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Smooth degeneration of complete intersection curves in positive characteristic

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0 Introduction

In this paper we study the limit of smooth complete intersection curves in $\mathbb{P}^3$. Lazarsfeld told me of the following problem, which is raised by Peskine [BC] and Kollar:

Question. In a family of smooth curves in $\mathbb{P}^3$, if the general member is a complete intersection, is the special member also a complete intersection?

The smoothness of the special member is necessary. It is easy to construct examples, where the special member is projectively Cohen-Macaulay, but not a complete intersection. When the special member is smooth, the above problem is equivalent by Serre's construction, to the following question:

Question. In a family of rank two bundles on $\mathbb{P}^3$, if the general member is a direct sum of line bundles, is the special member also a direct sum of line bundles?

Remark. Of course, one can ask the same question for $\mathbb{P}^n$ and arbitrary rank bundles on $\mathbb{P}^n$. It is easy to construct examples, which give a negative answer if rank is $n+1$.

In this paper, we construct a family of rank two bundles which gives a negative answer to the above question in all positive characteristics. In zero characteristic, no easy modification of these examples can work, since they depend on very special curves in positive characteristic.

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1 A criterion for deformability

In this section we will work over a field k of arbitrary characteristic. For general facts like *Serre's construction* we refer the reader to [OSS]. If S is a parameter scheme, we will denote by η its generic point and ξ its special point. (Usually S will be the spectrum of a power series ring in t over k.) If X is any scheme let $\mathscr{X}=X\times S$, and let $p\colon \mathscr{X}\to X$ and $q\colon \mathscr{X}\to S$ be the two projections.

We give a brief sketch of why the two questions in the introduction are equivalent. Let us first assume that $\mathscr{E}$ is a rank two vector bundle on $\mathbb{P}^3\times S$ with $\mathscr{E}_\eta$ a direct sum of two line bundles and $\mathscr{E}_\xi$ indecomposable. After twisting $\mathscr{E}$ by $\mathscr{O}(n)$ for large n, we may assume that $H^0(\mathscr{E})\to H^0(\mathscr{E}_\xi)$ is surjective and $\mathscr{E}_\xi$ is globally generated. Lifting a general section of $\mathscr{E}_\xi$ to $\mathscr{E}$, one sees that the zeroes of this lifted section gives a family of smooth curves in $\mathbb{P}^3$. Such a curve is a complete intersection or not depends on whether the corresponding vector bundle is decomposable or not (see [OSS], lemma 5.2.1., p. 101). By the assumptions on $\mathscr{E}$, we see that, the general member of this family is a complete intersection and the special member is not a complete intersection.

Conversely assume that a family of curves $\mathscr{C}$ contradicting the question in the introduction exists. Since $\mathscr{C}$ is smooth over S, $\omega_{\mathscr{C}/S}$, the relative canonical sheaf is a line bundle. By hypothesis, $(\omega_{\mathscr{C}/S})_\eta=\mathscr{O}(l)$, for some l. Let $L=\omega\otimes\mathscr{O}(-l)$. Then L is trivial at η. This shows that $L=\mathscr{O}(D)$, for a divisor D, supported on the special fibre. Since the special fibre is irreducible, D is a multiple of the special fibre. But the special fibre is principal and thus L is the trivial bundle. So $\omega=\mathscr{O}(l)$ on all of $\mathscr{C}$. Now one appeals to Serre's construction to obtain an example of a family of rank two vector bundles with the desired properties.

Next we prove the following general result:

Proposition. *Let A, B be two vector bundles of rank r on a scheme X. Let S be the spectrum of a power series ring in t over k. Let $\mathscr{X}=X\times S$. Then there exists a rank r bundle $\mathscr{E}$ on $\mathscr{X}$ which is isomorphic to A at the generic point and is isomorphic to B at the special point if and only if the following happens – there exists a vector bundle E (of arbitrary rank) on X, a nilpotent endomorphism ϕ of E and a homomorphism $\psi\colon A\to E$ such that the map $\theta=(\phi,\psi)\colon E\oplus A\to E$ is surjective and* Ker $\theta\cong B$.

Proof. Assume such an E exists. Define $\Theta\colon p^*E\oplus p^*A\to p^*E$ as follows: $\Theta=(tI+\phi,\psi)$. Then Θ is surjective. So Ker $\Theta=\mathscr{E}$ is a rank r vector bundle on $\mathscr{X}$. If $t\neq 0$, since ϕ is nilpotent, $tI+\phi$ is an isomorphism. So $\mathscr{E}_\eta\cong A_\eta$. By assumption, $\mathscr{E}_\xi\cong B_\xi$.

Conversely assume that such an $\mathscr{E}$ exists over $\mathscr{X}$. Since $\mathscr{E}_\eta\cong A_\eta$, we can find a map $\mathscr{E}\to p^*A$ whose cokernel is annihilated by some power of t. Further, replacing $\mathscr{E}$ by $t\mathscr{E}$, we may assume $\mathscr{E}\subset t\cdot p^*A$. So we have an exact sequence,

$$(*)\qquad 0\to\mathscr{E}\to p^*A\to\mathscr{G}\to 0$$

Since $t^n\mathscr{G}=0$ for some n, $\mathscr{G}$ is a sheaf over $X\times S_n$, where $S_n=\{t^n=0\}$ in S. Let $\pi\colon X\times S_n\to X$ be the map got by base-changing the structure map $S_n\to Spec$ k. Then π is finite and flat. Let $E=\pi_*\mathscr{G}$. Since $\mathscr{G}$ has homological dimension one over $X\times S$, one sees that E is a vector bundle over X. Multiplication by t induces a nilpotent endomorphism ϕ of E. The map $p^*A\to\mathscr{G}$ induces a map ψ: $A\to E$. It is immediate that $\phi(E)+\psi(A)=E$. So we have an exact sequence,

$$0\to E'\to E\oplus A\to E\to 0$$

where the map on the right is given by (ϕ,ψ). Since $\mathscr{E}\subset t\cdot p^*A$, we see that $A\to E/\phi(E)$, induced by ψ is an isomorphism. Hence $E'\subset E$ and thus E'

$=\operatorname{Ker}(E \xrightarrow{\phi} E)$. On the other hand, restricting (*) to the special point and noting again that $\mathscr{E} \subset t \cdot p^* A$, we see that $\mathscr{E}_\xi = B = \operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_\xi, \mathscr{G})$. Using the resolution

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \to \mathcal{O}_\xi \to 0,$$

to compute the Tor, one sees that, $B = \operatorname{Ker}(\mathscr{G} \xrightarrow{t} \mathscr{G}) = \operatorname{Ker}(E \xrightarrow{\phi} E) = E'$. □

Corollary. *Let $X = \mathbb{P}^n$ and A a direct sum of line bundles in the above notation. Then B (the special member) is not a direct sum of line bundles if E constructed in the lemma is not a direct sum of line bundles.*

Proof. This follows easily from Horrock's criterion that a vector bundle is a direct sum of line bundles if and only if all its middle cohomologies vanish [OSS] and the fact that ϕ (as in the Proposition) is nilpotent. □

The fact that there are families of rank two bundles on $\mathbb{P}^2$ with general member direct sum of line bundles and special member indecomposable seems well known. The following example, though not the most pleasant, I include here, only to prepare the reader for the next section.

Example. We will assume that k is algebraically closed. Let P_1, P_2, P_3 be three non-collinear points in $\mathbb{P}^2$. Let I be the ideal sheaf defining these points. Let $C = \{G = 0\}$ be a smooth conic passing through these points. Let $D = \{H = 0\}$ be a cubic touching C at these points. Let $Q = \{F = 0\}$ be a quintic intersecting C at the P_i's transversally. Let J be the ideal sheaf generated by G and H. Then scheme-theoretically $I = (F, G, H) = J + (F)$. Since J is a complete intersection, we have an exact sequence,

(i) $$0 \to \mathcal{O}(-5) \xrightarrow{p} \mathcal{O}(-2) \oplus \mathcal{O}(-3) = A \xrightarrow{q} J \to 0$$

The map $\mathcal{O}(-5) \xrightarrow{F} \mathcal{O}$, induces a map,

$$k = \operatorname{Ext}^1(J, \mathcal{O}(-5)) \to \operatorname{Ext}^1(J, \mathcal{O}) = H^0(\underline{\operatorname{Ext}}^1(J, \mathcal{O})) = H^0(\mathcal{O}/J).$$

The fact that Q meets C transversally implies that the $\mathcal{O}$-module generated by k in $\mathcal{O}/J$ is the natural submodule $\mathcal{O}/I \hookrightarrow \mathcal{O}/J$. The inclusion $J \hookrightarrow I$ induces a map,

$$\operatorname{Ext}^1(I, \mathcal{O}) = H^0(\underline{\operatorname{Ext}}^1(I, \mathcal{O})) \hookrightarrow H^0(\underline{\operatorname{Ext}}^1(J, \mathcal{O})) = \operatorname{Ext}^1(J, \mathcal{O}).$$

This is also induced by the natural inclusion of $\mathcal{O}/I \hookrightarrow \mathcal{O}/J$. Thus the element χ in $\operatorname{Ext}^1(J, \mathcal{O})$ corresponding to (i) gives an element in $\operatorname{Ext}^1(I, \mathcal{O})$, which generates $H^0(\underline{\operatorname{Ext}}^1(I, \mathcal{O}))$. Thus we have an extension, by Serre's construction,

(ii) $$0 \to \mathcal{O} \xrightarrow{\alpha} M \xrightarrow{\beta} I \to 0,$$

M a rank two bundle. Also by our choice we have a commutative diagram,

(iii)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-5) & \xrightarrow{p} & A & \xrightarrow{q} & J & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \eta} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\alpha} & \mathcal{M} & \xrightarrow{\beta} & I & \longrightarrow & 0
\end{array}$$

Using the natural inclusion of I in $\mathcal{O}$, we get a nilpotent endomorphism $\phi=\alpha\cdot\beta$ of M. $H^0(M(*))\to H^0(I(*))$ is surjective, since $H^1(\mathcal{O}(*))=0$. So we may lift $F\in H^0(I(5))$ to a section of $M(5)$. Thus we get a map $g\colon \mathcal{O}(-5)\to M$, such that $\beta\cdot g$ corresponds to F. Thus we have,

(iv) $$\phi\cdot g=\alpha\cdot F$$

Near P_i's, since F and G generate I, we see that,

(v) $$g(\mathcal{O}(-5))+\eta(A)=M.$$

Away from P_i's, $\alpha(\mathcal{O})=\phi(M)$ and the inclusion $J\hookrightarrow I$ is an isomorphism. So,

(vi) $$\phi(M)+\eta(A)=M$$

Now consider the map $\mathcal{O}(-5)\xrightarrow{f} M\oplus A$, given by $f(a)=(g(a),\,-p(a))$. Since outside P_i's $p(\mathcal{O}(-5))$ is a subbundle of A and near P_i's $g(\mathcal{O}(-5))$ is a subbundle of M, $f(\mathcal{O}(-5))$ is a subbundle of $M\oplus A$. Let E be its cokernel. E is a rank three vector bundle. Consider the map $\theta\colon M\oplus A\to M\oplus A$, given by $\theta(a,b)=(\phi(a)+\eta(b),0)$. Then $\theta^3=0$ since $\phi^2=0$. So θ is a nilpotent endomorphism. Using (iii) and (iv) one sees that $\theta\cdot f=0$. So θ descends to a nilpotent endomorphism $\tilde{\theta}$ of E. Also we have the natural map $\psi\colon A\to E$.

We will show that $\tilde{\theta}(E)+\psi(A)=E$. Since M is not a direct sum of line bundles, nor is E. Thus the corollary will furnish an example as required.

To show $\tilde{\theta}(E)+\psi(A)=E$, it clearly suffices to show that, $\phi(M)+\eta(A)+g(\mathcal{O}(-5))=M$. But this follows from (v) and (vi).

2 Examples in $\mathbb{P}^3$

Now we assume that k is a field of characteristic $p>0$.

Let $x,y\in\Gamma(\mathcal{O}_{\mathbb{P}^3}(p-1))$ and $x',y'\in\Gamma(\mathcal{O}_{\mathbb{P}^3}(p+1))$ be chosen, so that, these four have no common zeroes in $\mathbb{P}^3$. Let $F=xx'+yy'$. If we denote by $L=\mathcal{O}(-2p)$, then $F\in\Gamma(L^{-1})$. For $1\leqq i\leqq p$, let I_i be the ideal sheaf generated by x^p, y^p and F^i. Let C_i be the corresponding curves. Let $C=\operatorname{Supp} C_i$ (this of course is independent of i). One first observes that, C_i's are local complete intersection curves, since the characteristic is p. In fact at any point near C, F^i and one of x^p or y^p define I_i. Also C_p is a complete intersection curve defined by x^p and y^p. Using these facts, it is easy to check that,

$$\omega_{C_i}(4)=L^{-i+1}\otimes\mathcal{O}_{C_i},$$

where ω_{C_i} denotes the dualising sheaf of C_i. So by Serre's construction, one has exact sequences,

(i) $$0 \longrightarrow L^{i-1} \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} I_i \longrightarrow 0$$

with M_i, rank two bundles on $\mathbb{P}^3$.

Let $F: L^i \to L^{i-1}$ be multiplication by F. We may arrange the extensions in (i) so that one has commutative diagrams,

(ii) $$\begin{array}{ccccccccc} 0 & \longrightarrow & L^i & \xrightarrow{\alpha_{i+1}} & M_{i+1} & \xrightarrow{\beta_{i+1}} & I_{i+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle .F} & & \downarrow{\scriptstyle \eta_{i+1}} & & \downarrow & & \\ 0 & \longrightarrow & L^{i-1} & \xrightarrow{\alpha_i} & M_i & \xrightarrow{\beta_i} & I_i & \longrightarrow & 0 \end{array}$$

For $i=1$, we have $\phi=\alpha_1\beta_1$, a nilpotent endomorphism of M_1.

Claim. *$M_i/\eta_{i+1}(M_{i+1})$ is annihilated by F, $1 \leqq i < p$.*

Notice that outside $\{F=0\}$, since $\cdot F$ and natural inclusions of ideal sheaves are isomorphisms, η_{i+1} is also an isomorphism. So we need to verify the claim at points on $F=0$. For such a point, which is not on C, $I_{i+1} \hookrightarrow I_i$ is an isomorphism. So the cokernel of η_{i+1} is the same as the cokernel of $\cdot F$, so claim is proved for such points. Now let $p \in C$. Then near p, $I_i=(z, F^i)$, where $z=x^p$ or y^p at p. Also $I_{i+1}=(z, F^{i+1})$. Easy to check using (ii) that cokernel of η_{i+1} is annihilated by F.

Thus the map $M_i \otimes L \to M_i$, got by multiplication by F, factors through η_{i+1}.

Claim. *There exists g_i: $L^i \to M_i$, such that $\beta_i \cdot g_i$ corresponds to $F^i \in I_i$ and*

(iii) $$\phi \cdot g_1 = \alpha_1 \cdot F, \qquad \eta_i \cdot g_i = g_{i-1} \cdot F.$$

We will construct g_i's inductively. Since $H^0(M_1(*)) \to H^0(I_1(*))$ is surjective, we may lift $F \in H^0(I_1 \otimes L^{-1})$ to $H^0(M_1 \otimes L^{-1})$. This gives $g_1: L \to M_1$ as desired.

Assume we have constructed g_i. So we have $g_i \cdot F$: $L^{i+1} \to M_i$. Clearly this is the same as the composite, $L^i \otimes L \xrightarrow{g_i \otimes 1} M_i \otimes L \xrightarrow{\cdot F} M_i$. But the latter map factors through η_{i+1}. So we get g_{i+1}: $L^{i+1} \to M_{i+1}$ such that, $\eta_{i+1} g_{i+1} = g_i \cdot F$. To compute $\beta_{i+1} g_{i+1}$ we may compose it with the inclusion $I_{i+1} \hookrightarrow I_i$. But β_{i+1} followed by the inclusion is $\beta_i \eta_{i+1}$. So suffices to show that $\beta_i \eta_{i+1} g_{i+1}$ corresponds to F^{i+1}. But $\eta_{i+1} g_{i+1} = g_i \cdot F$ and $\beta_i g_i$ corresponds to F^i by induction. This proves the claim.

Properties of these maps

(a) At $P \in C$, $g_i(L^i)$ is a subbundle of M_i and $g_i(L^i)+\eta_{i+1}(M_{i+1})=M_i$ for $1 \leqq i < p$.

At such points F^i is a minimal generator of I_i and $\beta_i g_i$ corresponds to F^i. So $g_i(L^i)$ is a subbundle of M_i. At $P \in C$, $M_i \xrightarrow{\beta_i} I_i$ is a minimal resolution, $\beta_{i+1}(M_{i+1})=I_{i+1}$ and $I_i=I_{i+1}+(F^i)$. So $g_i(L^i)+\eta_{i+1}(M_{i+1})=M_i$.

(b) At P not on C, $\alpha_i(L^{i-1})$ is a subbundle of M_i and $\alpha_i(L^{i-1})+\eta_{i+1}(M_{i+1})=M_i$ for $1\leqq i<p$.
The first part is clear. For the second part, since the inclusion $I_{i+1}\hookrightarrow I_i$ is an isomorphism at such points $\beta_i\eta_{i+1}$ is surjective and then the assertion is clear.

(c) At P not in C, $\phi(M_1)+\eta_2(M_2)=M_1$.
By (b), $\alpha_1(\mathcal{O})+\eta_2(M_2)=M_1$. But if P is not in C, $\alpha_1(\mathcal{O})=\phi(M_1)$.
Now we define a map f as follows: $f\colon \oplus_1^{p-1} L^i=A\to\oplus_1^p M_i=G$ given by,

(iv) $$f(a_1,\dots,a_{p-1})$$
$$=(g_1(a_1), g_2(a_2)-\alpha_2(a_1),\dots, g_{p-1}(a_{p-1})-\alpha_{p-1}(a_{p-2}), -\alpha_p(a_{p-1}))$$

Claim *f makes A a subbundle of G.*

We must show that f is injective when evaluated at any point. So let $P\in C$. Then α_i's are zero. So if $f(a_1,\dots,a_{p-1})=0$ then $g_i(a_i)=0$. Property a) implies that all a_i's must be zero. If P is not in C, by Property b) α_i's are injective. We will show that if $f(a_1,\dots,a_{p-1})=0$ then $\alpha_i(a_{i-1})=0$ by descending induction. $\alpha_p(a_{p-1})=0$ by definition of f. If $\alpha_{i+1}(a_i)=0$ then $a_i=0$ since the point is not on C. But $g_i(a_i)-\alpha_i(a_{i-1})=0$ since f is zero. Hence $\alpha_i(a_{i-1})=0$. This proves the claim.

Denote by E the cokernel of f. Then E is a rank $p+1$ vector bundle. Next we define an endomorphism θ of G as follows:

(v) $$\theta(b_1,\dots,b_p)=(\phi(b_1)+\eta_2(b_2),\eta_3(b_3),\dots,\eta_p(b_p),0).$$

Claim *θ is nilpotent. In fact $\theta^{p+1}=0$.*

It is clear that $\theta^{p-1}(G)\subset M_1$. Thus it suffices to show that $\theta^2(M_1)=0$.

$$\theta^2(b,0,\dots,0)=\theta(\phi(b),0,\dots,0)=(\phi^2(b),0,\dots,0)=0$$

Claim *θ descends to an endomorphism φ of E.*

We should show that *Image* $\theta f\subset$ *Image* f. One checks using (iv) and (v) that,

$$\theta f(a_1,\dots,a_{p-1})=f(Fa_2,Fa_3,\dots,Fa_{p-1},0).$$

Denote by ψ the natural map from M_p to E.

Claim *The map $E\oplus M_p\xrightarrow{(\varphi,\psi)}E$ is surjective.*

Clearly it suffices to prove that, $G':=\operatorname{Im} f+\operatorname{Im}\theta+M_p$ is equal to G. Again let P be a point not in C. We will show that if $b=(b_1,\dots,b_p)\in G$ then there exists $c_i\in G'$ such that the j^{th} cordinate of $b-c_i$ is zero for every $j\geqq i$. We may clearly take $c_p=(0,\dots,0,b_p)$. By Property b) we may write $b_{p-1}=\alpha_{p-1}(s)+\eta_p(t)$. Let $c_{p-1}=f(0,\dots,0,-s,0)+\theta(0,\dots,0,t)+c_p$. Easy to see that $b-c_{p-1}$ has its p^{th} and $(p-1)^{st}$ cordinates zero. Assume we have found c_{i+1}. Let us look at the case $i>1$. Let u be the i^{th} cordinate of $b-c_{i+1}$. Again by Property b) one may write $u=\alpha_i(v)+\eta_{i+1}(w)$. Now put $c_i=f(0,\dots,-v,\dots,0)+\theta(0,\dots,w,\dots,0)+c_{i+1}$. This element has the required property. Finally assume $i=1$. If $b-c_2=(l, 0, \ldots, 0)$ then by Property c) $l=\phi(m)+\eta_2(n)$. Let $c_1 = c_2+\theta(m, n, 0, \ldots, 0)$ and then $b=c_1$.

Now let $P \in C$ and b be as before. Now we will show that there exists $c_i \in G'$ such that $b-c_i$ has its j^{th} cordinate zero for every $j \leqq i$. By Property a) $b_1=g_1(s) + \eta_2(t)$. Take c_1 to be $f(s, 0, \ldots, 0)+\theta(0, t, 0, \ldots, 0)$. Assume we have found c_{i-1}. First look at the case $i<p$. If the i^{th} cordinate of $b-c_{i-1}$ is u, then by Property a) we may write $u=g_i(v)+\eta_{i-1}(w)$. Take $c_i=f(0, \ldots, v, \ldots, 0) + \theta(0, \ldots, w, \ldots, 0)+c_{i-1}$. One easily checks that c_i has the required property. Finally assume $i=p$. Then $b-c_{p-1}=(0, \ldots, 0, x_p)$. But this element clearly belongs to G'. So $G=G'$.

Notice that since M_1 is a non-trivial bundle, E is also not a direct sum of line bundles. So by Corollary we have constructed a family of rank two vector bundles on $\mathbb{P}^3$ with general member direct sum of line bundles and special member indecomposable.

Remark. In the above example M_p is a direct sum of the same line bundle. So for the question about curves one sees that for all large n there exists a family of smooth curves in $\mathbb{P}^3$ with general member a complete intersection of type (n, n) and special member not a complete intersection (in positive characteristic).

References

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