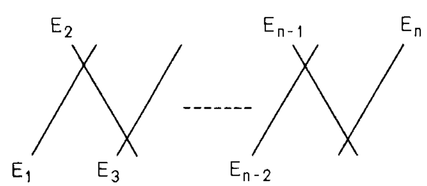

Fig. 4.

and the fundamental cycle $E = \sum_{i=1}^{n} E_i$. Now blow down all exceptional curves which do not meet E.

Step I. There exists a non-singular rational curve C on X with $C^2 \geqq -1$ and $C.E = 1$.

Proof. i) $K^2 \leqq 0$. Let L be any exceptional curve on X. There exists by (I) an $m \geqq 0$ such that $|L+m(E+K)| \neq \emptyset$ and $|L+(m+1)(E+K)| = \emptyset$. Choose the least integer r such that $|L+rE+mK| \neq \emptyset$ and $|L+(r+1)E+(m+1)K| = \emptyset$, which implies $|L+mK+(r-1)E| = \emptyset$. Since $K^2 \leqq 0$, $K.(L+rE+mK) \leqq -1$ and so choose any curve $C \in \operatorname{Supp}|L+rE+mK|$ such that $K.C<0$. So $C^2 \geqq -1$. Since $|C+E+K| \subset |L+(r+1)E+(m+1)K| = \emptyset$, by (II), we get that C is non-singular rational and $C.E \leqq 1$. If $C.E=0$, then by our minimality hypothesis on X, C cannot be exceptional and so $C^2 \geqq 0$.
Now we have an exact sequence:

$$0 \to O(C-E) \to O(C) \to O_E \to 0$$

which gives $l(C-E) \geqq l(C)-1$. By Riemann-Roch inequality, $l(C) \geqq 2$ and so $|C-E| \neq \emptyset$. But then $|L+(r-1)E+mK| \neq \emptyset$ which is a contradiction.
ii) $K^2>0$. If our surface X is a relatively minimal model, then $n=1$ and $E_1$is the unique section with negative self-intersection. Here any fibre can be taken as C. So we may assume that X is not minimal. Assume that every exceptional curve meet E more than once. Let L be any such. Then by (II), $|(L+E+K)| \neq \emptyset$. By Riemann-Roch, $|-E-K| \neq \emptyset$. Since $L=(L+E+K)+(-E-K)$ and $l(L)=1$, we immediately get $L+E+K=0$. So $K^2=1$ and $rk \operatorname{Pic} X=9$. Since the E_i's are linearly independent in $\operatorname{Pic} X$ and do not generate $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}$, we see that $n+1 \leqq 9$ or $n \leqq 8$. It easy to see that we can blow down exactly n curves, namely, $L, E_1, \ldots, E_{n-1}$ successively and in the image E_n has positive self intersection. If $n \leqq 6$, then we cannot have reached a relatively minimal model. Taking the proper transform of any exceptional curve, we get a non-singular rational curve C in X with $C^2=-1$ or $C^2 \leqq -2$ and $C.(L+E)>0$. The second case clearly cannot happen and hence we get another exceptional curve C in X. But $C.(L+E+K)=0$ gives $C.E \leqq 1$ which by minimality of X gives $C.E=1$. So assume $n=7$ or 8. By the equation $L+E+K=0$, we get $E_1=E_n=1$ and $(E_i \cdot L)=0$ for $1<i<n$ $|E_1+2L+E_n|$ gives a ruling of X and the member which contains $E_3, \ldots, E_{n-2}$ will contain an exceptional curve by (III), which meets E exactly once. QED

Now blow up this curve C away from E, so that the proper transform of C is an exceptional curve. Then blow down all exceptional curves which do not meet C or E. Thus we have a surface X where every exceptional curve meet E or C.

Step II. If $K^2 \leqq 0$, there exists a non-singular rational curve L, with $L^2 \geqq -1$, $L.E.=1$ and $L.E.$ or $L.E_n=1$.

Proof. If $n=1$ or 2 Step I implies Step II. If $n=3$ and if C meets E_2, then by (III), c), $|E_1+2E_2+2C+E_3|$ gives a ruling of X and one can always choose a section which meets only E_1 and has positive self intersection. If $n=4$ and $C.E_2$ (or $C.E_3)=1$ then $|E_1+2E_2+2C+E_3|$ gives a ruling of X and a general member will meet only E_4 exactly once. So now onwards let us assume that $n \geqq 5$.

If C meets E_2 (or E_{n-1}), then $F \sim E_1 + 2E_2 + 2C + E_3$ gives a ruling of X, E_4 is a section and $E_5, \ldots, E_n$ belong to another member of this linear system. By (III)a), there exists an exceptional curve in this member. If M is any exceptional curve in this member and if it met E_n we are done. If M met E_i, $5 < i < n$, then $E_{i-1} + 2E_i + 2M + E_{i+1}$ is another member of this system thereby implying that, $i-1=5$, $i+1=n$. Thus $n=7$. So $E_1 + 2E_2 + 2C + E_3 \sim E_5 + 2E_6 + 2M + E_7 \sim C$. If all other members of this pencil is irreducible then we must be able to blow down the exceptional divisors in these members to reach a minimal ruling [6]. But for such a minimal surface $K^2 = 8$ and since we can blow down exactly 6 curves, we reach a contradiction to the fact that $K^2 \leqq 0$. So there exists another reducible member of C and let N be any exceptional curve of that member. Then N has to meet E_4. But then by (III)a) N must occur with multiplicity one in this member and hence there exists another exceptional curve in this member which cannot meet any of the E_i's or C, contradicting our hypothesis.

So we may assume that any such M meets E_5. If $5 = n$ we are done and hence may assume that $n > 5$. By (III)a), E_5 occurs with multiplicity one in this member since it meets the section E_4. Write, $F \sim E_5 + \alpha M + M'$. Then $0 = E_5 . F = -2 + \alpha + (E_5 . M')$. Since E_6 is contained in the support of M', $(E_5 . M') > 0$ and therefore $\alpha = 1$. But then, $-2 = K.F = -1 + (K.M')$ and so there exists another exceptional curve $N \in \operatorname{Supp} M'$. But N also has to meet E_5. Then $M + N + E_5 \sim F$ by (III)a) and $n = 5$.

So we may assume that C meets E_i for $3 \leqq i \leqq n-2$, by symmetry. Also we may assume that i is the largest integer such that an exceptional curve meets E only in E_i exactly once.

Now let F be a general member of the linear system $E_{i-1} + 2E_i + 2C + E_{i+1}$. Then $F \approx \mathbb{P}^1$ and $F^2 = 0$ by (III)a) and E_{i-2} and E_{i+2} are sections for this ruling. Let $\mathring{E}' = E_{i-2} + E_{i-1} + E_i + E_{i+1} + E_{i+2}$. Then by (II), $|F + \mathring{E} + K| \neq \emptyset$. Also by [5, Lemmas 1.1 and 1.2] no curve in the support of E is in the support of $|F + \mathring{E} + K|$. If $i \geqq 4$, then $E_{i-2} . (F + \mathring{E} + K - E_{i-3}) = -1$ implies that E_{i-2} is in the support of $|F + \mathring{E} + K|$, if E_{i-3} is, which is contradictory to the previous statement. So we have shown that: E_j does not belong to the support of $|F + \mathring{E} + K|$ for $i-3 \leqq j \leqq i+3$. Since $F.(F + \mathring{E} + K) = 0$ and $|\mathring{E} + K| = \emptyset$, we see by (III)a), that every curve in the support of $|F + \mathring{E} + K|$ has negative self-intersection. Since $K.(F + \mathring{E} + K) \leqq -2$, we see that there exists at least one exceptional curve in the support of $|F + \mathring{E} + K|$. Also since $E_j . (F + \mathring{E} + K) = 0$ for $i-2 \leqq j \leqq i+2$, we get that no curve in the support of $F + \mathring{E} + K$ intersect F or any of the E_j's with $i-2 \leqq j \leqq i+2$. Also if M is any exceptional curve in the support of $|F + \mathring{E} + K|$ which meets E_k, then since M and E_k belong to the same member of the linear system $|F|$, we get that $M.E_k = 1$. Thus we have, $M . \sum_{k=1}^{i-3} E_k \leqq 1$ and $M . \sum_{i+3}^{n} E_k \leqq 1$. If $M . \sum_{k=1}^{i-3} \cdot E_k = 0$, then M has to meet E_k for $k \geqq i+3$ and this contradicts the maximality of i. So M meets E_k for $1 \leqq k \leqq i-3$. If M meets E_k with $1 < k < i-3$, then $F \sim E_{k-1} + 2E_k + 2M + E_{k+1}$ and then $F.E_{i+2} = 1$ and $(E_{k-1} + 2E_k + 2M + E_{k+1}) . E_{i+2} = 0$, leading to a contradiction. So M meets E_1 or E_{i-3}. If M met E_1, we may as well assume that M meets E_k

for $k \geqq i+3$ or the result is proved. But then $F \sim E_1 + 2M + E_k$ and since E_{i-3} must belong to this member, $i-3=1$. In other words we may assume that all such exceptional curves meet E_{i-3}.

Write $|F+\mathring{E}+K| = \alpha M + M'$,

$$1 = E_{i-3}.(F+\mathring{E}+K) = \alpha + (E_{i-3} \cdot M')$$

implies $\alpha = 1$ and $E_{i-3}.M' = 0$. But

$$-2 = K.(F+E+K) = -1 + (K.M')$$

shows that there exists an exceptional curve in support of M', which does not meet E_{i-3} contradicting the above deduction.

Step III. Now we analyse the case $K^2 > 0$. From the first paragraph of the Proof of Step II, we see that there exists a non-singular rational curve L with $L^2 \geqq -1$ and $L.E = L.E_1 = 1$ if $n \leqq 4$. So let us assume that $n > 5$. Since rank of Pic X is equal to $10 - K^2$ and the E_i's are linearly independent in Pic X and do not generate it, we get that $n \leqq 9 - K^2$ or $n \leqq 8$. Assume that the exceptional curve C from Step I meets E_2 (or E_{n-1}). Then $F \sim E_1 + 2E_2 + 2C + E_3$ gives a ruling of X and then there exists an exceptional curve meeting only E_n or $n=7$, $K^2 = 2$ and there exists an exceptional curve meeting only E_6.
$n=5$: The only case to be dealt with is when C meets E_3. $F \sim E_2 + 2E_3 + 2C + E_4$. If we blow down C, E_3 and E_4, the image of E_1 and E_5 both are sections for this ruling and both have negative self intersection. So there must exist at least one more reducible member in F, and it is easy to see that there exists an exceptional curve in this member which meets E only in E_1, exactly once.

$n=6$: By symmetry, again we may assume that C meets E_3. As before taking, $F \sim E_2 + 2E_3 + 2C + E_4$, easy to see that there exists an exceptional curve in the member of $|F|$ which contains E_6 and meeting only E_6.

$n=7$: If C met E_3 (or E_5) the argument is similar to the one above and we eventually get an exceptional curve meeting only E_1 or E_7, or exceptional curves meeting E_2 and E_6 and blowing down away from E, if necessary, $K^2 = 2$. If C met E_4, then by similar arguments one can check that there exists an exceptional curve M meeting E_1 and E_7 and no other E_i's or C. Blowing down exceptional curves in the linear system

$$|E_3 + 2E_4 + 2C + E_5| = |E_1 + 2M + E_7|,$$

one can easily see that there must be another reducible member, which will provide two exceptional curves meeting E_2 and E_6 as before and blowing down if necessary, we may further assume $K^2 = 2$.

$n=8$: By similar analysis one may show that either there exists a non-singular rational curve L with $L^2 \geqq -1$, $L.E = 1$ and $L.E_1$ or $L.E_8$ equals 1 or $K^2 = 1$ and there exists an exceptional curve L with $L + E + K = 0$.

So the upshot of all the above analysis is:

a) Either there exists a non-singular rational curve L with $L.E=1$ and $L.E_1$ or $L.E_n=1$ and $L^2=-1$.

b) or $n=7$, $K^2=2$ and there exists L, M disjoint exceptional curves with $L.E=L.E_2=1$ and $M.E=M.E_6=1$.

c) or $n=8$, $K^2=1$, and there exists an exceptional curve L with $L+E+K=0$.

Proposition 2.1. *Let A be the local ring of an A_n-singularity on a rational surface and $R=k[X, Y, Z]_{(X,Y,Z)}$. X be a complete surface as before which is "the" desingularisation of such a singularity. Then the following are equivalent:*

i) $A \simeq R/(Z^{n+1}-XY)$, $n \geqq 1$.

ii) *There exists a pencil of curves on X whose general member is non-singular rational and a special member contains E and it has no base points on E, where we are allowed to blow up points or blow down exceptional curves away from E.*

iii) *There exists an exceptional curve on X, if necessary after blowing up or blowing down away from E, meeting E exactly once either in E_1 or E_n.*

Proof. i) $\Rightarrow$ ii). Consider the family given by $Z=$constant.

ii) $\Rightarrow$ iii). By blowing up and removing base points of the pencil, the result is clear. [Basically one has to only use (III), except when $n=3$, and then a suitable section has to be used.]

iii) $\Rightarrow$ i). The result is clear if we show the following:

If X and Y are two non-singular surfaces containing E, the configuration of an A_n-singularity and in both if we have exceptional curves meeting once, in only the extreme components of E, then we can find a birational map φ: $X \to Y$ which is an isomorphism in a neighbourhood of $\operatorname{Supp} E$ and matches these configurations correctly.

We proceed to prove this statement: Let L be the exceptional curve meeting say E_1. Then since $E+L$ is an exceptional divisor, we may blow this down to a point. By [12] this can be identified with any point on any non-singular rational surface and hence blowing up or blowing down away from this point, we can obtain a non-singular rational curve M with $M^2=0$ and whose proper transform in X is an exceptional curve meeting only E_n. We call this proper transform also by M. Now $E+L+M$ gives a ruling of X without base points, whose general member is a non-singular rational curve. By blowing down exceptional curves in other reducible members of this linear system we may also assume that all other members are non-singular rational curves. We can also find a section S meeting only L. By elementary transformations we may further assume that $S^2=-n-3$.

So if we have two surfaces X and Y as in the statement above, we may further assume E, L, M, S in X exists as above and E', L', M', S' in Y. Now we will construct an isomorphism of X and Y of the required kind.

Blowing down $L, E_1, \ldots, E_n$ we get a relatively minimal surface isomorphic to $\mathbb{F}_2$ and image of S has become the unique section of negative self-intersection and image of M has become a fiber. Similarly for Y. Now taking an isomorphism of these surfaces which take the image of M to image of M', one can easily check that it lifts to an isomorphism of X and Y of the required type.

This proves part of Theorem II and settles the case a). Now we will analyse the cases b) and c).

b) First we will show that this case cannot lead to the case in the above proposition. For this it clearly suffices to show that there exists no curve D which meets E_1 once and does not meet any other E_i's. If such a curve existed let $D.L=r$ and $D.M=s$. Since

$$E_1+2E_2+2L+E_3 \sim E_5+2E_6+2M+E_7$$

we get

$$D.(E_1+2E_2+2L+E_3)=D.(E_5+2E_6+2M+E_7)$$

implying $2r+1=2s$ which is a contradiction. Thus we see that this A_7-singularity is not isomorphic to the one in the Proposition. Now we shall show that there is only one such and prove ii) of Theorem 2.

The following statements are easy to verify using Riemann-Roch theorem and intersection theory:

i) $-K=E_1+E_7+L+M+2E_2+2E_3+2E_4+2E_5+2E_6$.

ii) $l(-K)=3$, $|-K|$ has no base points and $K.D \neq 0$ for any curve D, not equal to one of the E_i's.

So one gets a morphism $X \to \mathbb{P}(H^0(-K))=\mathbb{P}^2$ and this factors through Y, which is the blown down of E. Also the map f: $Y \to \mathbb{P}^2$ is finite and of degree 2, since $K^2=2$.

Let $P \in Y$ be the A_7-singular point and $Q=f(P)$. We have $f|_{X-E}$: $X-E \to \mathbb{P}^2-Q$ a double cover and a line l in $\mathbb{P}^2$ pulls back to $-K$ in $X-E$. Using general formulas about double covers, one gets, $-K=f^*l=f^*(3l)$ $-\frac{1}{2}f^*B$ where $B=$branch locus of f. Let $\deg B=2d$. Then $f^*(l)=f^*((3-d)\,l)$ on $X-E$ or $f^*((d-2)\,l)=0$. But $f^*(l)$ is not a torsion element in $\mathrm{Pic}(X-E)$ because $nK \nsim \sum_i p_i E_i$ for any n, p_i, on X: Therefore $d=2$. Thus f: $Y \to \mathbb{P}^2$ is a double cover branched along a quartic. Also the singular points of this curve will give rise to singular points of Y and hence it has exactly one singular point.

If $f(x,y)=0$ define the branch curve near the singular point, $Z^2=f(x,y)$ gives the equation of the rational double point. Since we know the initial form of such an equation to be product of two distinct "variables", we see that order of f must be exactly two. i.e. B has a double point. We may blow up $\mathbb{P}^2$ at this point, successively and resolve singularities of B. By taking the double cover of the desingularisation, branched along the proper transform, we get a desingularisation of the rational singularity. But, since there are 7 components in this special fibre and each exceptional curve of the blown up of $\mathbb{P}^2$ can have at most two curves in its inverse image, we see that the number of times we need to blow up is bigger than or equal to 7/2. In other words, the quartic must have at least 4 singular points (including the infinitely near). If the quartic were irreducible, then it can have at most 3 singular points. So it must be reducible. If the quartic is a cubic and the tangent at a flex, it is easy to see that it has only 3 singular points. So the quartic must be two conics touching 4 times at a point. Since every pair of such conics can be taken by an automorphism of $\mathbb{P}^2$ to the ones given by $y-x^2=0$ and $y-x^2-y^2=0$, we have proved Theorem 2, ii).

c) In this case again we will show that the singularity we get is not the same as in the proposition. If not, there must be a curve F on X such that $F.E.=1$ and $F.E_1$ or $F.E_8=1$. Without loss of generality, assume that $F.E_8=1$. Now we may blow down $L, E_1, \ldots, E_7$ to get $\mathbb{P}^2$ and the image of E_8 is a cubic, with one ordinary double point: If l is the pull back of a general line of $\mathbb{P}^2$ in X, then it is easy to see that, $E_8 \sim 3l-2E_7-3E_6 \ldots 8E_1-9L$. So $F.E_8 = F.(3l-9L)=3.F.(l-3L) \neq 1$. Any such A_8-singularity can be obtained from $\mathbb{P}^2$ by blowing up points on a nodal cubic (also infinitely near) and the choice of these points is completely determined once we fix the cubic and the two tangents at the singular point. But any such cubic with prescribed tangents can be taken to any other by an automorphism of $\mathbb{P}^2$, we see that any such A_8-singularity is isomorphic to any other of the same kind. So it suffices to study any one such.

Let E_4, E_5 and L be three lines in $\mathbb{P}^2$ forming a triangle. Also let σ: $\mathbb{P}^2 \to \mathbb{P}^2$ be an involution which takes E_4 to E_5 and leaves L fixed. Now blow up $E_4 \cap L$ and $E_5 \cap L$ and call the exceptional curves E_1 and E_8 respectively. Also denote by the same names the proper transforms of our original curves. Then σ lifts to an involution of this new surface and $E_1 \cap E_4$ is taken to $E_5 \cap E_8$ by this. Blow up these two points and call the new exceptional curves E_2 and E_7. The involution still liftes and blow up $E_2 \cap E_4$ and $E_5 \cap E_7$ and call the exceptional curves E_3 and E_6 respectively. The involution still lifts. Let P be some point of E_3 away from $E_2 \cap E_3$ and $E_3 \cap E_4$ and let Q be its image under the involution on E_6.

Blow up P and Q and we have a surface X of the type described in c) and in addition it has an involution as above. Call the last two exceptional curves M and N and let us call this involution also σ. For the involution of $\mathbb{P}^2$ it is easy to see that the fixed point set is a line through $E_4 \cap E_5$ and an isolated point on L away from E_3 and E_4. So the fixed point set of σ: $X \to X$ is a nonsingular rational curve S through $E_4 \cap E_5$, not meeting any other E_i's, $S^2=1$, $S.L=1$ and a point on L away from E. Blowing up this point and calling this new surface still X, the fixed point set is $S \cup \{$the new exceptional curve$=T$ say$\}$. If we take the quotient of X by this involution, we get a smooth surface Y and call the map from X to Y, f. One can verify the following easily:

$f(E_4)=f(E_5)=F_4$, an exceptional curve.

$f(E_3)=f(E_6)=F_3$ a non-singular rational curve with $F_3^2=-2$.

$f(E_2)=f(E_7)=F_2 \approx \mathbb{P}^1$, $F_2^2=-2$.

$f(E_1)=f(E_8)=F_1 \approx \mathbb{P}^1$, $F_1^2=-2$.

$f(S)=S'$, $S'^2=2$, S' is tangent to F_4.

$f(L)=L'$, $L'^2=-1$, $f(T)=T'$, $T'^2=-2$,

$f(M)=f(N)=M'$, $M'^2=-1$.

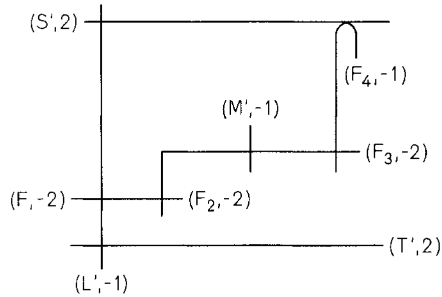

Fig. 5.

Now we blow down E and the corresponding F_i's. Since F_i's form a reducible exceptional divisor, the blown down is a smooth surface say Y'. Let X' be the blown down of X and $f'\colon X' \to Y'$, the corresponding double cover. Of course Y' is rational and it is easy to see that $K_{Y'}^2 = 8$. Also the image D of T' in Y' is a curve with self-intersection -2 and hence $Y' \approx \mathbb{F}_2$. Let f be a fibre of $\mathbb{F}_2$. The branch locus of f' is the disjoint union of the image of S' in Y' and D. The image of S' in Y' is a rational curve with a cusp and 3 consecutive cusps. By an elementary computation, one can show that the equation of this curve in $\mathbb{F}_2 \setminus (D \cup$ a general fibre) to be $(X+Y^3)^2+X^3$. So the equation of the X', near the A_8-singularity is, $Z^2=(X+Y^3)^2+X^3$ and this proves Theorem 2, iii).

§ 3. D_n-singularities

The D_n-configuration is:

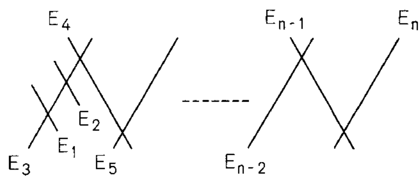

Fig. 6.

and the fundamental cycle $E=E_1+E_2+2E_3+\ldots+2E_{n-1}+E_n$. $E.E_i=0$ if $i \neq n-1$ and $E.E_{n-1}=-1$. Also if L is an exceptional curve which meets E in only E_n exactly once then $|E+E_n+2L|$ gives a ruling of X. So our attempt will be to show that such an exceptional curve exists in most cases.

Now blow down all exceptional curves not meeting any of the E_i's.

Let L be any exceptional curve. If $L.E_n=1$, we are done by the above remark. Also if $L.E=L.E_1=1$, then $|2L+2E_1+E_2+2E_3+E_4|$ gives a ruling and hence if $n \leqq 4$, we would have proved the theorem. If $n=5$, then a general member of the linear system meets only E_5, exactly once and blowing it up once we would be done. If $n>5$, then $E_6, \ldots, E_n$ belong to another member of the same linear system and by (III), this must contain an exceptional curve M which meets only one of the E_i's exactly once. If $M.E_n=1$, we are done. If $M.E_i=1$ for $6<i<n$, then $E_{i-1}+2E_i+2M+E_{i+1}$ is another member of this linear system giving us $n=8$ and M meeting only E_7. This case is the exceptional case. So if $n\neq 8$, we may assume M meets E_6 and then M occurs with multiplicity one in this member giving us one more exceptional curve N in this member. So N should also meet E_6, or we are done. But then $M+N+E_6$ is a member of this linear system, giving us $n=6$ and an exceptional curve meeting only E_6. So in all these cases we are done. So now on we may assume in addition that $L.E\geqq 2$ for every exceptional curve.

i) $K^2\leqq 0$. By (II) $|L+E+K|\neq\emptyset$ and by (I) we can find an $m\geqq 0$, such that $|L+m(E+K)|\neq\emptyset$ and $|L+(m+1)(E+K)|=\emptyset$. Since $K.(L+m(E+K))<0$, take any curve $C\in|L+m(E+K)|$ such that $K.C<0$. Then since $|C+E+K|=\emptyset$, we see by (II) that C is a non-singular rational curve and $C.E\leqq 1$. Since $K.C<0$, $C^2\geqq -1$. If $C^2=-1$ then $C.E=1$ and we are done. If $C^2\geqq 0$, since $C.E\leqq 1$, it is easy to see that this case quickly leads to the theorem.

ii) $K^2>0$. Again by (II). $|L+E+K|\neq\emptyset$ and by Riemann-Roch inequality, $|-E-K|\neq\emptyset$. But $l(L)=1$ and $L=(L+E+K)+(-E-K)$ implies that $L+E+K=0$. $E_{n-1}.(L+E+K)=0$ implies $E_{n-1}.L=1$. Also $L.E.=2$. But E_{n-1} occurs with multiplicity 2 in E and hence L meets only E_{n-1} amongst the E_i's. Now $|E_{n-2}+2E_{n-1}+2L+E_n|$ gives a ruling of X. ($n\geqq 3$). Using this ruling it is easy to verify that if $n\neq 8$, there is the required ruling. The only trouble comes when $n=8$ and there exists exceptional curves L and M meeting E in E_1 and E_7 respectively, exactly once.

In this case we will show that there is no rational pencil on X of the required kind, but an elliptic pencil. If there were such a rational pencil, then if necessary after blowing up or blowing down away from E, we must be able to find an exceptional curve meeting only E_8, exactly once. So in X, there must exist a curve C with $C.E_8=C.E=1$. But

$$2L+2E_1+E_2+2E_3+E_4\sim E_6+E_8+2E_7+2M$$

so $C.2L=1+(C.2M)$, which is impossible. $K^2=1$ and it is easy to see that $|-K|$ gives the required elliptic pencil.

§4. K_0 of Surfaces

In this section we will state some facts about K_0 of affine surfaces and see how the earlier theorems help us to compute certain K-groups. $K_0(A)$, where A is a ring, as usual, would mean the Grothendieck group of finitely generated projective modules upto stable equivalence or equivalently the abelian group generated by all finitely generated modules of finite homological dimension with equivalences defined by exact sequences. The results are all due to M.P. Murthy. A will denote the affine ring of a normal affine surface X.

Lemma 1. If there exists a rational curve on X, passing through a point $x\in X$ and not passing through any singular point of X, then the class $[k(x)]\in K_0(A)$ is zero.

Proof. Let P be the prime ideal defining the rational curve and $M \supset P$, define x. Let B = integral closure of A/P and let N be some maximal ideal of B sitting over M/P. Then $[B] \in K_0(A)$, by our assumption and $[k(x)] = [A/M] = [B/N]$. But B is a principal ideal domain and hence N is principal over B. $[B/N] = [B] - [N] = [B] - [B] = 0$.

Lemma 2. If X is a rational affine surface which has exactly one rational singularity of the type A_n, $n \neq 7, 8$ or D_n, $n \neq 8$, then $\tilde{K}_0(X) = \operatorname{Pic} X$.

Proof. Let the ring of functions of X be A. By the theorems, there exists a pencil of rational curves on X, which is a ruling. Let C be the member of this pencil, which contains the singular point. So for any point $x \in X$, $x \notin C$, $[k(x)] = 0$ in $K_0(A)$ by Lemma 1.

If P is any projective module of rank > 1, then $P = \text{Free} \oplus \text{rank}\, 2$. ($\because \dim A = 2$). So to prove the lemma, it suffices to consider projective modules of rank 2. Let P be projective of rank 2. Then $P|_C^*$ is a projective module of rank 2 on a curve and hence has a nowhere vanishing section. Lifting such a section, we get a map, $P \to I \to 0$, where the variety defined by the ideal I is disjoint from C. It is clear by checking locally that the kernel of this map is projective of rank 1. So it suffices to show that $[I]$ is the class of a projective module of rank 1. Filtering I by height one and two primes, and since $[k(x)] = 0$ for every $x \notin C$, we get $[A/I] = [A/J]$, where J is an ideal of pure height one and J is locally free of rank one at every point. Thus $[I] = [J] \in \operatorname{Pic} X$.

Corollary. Any projective module P over A as before is isomorphic to Free module $\oplus \overset{rk}{\bigwedge} P$.

Proof. Follows from Lemma 2 and the cancellation theorem of Murthy-Swan [11].

Remarks. The above results borrow heavily from the work of Miyanishi and Sugie [9] and what is done here is an attempt to obtain a $\mathbb{P}^1$-ruling of a certain kind whereas in [13], it is proved following closely the techniques of [9] that an "$\mathbb{A}^1$-ruling" can always be obtained. Also similar results can be proved for some non-rational surfaces, especially for ruled surfaces. These results will be published at a later stage.

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