

Experimental study of impact oscillator with one-sided elastic constraint

James Ing, Ekaterina Pavlovskaja, Marian Wiercigroch and Soumitro Banerjee

Phil. Trans. R. Soc. A 2008 **366**, 679-705

doi: 10.1098/rsta.2007.2122

References

[This article cites 27 articles](#)

<http://rsta.royalsocietypublishing.org/content/366/1866/679.full.html#ref-list-1>

Rapid response

[Respond to this article](#)

<http://rsta.royalsocietypublishing.org/letters/submit/roypta;366/1866/679>

Email alerting service

Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click [here](#)

To subscribe to *Phil. Trans. R. Soc. A* go to:

<http://rsta.royalsocietypublishing.org/subscriptions>

Experimental study of impact oscillator with one-sided elastic constraint

BY JAMES ING¹, EKATERINA PAVLOVSKAIA^{1,*}, MARIAN WIERCIGROCH¹
AND SOUMITRO BANERJEE²

¹*Centre for Applied Dynamics Research, School of Engineering, King's College, Aberdeen University, Aberdeen AB24 3UE, UK*

²*Department of Electrical Engineering, Indian Institute of Technology, Kharagpur 721 302, India*

In this paper, extensive experimental investigations of an impact oscillator with a one-sided elastic constraint are presented. Different bifurcation scenarios under varying the excitation frequency near grazing are shown for a number of values of the excitation amplitude. The mass acceleration signal is used to effectively detect contacts with the secondary spring. The most typical recorded scenario is when a non-impacting periodic orbit bifurcates into an impacting one via grazing mechanism. The resulting orbit can be stable, but in many cases it loses stability through grazing. Following such an event, the evolution of the attractor is governed by a complex interplay between smooth and non-smooth bifurcations. In some cases, the occurrence of coexisting attractors is manifested through discontinuous transition from one orbit to another through boundary crisis. The stability of non-impacting and impacting period-1 orbits is then studied using a newly proposed experimental procedure. The results are compared with the predictions obtained from standard theoretical stability analysis and a good correspondence between them is shown for different stiffness ratios. A mathematical model of a damped impact oscillator with one-sided elastic constraint is used in the theoretical studies.

Keywords: impact oscillator; experimental chaos; grazing; stability of limit cycles

1. Introduction

Engineering systems with impacts are very common and impacting behaviour may be a part of the original design, for example where an allowance for thermal expansion is given between joints, or may be the result of component wear during system operation. The simplest model used to investigate impacting systems is a harmonically forced linear oscillator with an amplitude constraint. This classical piecewise linear system has been intensively studied in the past both in the case of rigid (e.g. Peterka 1974*a,b*; Shaw 1985*a*; Whiston 1987, 1992; Nordmark 1991; Peterka & Vacik 1992) and soft (e.g. Shaw & Holmes 1983*a*; Peterka 2003; Pust & Peterka 2003; Peterka *et al.* 2004) constraints, and it is well known that the system exhibits intricate nonlinear dynamics including periodic and chaotic behaviour.

* Author for correspondence (e.pavlovskai@abdn.ac.uk).

One contribution of 8 to a Theme Issue 'Experimental nonlinear dynamics I. Solids'.

Bifurcations of the periodic orbits of this system have attracted special attention, in particular, when the variation of one of the system parameters causes the non-impacting trajectory to become an impacting one via grazing occurring when a part of the trajectory hits the constraint tangentially. In the case of a rigid constraint where impact is instantaneous and described by the restitution coefficient, Nordmark (1991) studied the grazing bifurcation and showed that the Jacobian of the Poincaré map contains a square root singularity in the first order Taylor expansion. Chin *et al.* (1995) classified the types of bifurcations possible in such square root maps. In particular, coexistence and hysteresis were found to be common, and the conditions for grazing bifurcation leading to a reverse period-doubling cascade were derived. A system with soft elastic constraint was investigated by means of a mapping technique by Shaw & Holmes (1983*a*), based upon an approximation of the time of flight which holds only for large penetrations into the high-stiffness region. They went on to consider also the limit case of a rigid stop in more detail. Period-doubling cascades and chaotic response were recorded. It was noted that singularities in the Jacobian existed for certain conditions. The distinction between impacting systems with soft and rigid constraints (also recognized as piecewise smooth systems with discontinuous vector fields and systems with discontinuity in the state, respectively) was addressed in di Bernardo *et al.* (1999, 2001) where it was shown that in the discrete-time representation the latter ones are associated with a square root singularity while the former ones yield maps with ‘power-of-3/2’ behaviour which is smooth at the borderline. Ma *et al.* (2006) investigated the properties of a soft impact with a prestressed spring, which also displays a square root singularity, and showed that the discontinuity is restricted to the trace of the Jacobian.

In general, experimental contributions have been rather limited. Shaw & Holmes (1983*b*) experimentally examined the response of a beam with a fixed amplitude constraint at one end, noting that a one degree of freedom approximation allowed prediction of the regions of periodic and chaotic motion. This was extended systematically in Shaw (1985*b*) where the subharmonic resonances predicted in Thompson *et al.* (1983) were observed along with period-doubling bifurcations. Wiercigroch & Sin (1998) designed a piecewise linear oscillator with symmetrical amplitude constraints and systematically varied a number of parameters to show a wide range of system responses. Wiercigroch *et al.* (1998) went on to analyse the power spectrum of the experimentally observed chaos. Various other experimental impact oscillators were studied by Hinrichs *et al.* (1997), Todd & Virgin (1997) and Wagg *et al.* (1999). Of note is Piironen *et al.* (2004) where a pendulum contacting with a rigid stop was shown to exhibit periodic windows in a period adding cascade up to period-5. Such a cascade was not observed experimentally in Ing *et al.* (2006) for a piecewise smooth oscillator due to either the soft impact or the presence of a second elastic constraint. The degree of asymmetry between the constraint positions was found to greatly affect the response.

The aim of this paper is to provide an experimental backing for the well-developed model of the soft impact oscillator, and, in particular, to study the system bifurcations close to grazing and to investigate the stability of the periodic orbits experimentally.

The paper is organized as follows. In §2, the details of the experimental rig and measurement procedures are discussed. It is shown that the use of recorded mass acceleration signal has significant advantage in detecting the contact with

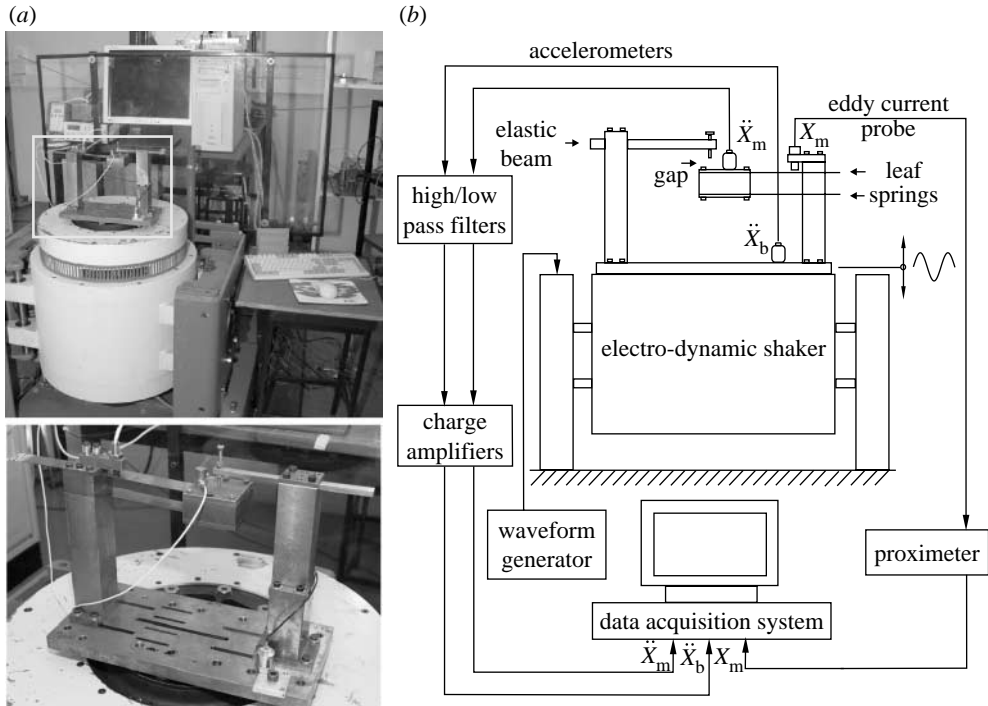


Figure 1. (a) Photographs of experimental setup. Parallel leaf springs prevent mass from rotation ensuring vertical displacement only. Harmonic excitation is provided to the oscillator from the shaker table. Since the oscillator mass is small when compared to the shaker armature, it is assumed that the oscillator does not interact with the shaker. (b) Schematic of experimental setup. Mass displacement X_m and accelerations of the mass and the base \ddot{X}_m , \ddot{X}_b are measured by an eddy current probe and two accelerometers, respectively, and then collected by the data acquisition system.

the elastic constraint in comparison with the mass displacement signal. Then different bifurcation scenarios under variation of the excitation frequency near grazing are shown and discussed in §3 for a number of values of the excitation amplitude. Section 4 is devoted to experimental evaluation of the stability of the recorded periodic orbits with and without impacts. Here a new experimental procedure is described, and the results under varying excitation frequency are presented for three values of stiffness ratio. In addition, the mathematical model used for the comparisons is discussed here, and the details of calculations of the Jacobian matrix for the stability analysis are given.

2. Experimental rig and measurement procedure

Extensive experimental investigations of an impact oscillator with one-sided amplitude constraint have been undertaken with the set-up shown in figure 1a. The oscillator rig developed at Aberdeen University (Wiercigroch & Sin 1998; Sin & Wiercigroch 1999; Ing *et al.* 2006) has been modified by taking off one of the symmetric amplitude constraints made of elastic beams. The oscillator shown in figure 1 consists of a block of mild steel supported by parallel leaf springs providing the primary stiffness and preventing the mass from rotation. The

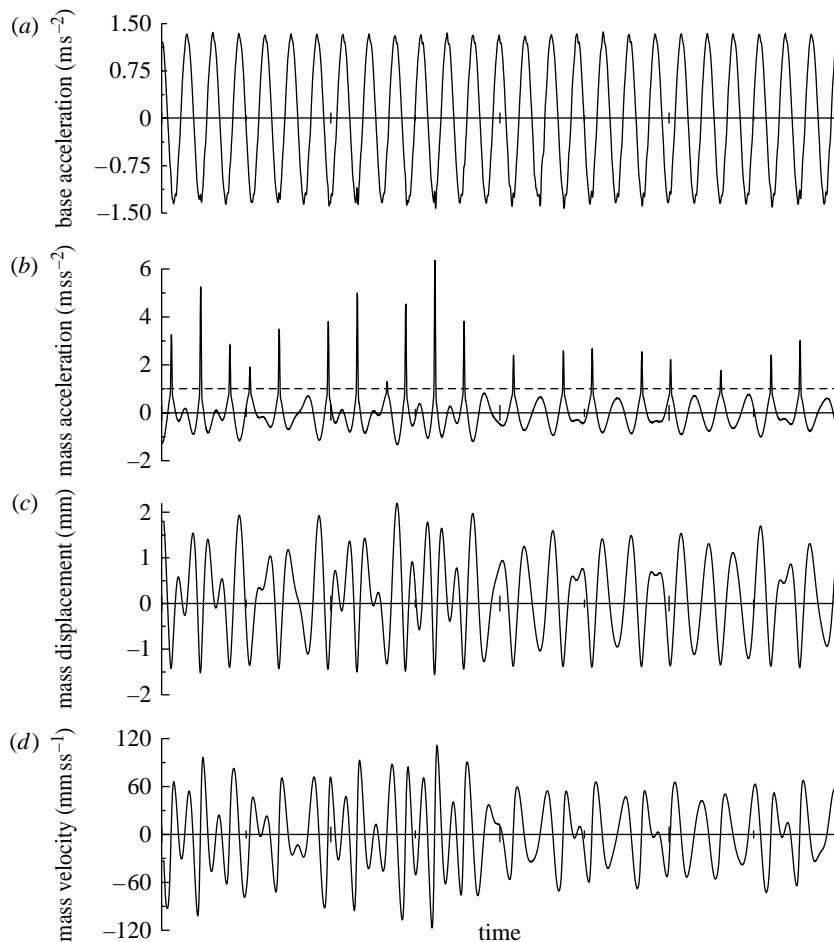


Figure 2. A sample of the recorded experimental time histories for $f_n=9.38$ Hz, $f=6.5$ Hz, $c=1.3$ kg s $^{-1}$ and $\beta=29$; (a) base acceleration \ddot{X}_b , (b) mass acceleration \ddot{X}_m , (c) mass displacement X_m and (d) mass velocity obtained by differentiation of the mass displacement.

secondary stiffness provided by the elastic beam is mounted on a separate column. Contact between the mass and the beam is made when their relative displacement is equal to zero. In practice, the contact is through a bolt that is attached to the beam. The length of the bolt can be adjusted to control the gap g . The secondary stiffness can be varied by changing the length of the beam.

The natural frequency of the oscillator f_n is controlled by varying the length of two parallel leaf springs, and for the experiments presented here to be such that both sub- and super-harmonic oscillations can be executed. The damping of this system was found to be well approximated by a linear viscous damping model (Ing *et al.* 2006). Although in principle times when the mass makes contact with the restraint can be determined by monitoring their relative displacement, a much more accurate estimation can be made by analysing the oscillator acceleration; hence this method was used. The stiffness of the secondary spring was determined through static tests. The harmonic excitation was provided by an electro-dynamic shaker. Displacement of the oscillator X_m was measured with an eddy current probe displacement

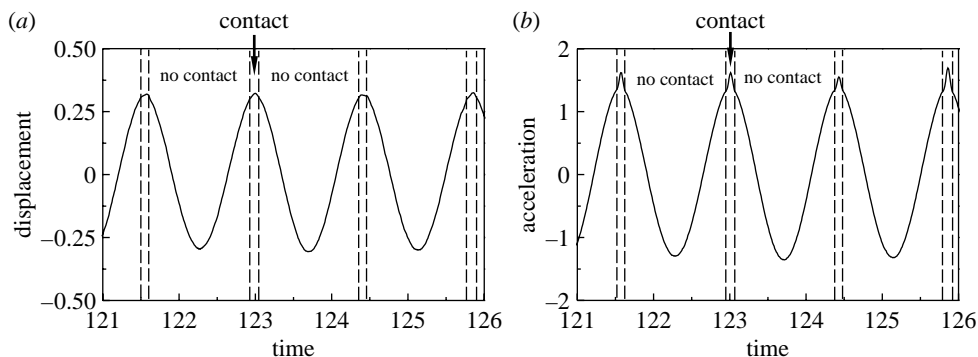


Figure 3. Time histories of (a) oscillator displacement and (b) acceleration. It can be clearly seen that while the displacement history is not sensitive enough to detect contacts, the acceleration signal changes abruptly when the contact is made and lost (Ing *et al.* 2006).

transducer mounted over one leaf spring. The acceleration of the oscillator \ddot{X}_m was measured using an accelerometer mounted directly on the mass. The signals from these three devices \dot{X}_b , \ddot{X}_m , X_m were captured and observed in real time using the data acquisition system. This allowed variation of a parameter until a change in the response was observed. The responses of the oscillator in the frequency range from 6 to 10 Hz were measured for different values of forcing amplitude. The choice of frequency was made to maximize the frequency intervals for which attractors remained topologically similar. A Savitzky–Golay algorithm was used to smooth the data, where a second order polynomial fitted to the eight surrounding data points gave the best results. This preserved the shape and height of the peaks in the time histories, while removing much of the background noise. The velocity was obtained from the smoothed displacement data. By mounting an accelerometer directly onto the mass and observing the acceleration time histories, the points of contact are easily recognizable in the form of sharp spikes.

A sample of the obtained chaotic time series for $f_n=9.38$ Hz, the excitation frequency $f=6.5$ Hz, damping coefficient $c=1.3$ kg s⁻¹ and stiffness ratio $\beta=29$ is presented in figure 2, where the smoothed signals of the recorded base acceleration (figure 2a), and acceleration and displacement of the mass (figure 2b,c) are shown together with the processed signal for the mass velocity (figure 2d), which was obtained by differentiation of the recorded displacement signal. As can be seen from figure 2b, short time intervals when the mass is in contact with the beam are clearly visible in the form of sharp spikes above the 1 m s⁻² level. Since this study was focused on grazing, contacts with the beam were short and hard to detect, especially by looking at the time histories of the mass displacement alone. Figure 3 presents an example of the case where contacts occur very close to grazing and they are not detectable from the displacement (figure 3a) but apparent from the acceleration (figure 3b).

Another method by which the mass acceleration signal can be used to detect the contact with the secondary spring is shown in figure 4. Here instead of a standard displacement–velocity phase plane, velocity–acceleration and displacement–acceleration planes are used. Figure 4a,d present trajectories without contact with the beam, whereas figure 4b,e together with figure 4c,f depict responses with contacts. As can be seen from these figures, the contact regimes are characterized by sharp spikes of acceleration.

Figure 4. (*a–c*) Acceleration–velocity curves; (*d–f*) acceleration–displacement curves. All results were recorded for $f_n=9.38$ Hz, $c=1.3$ kg s⁻¹, $\beta=29$ and the excitation frequency f equal to 7.75, 8.25 and 9.00 Hz for (*a,d*), (*b,e*) and (*c,f*), respectively.

3. Study of the grazing bifurcations

In this section, the behaviour of the system near grazing is investigated and the results are presented in the form of bifurcation diagrams, which were obtained in the following way. First for each chosen value of frequency, the excitation amplitude was slowly increased until the first contact occurred; then the amplitude was fixed at that value and the bifurcation scenario under varying frequency was studied. Starting with the non-impacting period-1 solution, the excitation frequency was slowly increased through the grazing and beyond. The recorded data were smoothed as described above in §2. For each chosen set of parameters, a steady state response was reached prior to data capture. Next, a first return Poincaré map was constructed and projected onto the displacement axis. The Poincaré plane was placed at the various phases (constant for different bifurcation diagrams) in order to maximize the separation between the points appearing on the diagram. This was done because for the original choice of the positive slope zero crossing of the excitation signal, the recorded points were too close and there was no difference between period-1 and period-2 regimes, etc. The procedure was repeated by increasing the frequency f in small steps over a reasonably large range.

The results presented in this section were obtained for a natural frequency of 9.38 Hz, gap equal to 1.26 mm, the secondary spring stiffness 29 times higher than that of the main one and damping coefficient equal to 1.3 kg s⁻¹. Bifurcation scenarios observed for different values of excitation amplitude are

shown in figures 5, 8 and 10. Figure 5a shows the changes in the system response under increasing frequency for the amplitude of the base excitation equal to 0.25 mm in the form of a bifurcation diagram and two phase portraits. As can be seen from the bifurcation diagram, the period-1 responses were obtained before and after grazing (see additional phase plane windows). The grazing frequency was determined at the point $f=8.40$ Hz where the slope of the bifurcation curve changed. Close to this point, a small window of chaotic behaviour with a small spread was observed.

For the excitation amplitude equal to 0.38 mm, a quite different bifurcation scenario was observed, which is shown in figure 5b. The grazing occurs at $f=9.75$ Hz, where a period-4 response shown in an additional window (with period equal to 0.503 s) was recorded. As the excitation frequency increases, period-3 oscillations with one impact per period are obtained for $f \in [8.00, 8.45]$ Hz and their sample trajectory on the phase plane is shown in an additional window for $f=8.25$ Hz (here period is equal to 0.364 s). Numerical simulation reveals the nature of the atypical transitions from period-1 to period-4 to period-3 as resulting from a complex interplay between coexisting orbits and grazing-induced bifurcations. Simulation shows that a period-4 orbit coexists with the non-impacting period-1 orbit before it makes the first contact with the switching surface and then with impacting period-1 orbit after grazing for a narrow range of frequency. At about $f=7.95$ Hz, there is a boundary crisis, and the state moves to the period-4 orbit. At a larger value of the parameter, another coexisting period-3 orbit (with one impact) is numerically found to occur. As the parameter is increased, there is another boundary crisis, and the state jumps from the period-4 to the period-3 orbit. The sequence of experimentally recorded phase plots and Poincaré maps showing these orbits for $f=7.95$, 8.00 and 8.05 Hz are given in figure 6.

As the parameter is further increased, another grazing event occurs around $f \approx 8.45$ Hz, and the period-3 orbit ceases to exist. After this event, the state jumps to a coexisting period-2 orbit with two impacts at $f \approx 8.50$ Hz. Following this, we observe a reverse period doubling (which is a smooth bifurcation) resulting in a period-1 orbit. Figure 7 presents the sequence of phase plots and Poincaré maps for $f=8.45$, 8.50 and 8.55 Hz, showing the orbits that occur in quick succession. For $f > 8.5$ Hz the impacting period-1 response was recorded and sample trajectory for $f=9.00$ Hz is presented in an additional window in figure 5b.

Another example of a possible bifurcation scenario is shown in figure 8a for excitation amplitude equal to 0.44 mm. As can be seen from this figure, chaotic behaviour is observed close to grazing at $f=7.55$ Hz, which changes to period-2 oscillations as the frequency increases. The sample trajectory of this period-2 response is presented on the phase plane for $f=7.85$ Hz. Next, a window of chaotic behaviour is obtained and a typical discrete-time phase portrait is given for $f=8.35$ Hz. This is followed by a reverse period-doubling cascade, resulting in a period-1 response with one impact per period. A typical phase space trajectory for this condition is shown for $f=8.55$ Hz.

For the excitation amplitude equal to 0.53 mm (see figure 8b), grazing occurs at $f=7.15$ Hz which turns the period-1 orbit unstable. However, the system does not collapse, as there is another coexisting periodic orbit at that parameter value, and so the orbit discontinuously jumps from a non-impacting period-1 orbit to a

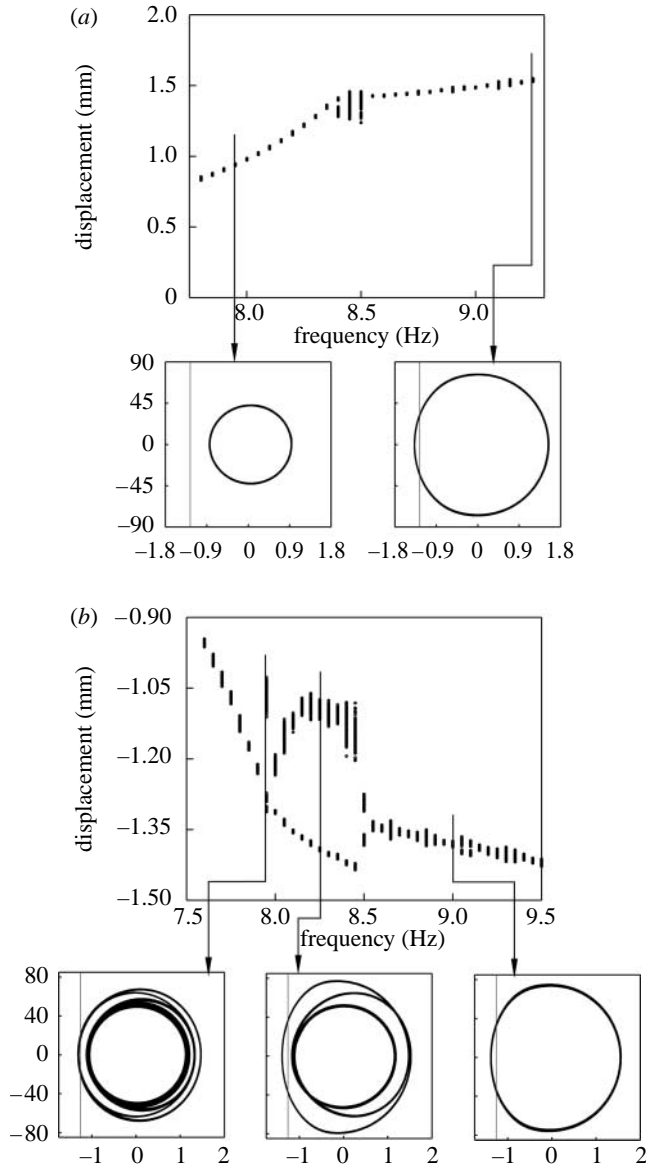


Figure 5. Bifurcation diagrams obtained for the mass displacement under varying frequency f at $f_n=9.38$ Hz, $c=1.3$ kg s $^{-1}$, $\beta=29$, $g=1.26$ mm and the excitation amplitude equal to (a) 0.25 mm and (b) 0.38 mm. Additional windows demonstrate the trajectories on the phase plane and obtained for (a) $f=7.95$ Hz and 9.34 Hz and (b) $f=7.95$, 8.25 and 9.00 Hz, respectively.

period-2 orbit with two impacts with period equal to 0.208 s (shown in the additional window). Simulation shows that this period-2 orbit coexists with the non-impacting period-1 orbit before the parameter value $f=7.15$ Hz, and there is a different period-2 orbit with one impact that begins to coexist with it after this value. At $f\approx 7.45$ Hz, there is a boundary crisis, due to which the system behaviour jumps from the period-2 orbit with two impacts to the coexisting period-2 orbit with one impact. As the parameter is further increased, one of the

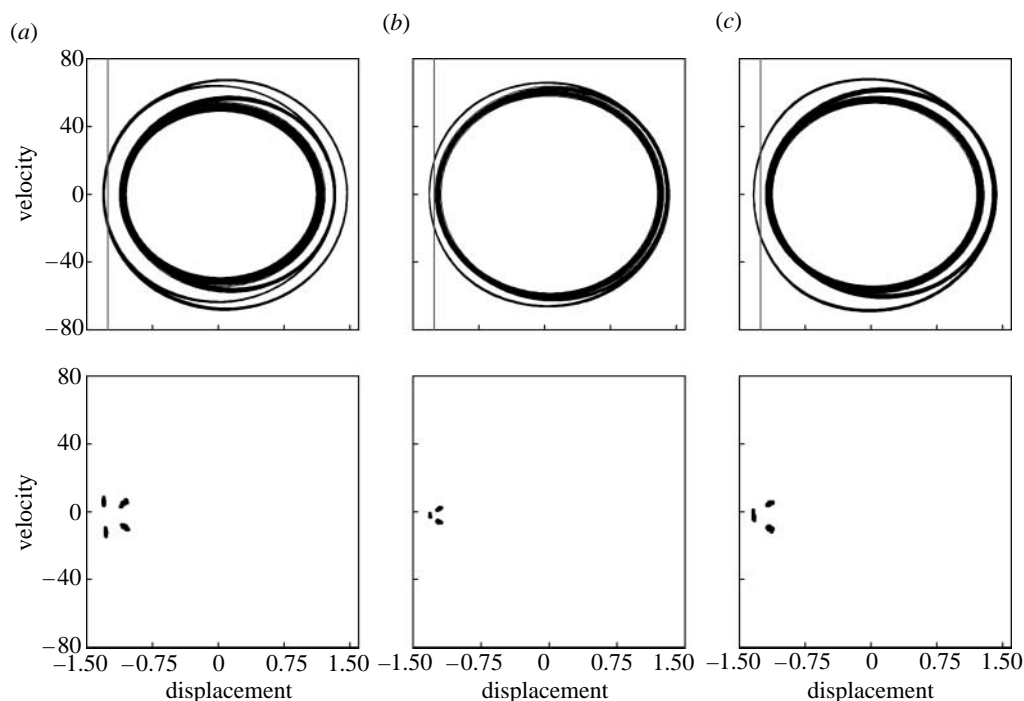


Figure 6. Trajectories on the phase plane and the Poincaré maps for three values of frequency f near the second bifurcation of figure 5*b* at $f_n=9.38$ Hz, $c=1.3$ kg s⁻¹, $\beta=29$, $g=1.26$ mm and the excitation amplitude equal to 0.38 mm; (a) $f=7.95$ Hz, (b) $f=8.00$ Hz and (c) $f=8.05$ Hz. Grey lines indicate the location of the secondary spring.

loops of the period-2 orbit approaches the switching manifold, and at $f \approx 8.45$ Hz another grazing occurs. This results in a period-2 orbit with two impacts. Following this, there is a smooth reverse period-doubling bifurcation giving rise to a period-1 orbit with one impact per cycle. To demonstrate these transitions the sequence of phase plots and Poincaré maps for $f=8.45$, 8.50, 8.55 and 8.60 Hz are given in figure 9. Representative phase space trajectories of different period-2 oscillations and the period-1 orbit for $f > 8.6$ Hz are shown for $f=7.45$, 8.00 and 9.10 Hz, respectively, in additional windows in figure 8*b*.

Two other examples of possible bifurcation scenarios are presented in figure 10*a,b* for excitation amplitudes equal to 0.66 and 0.70 mm, respectively. As can be seen from these figures, chaotic behaviour is observed at grazing frequencies $f=6.50$ and 6.25 Hz and the corresponding Poincaré maps are presented in additional windows. After a series of bifurcations in both cases the system settles down to impacting the period-1 response. However, the intermediate bifurcations are different. For the smaller excitation amplitude of 0.66 mm, the chaotic regime is followed by period-3 oscillations (with three impacts per period) shown for $f=6.70$ Hz. It jumps to a coexisting period-2 orbit through crisis at $f \approx 6.75$ Hz, and then on to another different period-2 orbit at $f=7.9$ Hz. Finally at $f=8.50$ Hz, we observe a bifurcation to a period-1 response with one impact per period. For the larger amplitude of 0.70 mm, the narrow range of chaotic behaviour is followed by impacting period-1 oscillations shown for $f=6.65$ Hz,

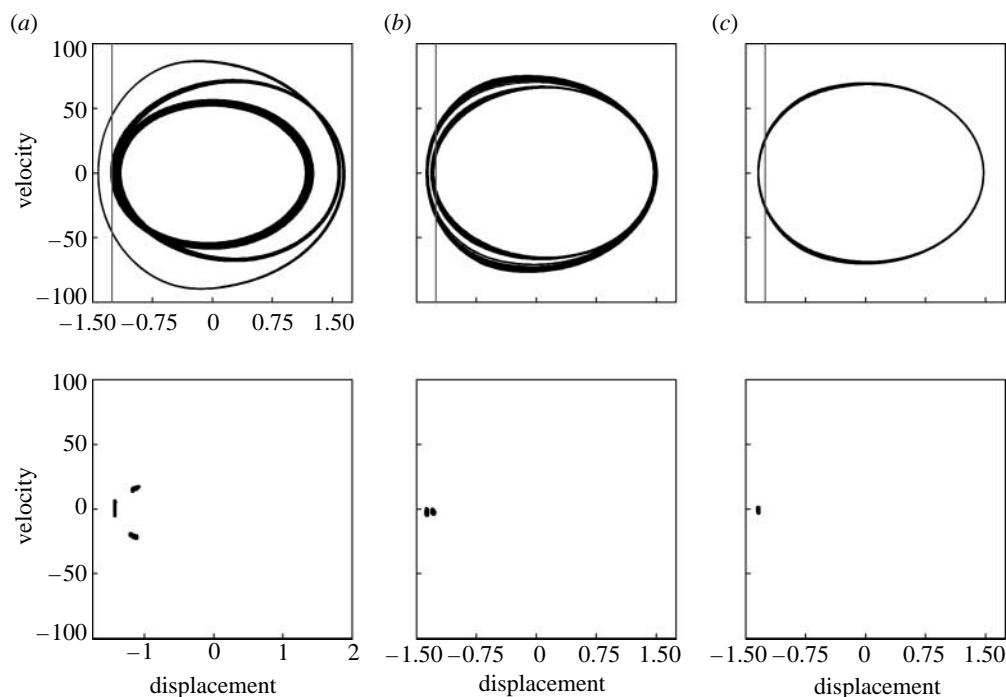


Figure 7. Trajectories on the phase plane and Poincaré maps for three values of frequency f near the third bifurcation of figure 5*b* at $f_n=9.38$ Hz, $c=1.3$ kg s⁻¹, $\beta=29$, $g=1.26$ mm and the excitation amplitude equal to 0.38 mm; (a) $f=8.45$ Hz, (b) $f=8.45$ Hz and (c) $f=8.55$ Hz. Grey lines indicate the location of the secondary spring.

which bifurcates into period-2 response at $f=8.05$ Hz. After two grazing events, it bifurcates back to a period-1 response with one impact at $f=8.55$ Hz.

4. Evaluation of the stability of periodic orbit

The experimental studies undertaken so far have confirmed that the most typical bifurcation scenario is when a non-impacting orbit bifurcates into an impacting one via the grazing mechanism as can be seen in figures 5*a* and 10*b*, for example. Such stability change is an important archetype in the dynamics of impacting systems. Therefore, in this section, the stability of these periodic orbits close to grazing conditions is investigated experimentally. However, before the experimental procedure is explained, a short summary of standard theoretical stability analysis for a periodic orbit, which goes back to the work of Floquet (1883), is given (Argyris *et al.* 1994), and also a mathematical model used for the comparison is very briefly discussed.

For a nonlinear periodic externally excited system described by a set of non-autonomous differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) \quad (4.1)$$

and having periodic solution \mathbf{x}^* of period length T , i.e. $\dot{\mathbf{x}}^* = \mathbf{F}(\mathbf{x}^*, t)$, to investigate the stability of the solution \mathbf{x}^* , a neighbouring state $\mathbf{x} = \mathbf{x}^* + \mathbf{u}$ is considered, where \mathbf{u} represents a small perturbation $|\mathbf{u}|/|\mathbf{x}^*| \ll 1$. The following

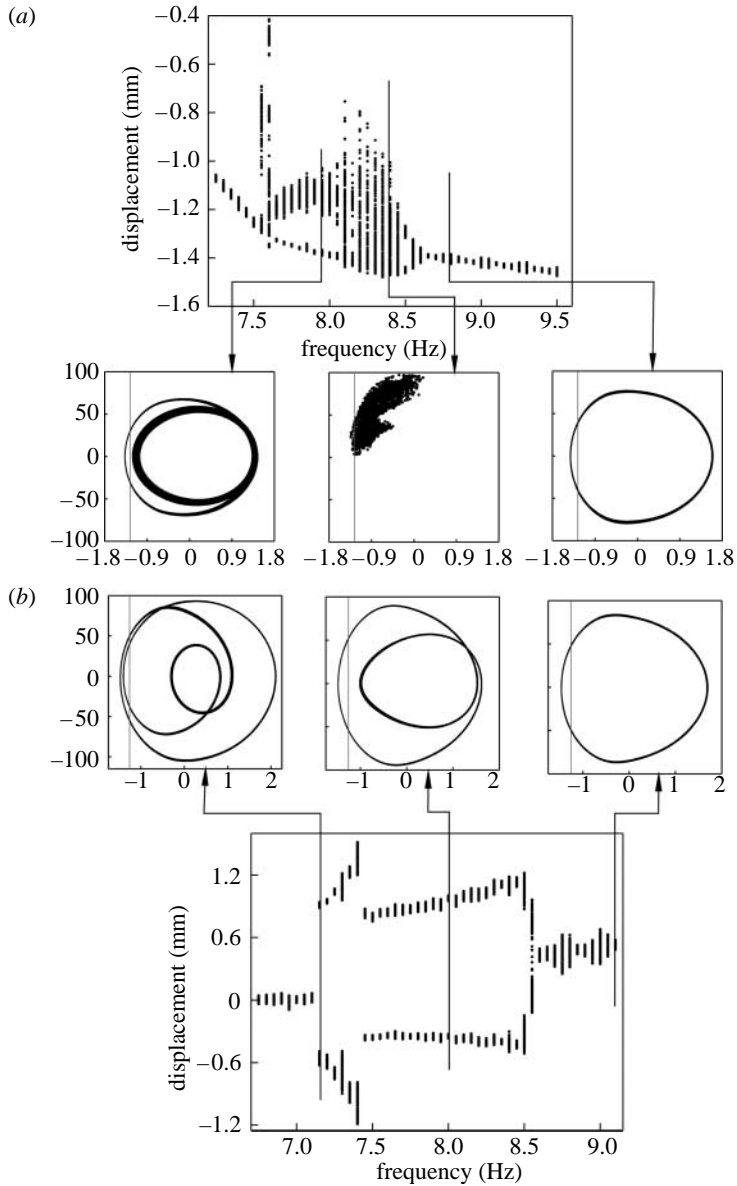


Figure 8. Bifurcation diagrams obtained for the mass displacement under varying frequency f at $f_n = 9.38$ Hz, $c = 1.3$ kg s⁻¹, $\beta = 29$, $g = 1.26$ mm and the excitation amplitude equal to (a) 0.44 mm and (b) 0.53 mm. Additional windows demonstrate the trajectories on the phase plane and obtained for (a) $f = 7.85, 8.35$ (Poincaré map) and 8.55 Hz; and (b) $f = 7.15, 8.00$ and 9.10 Hz, respectively.

equation can be easily obtained for the perturbation dynamics as each component of the function $\mathbf{F}(\mathbf{x}, t)$ in the immediate neighbourhood of \mathbf{x}^* can be expanded into a Taylor series:

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{x}, t) - \mathbf{F}(\mathbf{x}^*, t) = \left. \frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{u} + O(\|\mathbf{u}\|^2). \quad (4.2)$$

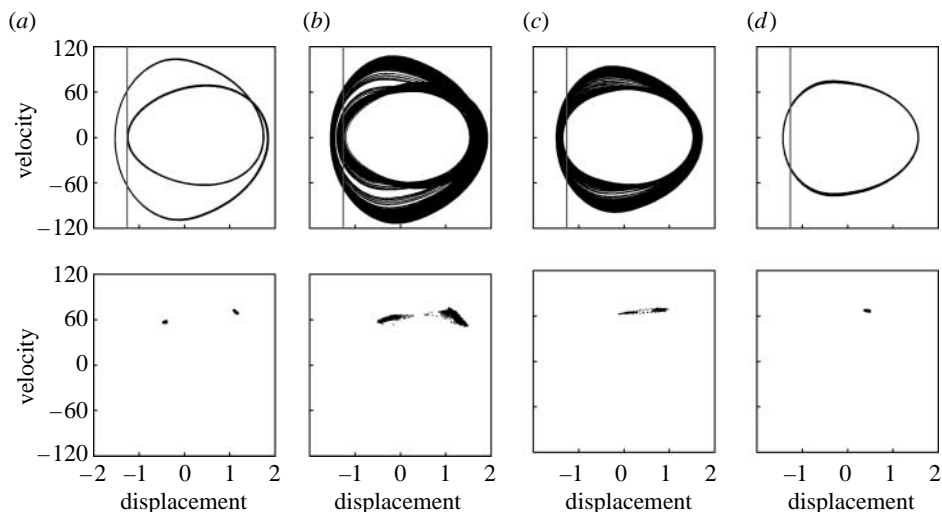


Figure 9. Trajectories on the phase plane and Poincaré maps for three values of frequency f near the third bifurcation of figure 8*b* at $f_n=9.38$ Hz, $c=1.3$ kg s⁻¹, $\beta=29$, $g=1.26$ mm and the excitation amplitude equal to 0.53 mm; (a) $f=8.45$ Hz, (b) $f=8.50$ Hz, (c) $f=8.55$ Hz and (d) $f=8.60$ Hz. Grey lines indicate the location of the secondary spring.

Thus neglecting higher order terms, to analyse the perturbation, a set of linear differential equations with a time-dependant coefficient matrix has to be considered

$$\dot{\mathbf{u}} = \left. \frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*} \mathbf{u}. \quad (4.3)$$

As explained in Argyris *et al.* (1994), the basic idea behind Floquet's stability theory is the conjecture that the periodicity of the matrix $\mathbf{D} = (\partial \mathbf{F}(\mathbf{x}, t)/\partial \mathbf{x})|_{\mathbf{x}=\mathbf{x}^*}$ induces a reduction to a system with constant coefficients, allowing an observation of the behaviour of \mathbf{u} only at the discrete points in time $t=0, T, 2T, 3T, \dots$. From a geometrical point of view, this corresponds exactly to a Poincaré section, thus allowing conclusions drawn from the characteristics of the appertaining Poincaré map to apply to the behaviour of the continuous system.

The periodic solution \mathbf{x}^* of the system (4.1) corresponds to the fixed point \mathbf{v}^* of the appertaining Poincaré map \mathbf{P} , i.e. $\mathbf{v}^* = \mathbf{P}(\mathbf{v}^*)$. So the linear equation with the constant coefficient Jacobian matrix $\mathbf{J} = (\partial \mathbf{P}/\partial \mathbf{v})|_{\mathbf{v}=\mathbf{v}^*}$ for the infinitesimal perturbation \mathbf{v}_n on the n th iteration of the Poincaré map can be obtained

$$\mathbf{v}_{n+1} = \mathbf{J}\mathbf{v}_n. \quad (4.4)$$

If all eigenvalues of the Jacobian matrix have modulus less than unity, then the fixed point \mathbf{v}^* and the corresponding periodic orbit \mathbf{x}^* are stable.

Theoretical investigations of the considered oscillator were conducted using the physical model shown in figure 11*a*. The non-dimensionalized equations of motion for this system are derived in Shaw & Holmes (1983*a,b*) and Wiercigroch & Sin (1998) and they are given in non-dimensional form as

$$\begin{aligned} \mathbf{x}' &= \mathbf{v} \quad \text{and} \\ \mathbf{v}' &= \Gamma \sin(\omega\tau) - 2\xi\mathbf{v} - \mathbf{x} - \beta(\mathbf{x} - e)H(\mathbf{x} - e), \end{aligned} \quad (4.5)$$

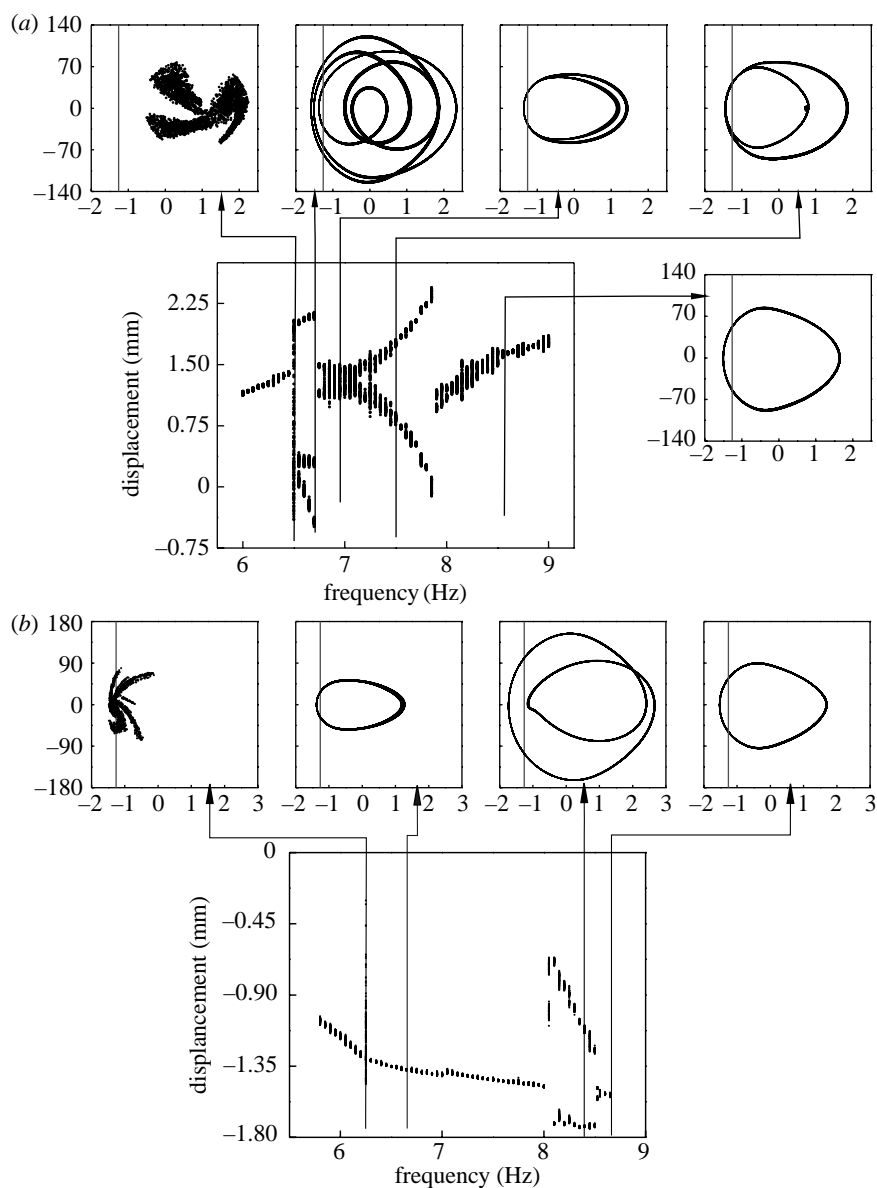


Figure 10. Bifurcation diagrams obtained for the mass displacement under varying frequency f at $f_n = 9.38$ Hz, $c = 1.3$ kg s $^{-1}$, $\beta = 29$, $g = 1.26$ mm and the excitation amplitude equal to (a) 0.66 mm and (b) 0.70 mm. Additional windows demonstrate the trajectories on the phase plane and obtained for (a) $f = 6.50$ (Poincaré map), 6.70, 6.95, 7.55 and 8.60 Hz; and (b) $f = 6.25$ (Poincaré map), 6.65, 8.35 and 8.60 Hz, respectively.

where

$$x = \frac{y}{x_0}, \quad v = \frac{dx}{d\tau}, \quad \tau = 2\pi f_n t, \quad f_n = \frac{1}{2\pi} \sqrt{\frac{k_1}{m}}, \quad (4.6)$$

and $H(\cdot)$ is the Heaviside step function, f_n is the natural frequency for the undamped linear system (measured in Hz as before), x_0 is some arbitrary

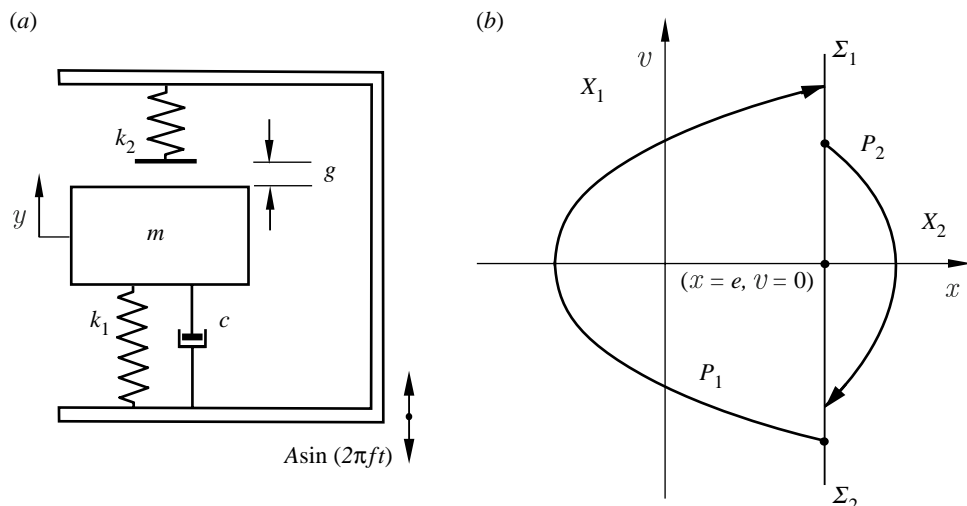


Figure 11. (a) Physical model of the oscillator. (b) Phase space is divided by the discontinuity boundaries Σ_1 and Σ_2 , and the locally valid maps P_1 and P_2 project the points from one boundary to another.

reference distance and the remaining parameters are

$$\beta = \frac{k_2}{k_1}, \quad \omega = \frac{f}{f_n}, \quad e = \frac{g}{x_0}, \quad \xi = \frac{c}{4m\pi f_n} \quad \text{and} \quad \Gamma = \frac{A}{x_0} \frac{f^2}{f_n^2}. \quad (4.7)$$

Here β is the stiffness ratio, ω is the non-dimensionalized frequency, e is the non-dimensionalized gap, ξ is the damping ratio and Γ is the non-dimensionalized forcing amplitude.

The considered oscillator operates in two different regimes, i.e. with and without contact with the secondary spring. Thus the phase space of the system is divided into two half-spaces X_1 and X_2 where the motion of the oscillator is governed by different linear ordinary differential equations. Switches between those two regimes occur on the discontinuity boundaries, Σ_1 and Σ_2 , which are defined as

$$\begin{aligned} \Pi X_{1,2} &\equiv \Sigma_1 = \{(\tau_i; x_i, v_i) | x_i = e, \quad v_i > 0\} \quad \text{and} \\ \Pi X_{2,1} &\equiv \Sigma_2 = \{(\tau_i; x_i, v_i) | x_i = e, \quad v_i < 0\}. \end{aligned} \quad (4.8)$$

Numerical integrations of this piecewise smooth system are connected with the principal difficulties of determining the moments when the mass hits the secondary spring, or in other words the moments when the trajectory crosses the discontinuity boundaries. The precise value of the crossing time τ_i has to be evaluated since the response can be very sensitive to any inaccuracy of the computed solution on the above-mentioned boundaries. Various procedures could be implemented in order to achieve the required precision of the crossing time (e.g. [Wiercigroch & de Kraker 2000](#)). To avoid these difficulties in this paper, it is proposed to construct a global solution of the system using local maps. These maps are defined as follows:

$$P_1 : \Sigma_1 \rightarrow \Sigma_2 \quad \text{and} \quad P_2 : \Sigma_2 \rightarrow \Sigma_1. \quad (4.9)$$

The local maps are two dimensional and project the point located on one of the discontinuity boundaries onto another as $P_i : (\psi_n, v_n) \rightarrow (\psi_{n+1}, v_{n+1})$ as shown in [figure 11b](#). As the external excitation is periodic, a variable $\psi = \omega\tau \bmod 2\pi$ is

introduced so that $\psi \in (0, 2\pi]$. The maps (4.9) can be calculated using the solutions of the equations of motion in each smooth half-space which are linear. These solutions are given in full in appendix A, and details of the procedure describing construction of the local maps are given in appendix B.

Experimentally measured and theoretically predicted trajectories are compared in figure 12, where periodic orbits of different complexity are shown. As can be seen from this figure, a good correspondence between theory and experiment is obtained for the periodic responses. Therefore, this mathematical model is used for theoretical prediction of the stability of the periodic orbits.

Next, we propose a new experimental method to study the stability of the periodic orbits based on the experimental evaluation of the Jacobian matrix. First, a periodic orbit of the experimental system needs to be established. Two examples of the steady state system response are shown in black in figure 13*a, b* and *c, d*. Then a perturbation from the established periodic orbit is applied to the system, which is practically realized by a gentle tap onto the oscillator mass. A suitable perturbation should be strong enough to be distinguishable from the background noise, but small enough to preserve the locally linear nature of the map (e.g. a perturbation should not change a non-impacting to an impacting trajectory or vice versa). Since the perturbations were randomly applied, care needed to be taken so that the results were repeatable since, for example, perturbations along one eigen direction would suppress the corresponding eigenvalue. In our case, all the observed maps showed complex conjugate eigenvalues, so in fact this was not an issue. Nevertheless, the data from at least five perturbations were averaged for each estimate of the stability. Figure 13*a, c* show the time histories of displacements for the periodic orbits with and without impacts, where the moments of perturbations are marked by arrows. Next, the Poincaré maps for the perturbed orbits are constructed as shown in figure 14, and their points are marked by numbers 1, 2, 3, ... and connected by solid lines for the orbits without (figure 14*a*) and with impacts (figure 14*b*). The choice of the coordinate system $(x - x^*, v - v^*)$ is made to demonstrate the behaviour of the perturbation around the point (0, 0) which corresponds to the Poincaré point of the original periodic orbit. Next, we make an assumption that any two subsequent points of the map are linearly dependant. This assumption is based on the fact that the perturbation of the original orbit was very small. The recorded data points of the perturbed map are then used to calculate the matrix \mathbf{A}

$$\begin{pmatrix} x_{n+1} - x^* \\ v_{n+1} - v^* \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_n - x^* \\ v_n - v^* \end{pmatrix}, \quad (4.10)$$

where (x^*, v^*) is the fixed point of the Poincaré map corresponding to unperturbed periodic orbit, A_{11} , A_{12} , A_{21} and A_{22} are unknown coefficients of the constant matrix \mathbf{A} to be determined and index n denotes the n th step of the iteration. Having the form of equation (4.10) describing the experimental perturbations of the mass displacement and velocity exactly the same as the theoretical linear equation for perturbation of the Poincaré map (4.4), we can conclude that the matrix \mathbf{A} is the experimental evaluation of the Jacobian matrix \mathbf{J} . In general to find the matrix \mathbf{A} only three consecutive data points are required, but in this case to satisfy the assumption about the constancy of \mathbf{A} the first eight data points were considered and a least-squares method was used to obtain the best fit. The results

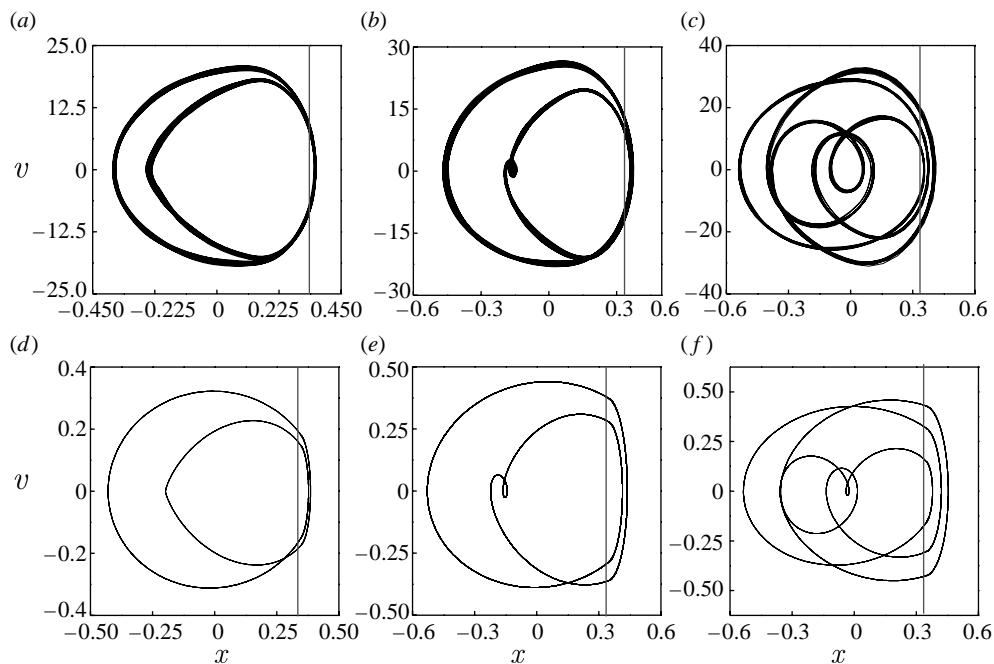


Figure 12. Comparison between the experimental results (*a–c*) and theoretical predictions (*d–f*) for $\beta=13$, $\xi=0.015822$, $e=0.35$ and (*a*) $\omega=0.78$, $\Gamma=0.35$, (*b*) $\omega=0.804$, $\Gamma=0.35$, (*c*) $\omega=0.70$, $\Gamma=0.45$, (*d*) $\omega=0.73$, $\Gamma=0.35$, (*e*) $\omega=0.80$, $\Gamma=0.35$ and (*f*) $\omega=0.70$, $\Gamma=0.45$.

are shown by dotted lines in [figure 14a,b](#) for the orbits without and with impacts, respectively, for $f_n=11$ Hz, $g=0.335$ mm, $c=1.12$ kg s⁻¹, $\beta=13$, the excitation amplitude of 0.552 mm, and $f=6.07$ and 7.34 Hz. These approximations were obtained by using the matrices \mathbf{A}_a and \mathbf{A}_b for the orbits without and with impacts, respectively, which were determined as

$$\mathbf{A}_a = \begin{pmatrix} 0.164 & -0.014 \\ 63.105 & 0.184 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_b = \begin{pmatrix} 0.014 & -0.010 \\ 79.483 & 0.499 \end{pmatrix}. \quad (4.11)$$

As can be seen from [figure 14](#), a good correspondence between the original experimental results and the linearizations is obtained.

Knowing the matrices \mathbf{A}_a and \mathbf{A}_b , the stability of the corresponding periodic orbits shown in [figure 13b,d](#) can be easily evaluated by calculating the appropriate eigenvalues. It was found that the fixed points in the two cases have complex conjugate eigenvalues $\lambda_{1,2a}=0.174 \pm i0.944$ for the regime without impacts and $\lambda_{1,2b}=0.256 \pm i0.865$ for the regime with impacts. That means for the specific set of parameters, at the grazing condition the eigenvalues abruptly jump from one position to another, but remain within the unit circle and hence there is a transition from a stable non-impacting period-1 orbit to a stable impacting period-1 orbit. We have seen earlier that it is also possible for the period-1 orbit to lose stability abruptly at the grazing condition.

If the eigenvalues jump abruptly at grazing, can they jump from any position to any other position? Or, is there any restriction or pattern in the jump? The procedure described above has been adopted to investigate this question

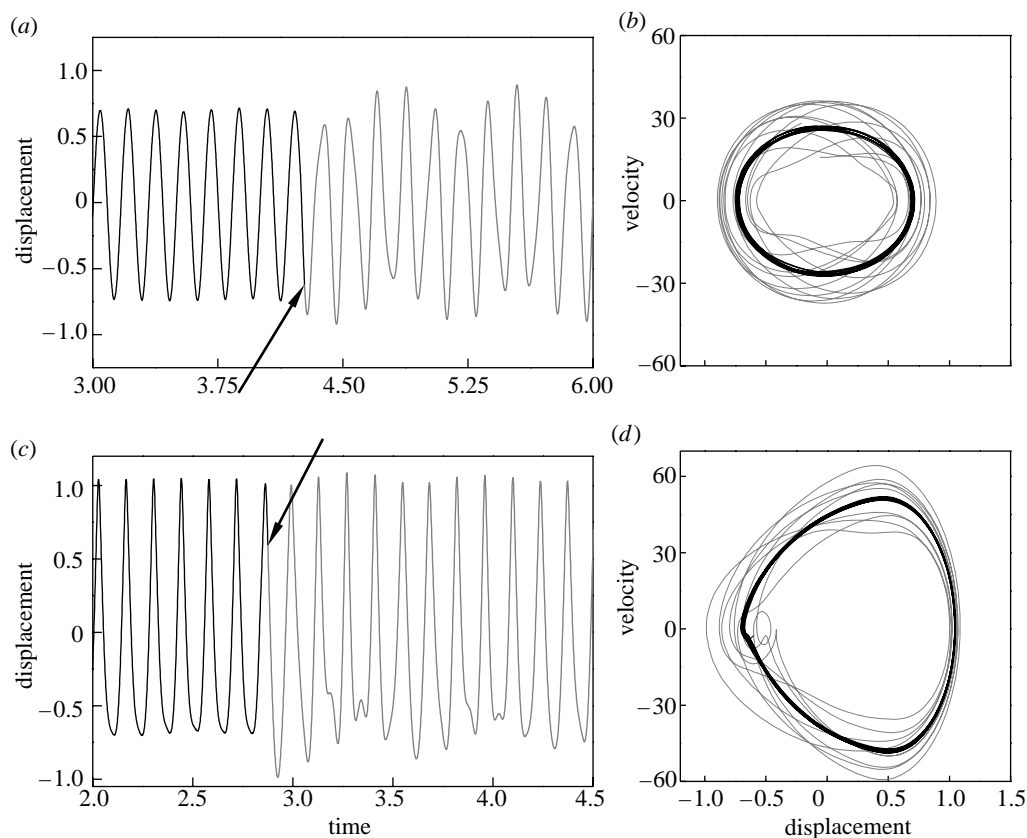


Figure 13. Trajectories of unperturbed and perturbed orbits are given in black and grey, respectively, for $f_n = 11$ Hz, $g = 0.335$ mm, $c = 1.12$ kg s⁻¹, $\beta = 13$, the excitation amplitude of 0.552 mm, and for the regimes (a,b) without contact, $f = 6.07$ Hz; and (c,d) with contact, $f = 7.34$ Hz.

experimentally. For this, we chose a set of parameters for which a stable non-impacting period-1 orbit changes to a stable impacting period-1 orbit as the excitation frequency was varied. It has been shown earlier (Ma *et al.* 2006) that patterns in the jump of the eigenvalues emerge only when seen in terms of the trace and the determinant of the Jacobian matrix. Following that lead, we obtained the Jacobian matrix for each parameter value, and then the trace and the determinant of that matrix—which are shown in figure 15. Three different values of the stiffness ratio were chosen, i.e. $\beta = 2.9$ (figure 15a), 13.0 (figure 15b) and 95.4 (figure 15c), whereas all other system parameters remained in close range. In practice, three different beams providing secondary stiffness were used, but all the other elements of the rig were the same. The figure also shows the corresponding values predicted from theory, showing a very close match between the two. Theoretical curves used for comparison were obtained using the mathematical model described above, whereas the details of the Jacobian matrix calculations for this piecewise linear model are given in appendix C.

The experimental results showed that the determinant of the Jacobian matrix does not change significantly under varying excitation frequency, but the trace seems to jump abruptly following the grazing condition. However, in all three cases

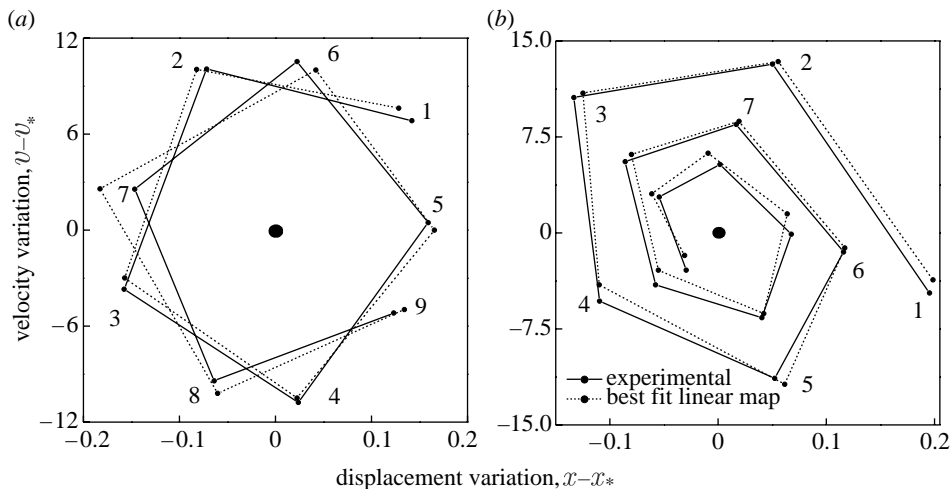


Figure 14. Iterations of the Poincaré maps for the perturbed orbits (a) without and (b) with contact for $f_n=11$ Hz, $g=0.335$ mm, $c=1.12$ kg s⁻¹, $\beta=13$, the excitation amplitude of 0.552 mm, and (a) $f=6.07$ Hz; and (b) $f=7.34$ Hz. Experimental data points and the points obtained by approximations are connected by solid and dotted lines, respectively.

shown in figure 15, there is a gap in the experimental curve near the grazing frequency, where, as mentioned before, it was difficult to gather reliable experimental data. The curves obtained from theory were used to fill the gap, to obtain a clearer understanding of the variation of the trace and the determinant very close to the grazing condition. It can be seen from the simulations that the trace remains continuous, but not smooth, during this transition. The variation of the trace curve is indicative of a square-root singularity. However, the determinant remains invariant as the system is moved from a non-impacting state to an impacting state.

5. Conclusions

In this paper, extensive experimental investigations of an impact oscillator with a one-sided elastic constraint have been presented. Different bifurcation scenarios have been shown for a number of values of the excitation amplitude, with the excitation frequency as the bifurcation parameter. Close to grazing, the mass acceleration signal is shown to be more effective in detecting contacts with the secondary spring. The most typical recorded scenario is when a non-impacting periodic orbit bifurcates into an impacting one via the grazing mechanism. In some cases the resulting orbit is stable, but in most cases it loses stability through grazing. Following such an event, the evolution of the attractor is governed by a complex interplay between smooth and non-smooth bifurcations. In some cases, the occurrence of coexisting attractors was manifested through discontinuous transition from one orbit to another through boundary crisis. One notable feature was the occurrence of boundary crises provoked by grazing events.

In this paper we also presented a new experimental technique to study the stability of periodic orbits in impact oscillators. The Jacobian matrix of the Poincaré map, obtained with the experimental technique, was compared with

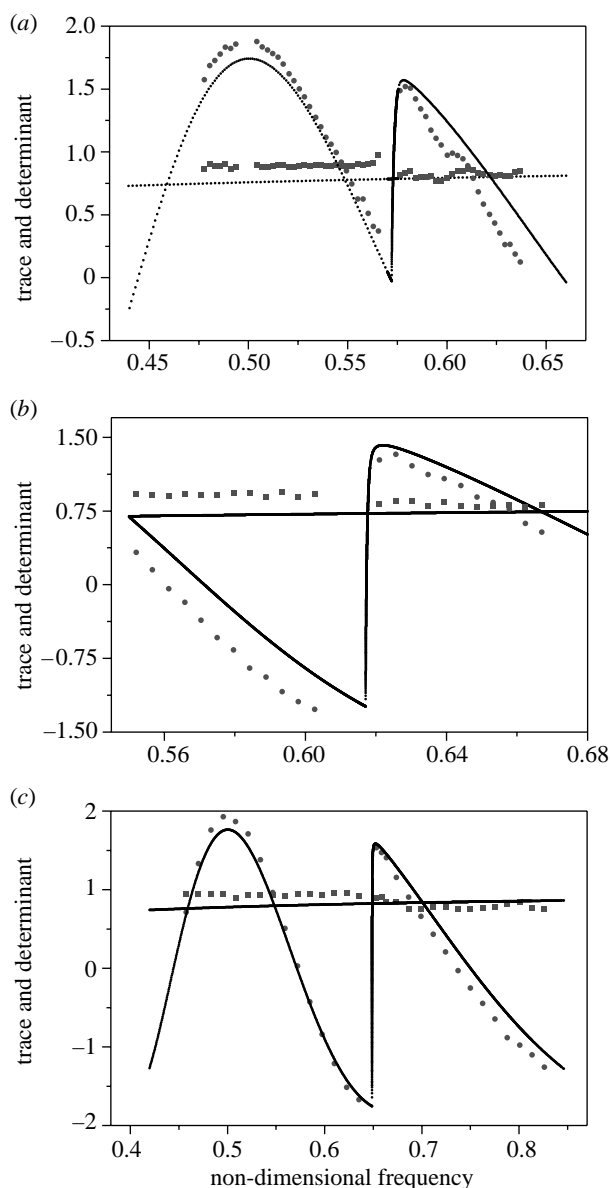


Figure 15. Trace and determinant of the Jacobian matrix obtained for (a) $\beta=2.922$, $\xi=0.011$, $A=1.580$, $e=0.769$, $f_n=18.84$ Hz; (b) $\beta=13$, $\xi=0.008$, $A=0.552$, $e=0.335$, $f_n=10.87$ Hz and (c) $\beta=95.4$, $\xi=0.01$, $A=2.08$, $e=1.511$, $f_n=7.87$ Hz. Experimental results (circles for the trace and squares for the determinant) are shown in grey and the theoretical curves are in black.

that obtained from theoretical analysis, and a good correspondence between them was observed for different stiffness ratios. Using this approach, we also studied how the eigenvalues of the Jacobian matrix change as the system is driven from a non-impacting state to an impacting state. It was concluded that the determinant remains invariant, while the trace varies rapidly at this transition, exhibiting a slope singularity.

The financial support from EPSRC DTA (J.I.), the Nuffield Foundation (E.P.), EPSRC (E.P.) and the Royal Society (S.B.) is gratefully acknowledged.

Appendix A

In the smooth half-space \mathbf{X}_1 , the solutions of the equation of motion (4.5) for initial conditions (x_n, v_n) at τ_n are

$$\begin{aligned} x^I(\tau) &= D_1 \sin(\omega\tau + \varphi_1) + \exp(-\xi(\tau - \tau_n))(A_1 \cos \gamma_1(\tau - \tau_n) \\ &\quad + B_1 \sin \gamma_1(\tau - \tau_n)) \end{aligned} \quad (\text{A } 1)$$

and

$$\begin{aligned} v^I(\tau) &= \omega D_1 \cos(\omega\tau + \varphi_1) + \exp(-\xi(\tau - \tau_n))((\gamma_1 B_1 - \xi A_1) \cos \gamma_1(\tau - \tau_n) \\ &\quad + (-\gamma_1 A_1 - \xi B_1) \sin \gamma_1(\tau - \tau_n)), \end{aligned} \quad (\text{A } 2)$$

where

$$D_1 = \frac{\Gamma}{\sqrt{(2\xi\omega)^2 + (1 - \omega^2)^2}}, \quad \gamma_1 = \sqrt{1 - \xi^2}, \quad \varphi_1 = \arctan\left(\frac{-2\omega\xi}{1 - \omega^2}\right) \quad (\text{A } 3)$$

and

$$\begin{aligned} A_1(\tau_n, x_n) &= x_n - D_1 \sin(\omega\tau_n + \varphi_1), \\ B_1(\tau_n, x_n, v_n) &= \frac{v_n - \omega D_1 \cos(\omega\tau_n + \varphi_1) + \xi A_1(\tau_n, x_n)}{\gamma_1}. \end{aligned} \quad (\text{A } 4)$$

In the second half-space \mathbf{X}_2 , the solutions for initial conditions (x_n, v_n) at τ_n are

$$\begin{aligned} x^{II}(\tau) &= \frac{\beta e}{1 + \beta} + D_2 \sin(\omega\tau + \varphi_2) + \exp(-\xi(\tau - \tau_n))(A_2 \cos \gamma_2(\tau - \tau_n) \\ &\quad + B_2 \sin \gamma_2(\tau - \tau_n)) \end{aligned} \quad (\text{A } 5)$$

and

$$\begin{aligned} v^{II}(\tau) &= \omega D_2 \cos(\omega\tau + \varphi_2) + \exp(-\xi(\tau - \tau_n))((\gamma_2 B_2 - \xi A_2) \cos \gamma_2(\tau - \tau_n) \\ &\quad + (-\gamma_2 A_2 - \xi B_2) \sin \gamma_2(\tau - \tau_n)), \end{aligned} \quad (\text{A } 6)$$

where the constants are given below:

$$D_2 = \frac{\Gamma}{\sqrt{((1 + \beta) - \omega^2)^2 + (2\omega\xi)^2}}, \quad \gamma_2 = \sqrt{(1 + \beta) - \xi^2}, \quad (\text{A } 7)$$

$$\varphi_2 = \arctan\left(\frac{-2\omega\xi}{(1 + \beta) - \omega^2}\right) \quad \text{and}$$

$$\begin{aligned} A_2(\tau_n, x_n) &= x_n - \frac{\beta e}{1 + \beta} - D_2 \sin(\omega\tau_n + \varphi_2) \\ B_2(\tau_n, x_n, v_n) &= \frac{v_n + \xi A_2(\tau_n, x_n) - \omega D_2 \cos(\omega\tau_n + \varphi_2)}{\gamma_2}. \end{aligned} \quad (\text{A } 8)$$

Appendix B

The map \mathbf{P}_1 associates the point $(\psi_i, v_i) \in \Sigma_2$ with the point $(\psi_{i+1}, v_{i+1}) \in \Sigma_1$, or in other words the initial moment $\tau_i = \psi_i/\omega$ of *no contact* regime characterized by initial conditions (e, v_i) with its final moment τ_{i+1} . Equation $\mathcal{G}_1(\tau_{i+1}) = x^I(\tau_{i+1}) - e = 0$ is an implicit equation that needs to be solved in order to determine the time τ_{i+1} . The formula for $x^I(\cdot)$ is given in equation (A 1), and thus we have an equation for the unknown τ_{i+1}

$$\begin{aligned} \mathcal{G}_1(\tau_{i+1}) = & D_1 \sin(\omega\tau_{i+1} + \phi_1) + \exp(-\xi(\tau_{i+1} - \tau_i))(A_1 \cos \gamma_1(\tau_{i+1} - \tau_i) \\ & + B_1 \sin \gamma_1(\tau_{i+1} - \tau_i)) - e = 0, \end{aligned} \quad (\text{B } 1)$$

where D_1 , γ_1 , ϕ_1 , A_1 and B_1 are described by equations (A 3) and (A 4) substituting $\tau_n = \psi_i/\omega$, $x_n = e$, $v_n = v_i$.

Once τ_{i+1} is calculated, the values of ψ_{i+1} and velocity v_{i+1} can be determined from

$$\psi_{i+1} = \omega\tau_{i+1} \bmod 2\pi \quad \text{and} \quad (\text{B } 2)$$

$$\begin{aligned} v_{i+1} = & v^I(\tau_{i+1}) = \omega D_1 \cos(\omega\tau_{i+1} + \phi_1) + \exp(-\xi(\tau_{i+1} - \tau_i)) \\ & \times ((\gamma_1 B_1 - \xi A_1) \cos \gamma_1(\tau_{i+1} - \tau_i) + (-\gamma_1 A_1 - \xi B_1) \sin \gamma_1(\tau_{i+1} - \tau_i)). \end{aligned} \quad (\text{B } 3)$$

Similarly, the map \mathbf{P}_2 relates the point $(\psi_{i+1}, v_{i+1}) \in \Sigma_1$ to the point $(\psi_{i+2}, v_{i+2}) \in \Sigma_2$, or in other words the initial moment $\tau_{i+1} = \psi_{i+1}/\omega$ of *contact* regime characterized by initial conditions (e, v_{i+1}) to its final moment τ_{i+2} . Equation $\mathcal{G}_2(\tau_{i+2}) = x^{II}(\tau_{i+2}) - e = 0$ is again implicit and needs to be solved in order to determine time τ_{i+2} , and now the formula for $x^{II}(\cdot)$ is given in equation (A 5)

$$\begin{aligned} \mathcal{G}_2(\tau_{i+2}) = & \frac{\beta e}{1 + \beta} + D_2 \sin(\omega\tau_{i+2} + \phi_2) + \exp(-\xi(\tau_{i+2} - \tau_{i+1})) \\ & \times (A_2 \cos \gamma_2(\tau_{i+2} - \tau_{i+1}) + B_2 \sin \gamma_2(\tau_{i+2} - \tau_{i+1})) - e = 0, \end{aligned} \quad (\text{B } 4)$$

where D_2 , γ_2 , ϕ_2 , A_2 and B_2 are described by equations (A 7) and (A 8) substituting $\tau_n = \psi_{i+1}/\omega$, $x_n = e$, $v_n = v_{i+1}$.

Once τ_{i+2} is calculated, the values of ψ_{i+2} and velocity v_{i+2} can be determined from

$$\psi_{i+2} = \omega\tau_{i+2} \bmod 2\pi \quad \text{and} \quad (\text{B } 5)$$

$$\begin{aligned} v_{i+2} = & v^{II}(\tau_{i+2}) = \omega D_2 \cos(\omega\tau_{i+2} + \phi_2) + \exp(-\xi(\tau_{i+2} - \tau_{i+1})) \\ & \times ((\gamma_2 B_2 - \xi A_2) \cos \gamma_2(\tau_{i+2} - \tau_{i+1}) \\ & + (-\gamma_2 A_2 - \xi B_2) \sin \gamma_2(\tau_{i+2} - \tau_{i+1})). \end{aligned} \quad (\text{B } 6)$$

Appendix C

Theoretical results for the Jacobian matrix were obtained using the solutions of the linear equations of motion constructed in both smooth half spaces of the system and given in appendix A. As it was difficult to obtain experimental results

on the discontinuity boundary, the earlier developed theoretical maps were not suitable for the comparison, and appropriate Poincaré maps were constructed and their stability was analysed. Similar to the earlier introduced local maps \mathbf{P}_1 and \mathbf{P}_2 described in appendix B, three additional maps \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 were introduced. Local map \mathcal{P}_1 associates a point in the first half space $(\psi_i, x_i, v_i) \in \mathbf{X}_1$ (assuming that $\psi_i=0$) with the point on the boundary of this half space $(\psi_{i+1}, e, v_{i+1}) \in \Sigma_1$

$$(\psi_{i+1}, v_{i+1}) = \mathcal{P}_1(x_i, v_i).$$

Local map \mathcal{P}_2 associates a point on the boundary $(\psi_i, e, v_i) \in \Sigma_1$ with a point on the other boundary of this half space $(\psi_{i+1}, e, v_{i+1}) \in \Sigma_2$ (and in fact \mathcal{P}_2 coincides with earlier introduced local map \mathbf{P}_2)

$$(\psi_{i+1}, v_{i+1}) = \mathcal{P}_2(\psi_i, v_i).$$

Finally, local map \mathcal{P}_3 associates a point on the boundary $(\psi_i, e, v_{i+1}) \in \Sigma_2$ with point $(2\pi - \psi_i, x_{i+1}, v_{i+1})$

$$(x_{i+1}, v_{i+1}) = \mathcal{P}_3(\psi_i, v_i).$$

Here again to use local maps \mathcal{P}_1 and \mathcal{P}_2 it is necessary to solve nonlinear algebraic equations, but the application of \mathcal{P}_3 is straightforward as in this case for known $\psi_{i+1} = 2\pi - \psi_i$ formulae of the displacement and velocity are explicitly defined as equation (A 1). Then a Poincaré map can be constructed as

$$(x_{n+1}, v_{n+1}) = \mathbf{P}(x_n, v_n) = \mathcal{P}_3 \circ \mathcal{P}_2 \circ \mathcal{P}_1(x_n, v_n). \quad (\text{C } 1)$$

A fixed point of the Poincaré map is expressed as $\mathbf{v}^* = \mathbf{P}(\mathbf{v}^*)$ and to calculate the stability of this point the Jacobian matrix is computed by the chain rule (Shaw & Holmes 1983*a,b*)

$$D\mathbf{P} = \left[\frac{\partial(x_{i+3}, v_{i+3})}{\partial(x_i, v_i)} \right]_{(x_*, v_*)} = \prod_{j=1}^3 D\mathcal{P}_j. \quad (\text{C } 2)$$

Here it should be noted that as shown in figure 16, local map \mathcal{P}_1 associates (x_i, v_i) with (ψ_{i+1}, v_{i+1}) as the displacement $x_{i+1} = e$ and the time (or variable $\psi = \tau\omega \bmod 2\pi$) vary with the change of initial conditions. Thus

$$D\mathcal{P}_1 = \left(\frac{\partial(\psi_{i+1}, v_{i+1})}{\partial(x_i, v_i)} \right). \quad (\text{C } 3)$$

Similarly, local map \mathcal{P}_2 associates (ψ_{i+1}, v_{i+1}) with (ψ_{i+2}, v_{i+2}) as again the displacement $x_{i+2} = e$, and

$$D\mathcal{P}_2 = \left(\frac{\partial(\psi_{i+2}, v_{i+2})}{\partial(\psi_{i+1}, v_{i+1})} \right). \quad (\text{C } 4)$$

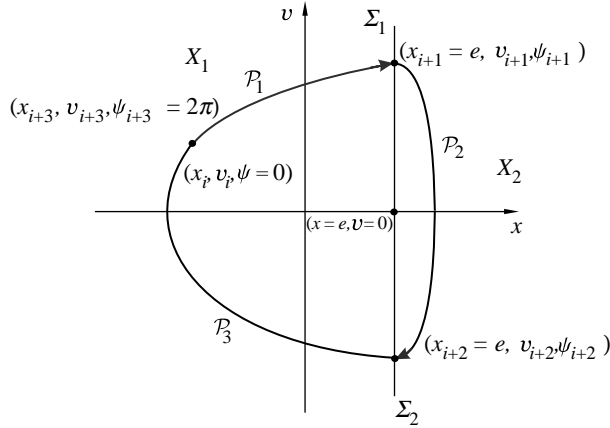


Figure 16. The Poincaré map is constructed using the locally valid maps \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 .

Finally, local map \mathcal{P}_3 associates (ψ_{i+2}, v_{i+2}) with (x_{i+3}, v_{i+3}) as in this case the final value of ψ is known as $\psi_{i+3} = 2\pi - \psi_{i+1} - \psi_{i+2}$, and

$$D\mathcal{P}_3 = \left(\frac{\partial(x_{i+3}, v_{i+3})}{\partial(\psi_{i+2}, v_{i+2})} \right). \tag{C 5}$$

Thus the Jacobian matrix $D\mathbf{P}$ is calculated as

$$D\mathbf{P} = \left[\frac{\partial(x_{i+3}, v_{i+3})}{\partial(x_i, v_i)} \right]_{(x_*, v_*)} = \left(\frac{\partial(x_{i+3}, v_{i+3})}{\partial(\psi_{i+2}, v_{i+2})} \right) \left(\frac{\partial(\psi_{i+2}, v_{i+2})}{\partial(\psi_{i+1}, v_{i+1})} \right) \left(\frac{\partial(\psi_{i+1}, v_{i+1})}{\partial(x_i, v_i)} \right) \Bigg|_{(x_*, v_*)}. \tag{C 6}$$

The partial derivatives given in equation (C 6) are calculated using implicit function differentiation as explained below.

The first matrix in the r.h.s. of the equation (C 6) contains four partial derivatives $\partial\psi_{i+1}/\partial x_i$, $\partial v_{i+1}/\partial x_i$, $\partial\psi_{i+1}/\partial v_i$ and $\partial v_{i+1}/\partial v_i$. The phase variable ψ_{i+1} is implicitly defined by equation (B 1) where the substitutions of $\tau_i=0$ and $\psi_{i+1}=\omega\tau_{i+1}$ are made. Thus using implicit differentiation, one can obtain partial differentials

$$\frac{\partial\psi_{i+1}}{\partial x_i} = - \frac{\frac{\partial\mathcal{G}_1}{\partial x_i}}{\frac{\partial\mathcal{G}_1}{\partial\psi_{i+1}}} \tag{C 7}$$

$$\frac{\partial\psi_{i+1}}{\partial v_i} = - \frac{\frac{\partial\mathcal{G}_1}{\partial v_i}}{\frac{\partial\mathcal{G}_1}{\partial\psi_{i+1}}} \tag{C 8}$$

and because $v_{i+1} = v^I(\tau_{i+1})$ (where function v^I is given by equation (A 1) substituting $\tau_{n+1} = \psi_{i+1}/\omega$, $\tau_n=0$, $x_n=x_i$ and $v_n=v_i$) is a function of ψ_{i+1} as well

as x_i and v_i , the other partial derivatives are calculated as

$$\frac{\partial v_{i+1}}{\partial x_i} = \frac{\partial v^I}{\partial x_i} + \frac{\partial v^I}{\partial \psi_{i+1}} \frac{\partial \psi_{i+1}}{\partial x_i} \quad \text{and} \quad (\text{C } 9)$$

$$\frac{\partial v_{i+1}}{\partial v_i} = \frac{\partial v^I}{\partial v_i} + \frac{\partial v^I}{\partial \psi_{i+1}} \frac{\partial \psi_{i+1}}{\partial v_i}. \quad (\text{C } 10)$$

Thus substituting

$$\begin{aligned} \mathcal{G}_1(\psi_{i+1}/\omega) &= D_1 \sin(\psi_{i+1} + \varphi_1) + \exp(-\xi \psi_{i+1}/\omega) (A_1(x_i) \cos(\gamma_1 \psi_{i+1}/\omega) \\ &\quad + B_1(x_i, v_i) \sin(\gamma_1 \psi_{i+1}/\omega)) - e = 0, \end{aligned}$$

$$\frac{\partial \mathcal{G}_1}{\partial \psi_{i+1}} = v^I(\psi_{i+1}/\omega)/\omega,$$

$$\begin{aligned} \frac{\partial \mathcal{G}_1}{\partial x_i} &= \exp(-\xi \psi_{i+1}/\omega) \left(\frac{\partial A_1(x_i)}{\partial x_i} \cos(\gamma_1 \psi_{i+1}/\omega) + \frac{\partial B_1(x_i, v_i)}{\partial x_i} \sin(\gamma_1 \psi_{i+1}/\omega) \right) \\ &= \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) + \frac{\xi}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{G}_1}{\partial v_i} &= \exp(-\xi \psi_{i+1}/\omega) \left(\frac{\partial A_1(x_i)}{\partial v_i} \cos(\gamma_1 \psi_{i+1}/\omega) + \frac{\partial B_1(x_i, v_i)}{\partial v_i} \sin(\gamma_1 \psi_{i+1}/\omega) \right) \\ &= \exp(-\xi \psi_{i+1}/\omega) \left(\frac{1}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial v^I}{\partial \psi_{i+1}} &= -\omega D_1 \sin(\psi_{i+1} + \varphi_1) + \exp(-\xi \psi_{i+1}/\omega) \left(\frac{1}{\omega} \cos(\gamma_1 \psi_{i+1}/\omega) (A_1(x_i) (\xi^2 - \gamma_1^2) \right. \\ &\quad \left. + B_1(x_i, v_i) (-2\xi \gamma_1)) + \frac{1}{\omega} \sin(\gamma_1 \psi_{i+1}/\omega) (A_1(x_i) (2\xi \gamma_1) + B_1(x_i, v_i) (\xi^2 - \gamma_1^2)) \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial v^I}{\partial x_i} &= \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) \left(-\xi \frac{\partial A_1(x_i)}{\partial x_i} + \gamma_1 \frac{\partial B_1(x_i, v_i)}{\partial x_i} \right) \right. \\ &\quad \left. + \sin(\gamma_1 \psi_{i+1}/\omega) \left(-\gamma_1 \frac{\partial A_1(x_i)}{\partial x_i} - \xi \frac{\partial B_1(x_i, v_i)}{\partial x_i} \right) \right) \\ &= \exp(-\xi \psi_{i+1}/\omega) \left(\sin(\gamma_1 \psi_{i+1}/\omega) \frac{-\xi^2 - \gamma_1^2}{\gamma_1} \right) \quad \text{and} \end{aligned}$$

$$\begin{aligned} \frac{\partial v^I}{\partial v_i} &= \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) \left(\gamma_1 \frac{\partial B_1(x_i, v_i)}{\partial v_i} \right) \right. \\ &\quad \left. + \sin(\gamma_1 \psi_{i+1}/\omega) \left(-\xi \frac{\partial B_1(x_i, v_i)}{\partial v_i} \right) \right) \\ &= \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) - \frac{\xi}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right), \end{aligned}$$

we obtained

$$\frac{\partial \psi_{i+1}}{\partial x_i} = -\frac{\omega}{v^I(\psi_{i+1}/\omega)} \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) + \frac{\xi}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right),$$

$$\frac{\partial \psi_{i+1}}{\partial v_i} = -\frac{\omega}{v^I(\psi_{i+1}/\omega)} \exp(-\xi \psi_{i+1}/\omega) \left(\frac{1}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right),$$

$$\frac{\partial v_{i+1}}{\partial x_i} = \exp(-\xi \psi_{i+1}/\omega) \left(\sin(\gamma_1 \psi_{i+1}/\omega) \frac{-\xi^2 - \gamma_1^2}{\gamma_1} \right) + \frac{\partial v^I}{\partial \psi_{i+1}} \frac{\partial \psi_{i+1}}{\partial x_i} \quad \text{and}$$

$$\frac{\partial v_{i+1}}{\partial v_i} = \exp(-\xi \psi_{i+1}/\omega) \left(\cos(\gamma_1 \psi_{i+1}/\omega) - \frac{\xi}{\gamma_1} \sin(\gamma_1 \psi_{i+1}/\omega) \right) + \frac{\partial v^I}{\partial \psi_{i+1}} \frac{\partial \psi_{i+1}}{\partial v_i}.$$

The other partial derivatives given in equation (C 6) are calculated similarly using function \mathcal{G}_2 given by equation (B 4) and formulae for the displacement and velocity in the second half-space equation (A 5).

References

- Argyris, J., Faust, G. & Haase, M. 1994 *An exploration of chaos*. Amsterdam, The Netherlands: Elsevier Science/North-Holland.
- Chin, W., Ott, E., Nusse, H. E. & Grebogi, C. 1995 Universal behavior of impact oscillators near grazing incidence. *Phys. Lett. A* **201**, 197–204. (doi:10.1016/0375-9601(95)00252-X)
- di Bernardo, M., Feigin, M. I., Hogan, S. J. & Homer, M. E. 1999 Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical system. *Chaos Soliton. Fract.* **10**, 1881–1908. (doi:10.1016/S0960-0779(98)00317-8)
- di Bernardo, M., Budd, C. J. & Champneys, A. R. 2001 Grazing and border-collision in piecewise-smooth systems: a unified analytical framework. *Phys. Rev. Lett.* **86**, 2553–2556. (doi:10.1103/PhysRevLett.86.2553)
- Floquet, G. 1883 Sur les équations différentielles linéaires à coefficients périodiques. *Ann. Sci. Ecole Norm. Sup. Ser. 2* **12**, 47.
- Hinrichs, N., Oestreich, M. & Popp, K. 1997 Dynamics of oscillators with impact and friction. *Chaos Soliton. Fract.* **8**, 535–558. (doi:10.1016/S0960-0779(96)00121-X)
- Ing, J., Pavlovskaia, E. & Wiercigroch, M. 2006 Dynamics of a nearly symmetrical piecewise linear oscillator close to grazing incidence: modelling and experimental verification. *Nonlinear Dyn.* **46**, 225–238. (doi:10.1007/s11071-006-9045-9)
- Ma, Y., Agarwal, M. & Banerjee, S. 2006 Border collision bifurcations in a soft impact system. *Phys. Lett. A* **354**, 281–287. (doi:10.1016/j.physleta.2006.01.025)
- Nordmark, A. B. 1991 Non-periodic motion caused by grazing incidence in an impact oscillator. *J. Sound Vib.* **145**, 279–297. (doi:10.1016/0022-460X(91)90592-8)
- Peterka, F. 1974a Laws of impact motion of mechanical systems with one degree of freedom, Part I—theoretical analysis of n -multiple ($1/n$)-impact motions. *Acta Technica ČSAV* **4**, 462–473.
- Peterka, F. 1974b Laws of impact motion of mechanical systems with one degree of freedom, Part II—results of analogue computer modelling of the motion. *Acta Technica ČSAV* **5**, 569–580.
- Peterka, F. 2003 Behaviour of impact oscillator with soft and preloaded stop. *Chaos Soliton. Fract.* **18**, 79–88. (doi:10.1016/S0960-0779(02)00603-3)
- Peterka, F. & Vacik, J. 1992 Transition to chaotic motion in mechanical systems with impacts. *J. Sound Vib.* **154**, 95–115. (doi:10.1016/0022-460X(92)90406-N)

- Peterka, F., Kotera, T. & Ciperá, S. 2004 Explanation of appearance and characteristics of intermittency chaos of the impact oscillator. *Chaos Soliton. Fract.* **19**, 1251–1259. (doi:10.1016/S0960-0779(03)00327-8)
- Piironen, P. T., Virgin, L. N. & Champneys, A. R. 2004 Chaos and period adding; experimental and numerical verification of the grazing bifurcation. *J. Nonlinear Sci.* **14**, 383–404. (doi:10.1007/s00332-004-0616-y)
- Pust, L. & Peterka, F. 2003 Impact oscillator with Hertz's model of contact. *Meccanica* **38**, 99–114. (doi:10.1023/A:1022075519038)
- Shaw, S. W. 1985a The dynamics of a harmonically excited system having rigid amplitude constraints, Part I: subharmonic motions and local bifurcations; Part II: chaotic motions and global bifurcations. *J. Appl. Mech.* **52**, 453–464.
- Shaw, S. W. 1985b Forced vibrations of a beam with one-sided amplitude constraint: theory and experiment. *J. Sound Vib.* **99**, 199–212. (doi:10.1016/0022-460X(85)90357-8)
- Shaw, S. W. & Holmes, P. J. 1983a A periodically forced piecewise linear oscillator. *J. Sound Vib.* **90**, 129–155. (doi:10.1016/0022-460X(83)90407-8)
- Shaw, S. W. & Holmes, P. 1983b Periodically forced linear oscillator with impacts—chaos and long-period motions. *Phys. Rev. Lett.* **51**, 623–626. (doi:10.1103/PhysRevLett.51.623)
- Sin, V. W. T. & Wiercigroch, M. 1999 The design of a symmetrically piecewise oscillator for measurement of chaotic vibration. *Proc. Inst. Mech. Eng. Part C* **213**, 241–249. (doi:10.1243/0954406991522617)
- Thompson, J. M. T., Bokaian, A. R. & Ghaffari, R. 1983 Subharmonic resonances and chaotic motions of a bilinear oscillator, IMA. *J. Appl. Math.* **31**, 207–234.
- Todd, M. D. & Virgin, L. N. 1997 An experimental impact oscillator. *Chaos Soliton. Fract.* **8**, 699–715. (doi:10.1016/S0960-0779(96)00112-9)
- Wagg, D. J., Karpodinis, G. & Bishop, S. R. 1999 An experimental study of the impulse response of a vibro-impacting cantilever beam. *J. Sound Vib.* **228**, 243–264. (doi:10.1006/jsvi.1999.2318)
- Whiston, G. S. 1987 Global dynamics of a vibro-impacting linear oscillator. *J. Sound Vib.* **118**, 395–429. (doi:10.1016/0022-460X(87)90361-0)
- Whiston, G. S. 1992 Singularities in vibro-impact dynamics. *J. Sound Vib.* **152**, 427–460. (doi:10.1016/0022-460X(92)90480-L)
- Wiercigroch, M. & Sin, V. W. T. 1998 Experimental study of a symmetrical piecewise base-excited oscillator. *J. Appl. Mech.* **65**, 657–663.
- Wiercigroch, M., Sin, V. W. T. & Li, K. 1998 Measurement of chaotic vibration in symmetrically piecewise linear oscillator. *Chaos Soliton. Fract.* **9**, 209–220. (doi:10.1016/S0960-0779(97)00174-4)
- Wiercigroch, M. & de Kraker, B. (eds) 2000. *Applied nonlinear dynamics and chaos of mechanical systems with discontinuities*. Nonlinear Science Series A, vol. 28. Singapore: World Scientific.

NOTICE OF CORRECTION

Figure 10 and equation (4.7) are now presented in their correct forms.

19 November 2007