Dynamics of a piecewise smooth map with singularity

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Abstract

Experiments observing the liquid surface in a vertically oscillating container have indicated that modeling the dynamics of such systems require maps that admit states at infinity. In this paper we investigate the bifurcations in such a map. We show that though such maps in general fall in the category of piecewise smooth maps, the mechanisms of bifurcations are quite different from those in other piecewise smooth maps. We obtain the conditions of occurrence of infinite states, and show that periodic orbits containing such states are superstable. We observe period-adding cascade in this system, and obtain the scaling law of the successive periodic windows.

Key words: Border collision bifurcation, piecewise smooth maps, chaos.
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1 Introduction

Recently a lot of research attention has been directed toward the dynamics of piecewise smooth maps (PWS), because they represent a large number of systems of practical interest including switching electrical circuits and impacting mechanical systems. In such systems the discrete phase space is divided into compartments

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within which the map is smooth, and the compartments are separated by borderlines at which the map is not differentiable. A one-dimensional piecewise smooth map has the general form

$$x_{n+1} = f(x_n) = \begin{cases} g(x_n, \mu), \text{ for } & x_n < \lambda \\ h(x_n, \mu), \text{ for } & x_n > \lambda \end{cases} \tag{1}$$

where μ is the bifurcation parameter and the compartments are separated by the borderline value λ. In such a map, there is the possibility that a fixed point may collide with the border with the change of a system parameter. When that happens, there is a sudden change in the stability of the fixed point, leading to a new type of nonlinear phenomenon called *border collision bifurcation* [1,2]. It has been shown that such border collisions may lead to atypical bifurcation phenomena like transition from period-2 to period-3 or a sudden onset of chaos without undergoing the usual period doubling cascade [3].

A few different forms of such maps have been investigated to date:

(1) The map f is continuous, not differentiable at λ, and both dg/dx and dh/dx are finite. Such maps represent a class of switching circuits, and have been investigated in detail [2,4].
(2) The map f is continuous, not differentiable at λ, and there is a square root singularity (i.e., dh/dx is infinite) at one side of the border. Such maps represent the impact oscillator [5,6,1].
(3) The map f is discontinuous at λ, the derivative df/dx is also discontinuous at λ, but the value of the derivative at both sides of the border are finite. Such maps represent a class of electronic circuits [7] including the Colpitts oscillator [8], and the bifurcation theory for such maps has been developed recently [9].

In 1997 an experiment was reported, where the oscillations in the surface of a liquid held in a vertically oscillating container were observed using a laser probe, and it was found that under some conditions narrow jets are ejected from the center of the surface. It was shown that representation of this system required a map with not only slope singularity but also magnitude singularity at the border [10]. The proposed map had the form

$$x_{n+1} = \gamma x_n + \frac{\alpha x_n}{(x_n - \lambda)^2} \quad \text{for} \quad x_n < \lambda \tag{2}$$

$$x_{n+1} = \beta + \frac{\rho x_n}{(x_n - \lambda)^2} \quad \text{for} \quad x_n > \lambda \tag{3}$$

where α, β, ρ and λ are constants and γ is the bifurcation parameter. The graph of the map is schematically shown in Fig. 1. In this paper we investigate the bifurcation phenomena in a map of the above form – for which no theory is currently available.

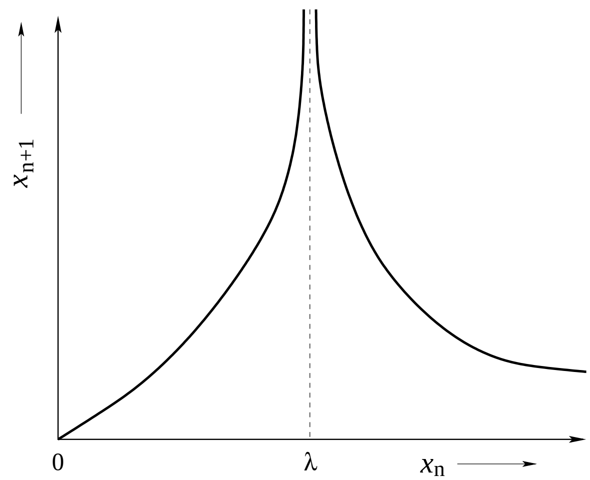

Fig. 1. The graph of the map given by (2) and (3).

In this map, the vertical line $x = \lambda$ forms an asymptote for the equations (2) and (3), and a singularity occurs at this value. This asymptotic behavior occurs due to the geometric considerations in the waves — that the waveheight/wavelength cannot exceed some ratio before the wave becomes self-intersecting. While obtaining the bifurcation diagram, if x takes a value close to λ, the value of x at the next iterate is very high. The program has to account for this possibility. The bifurcation diagram obtained this way is presented in Fig. 2. As both domain and range of the piecewise smooth map are represented by the half-open set $[0, \infty)$, Fig. 2 and all subsequent bifurcation diagrams have been truncated.

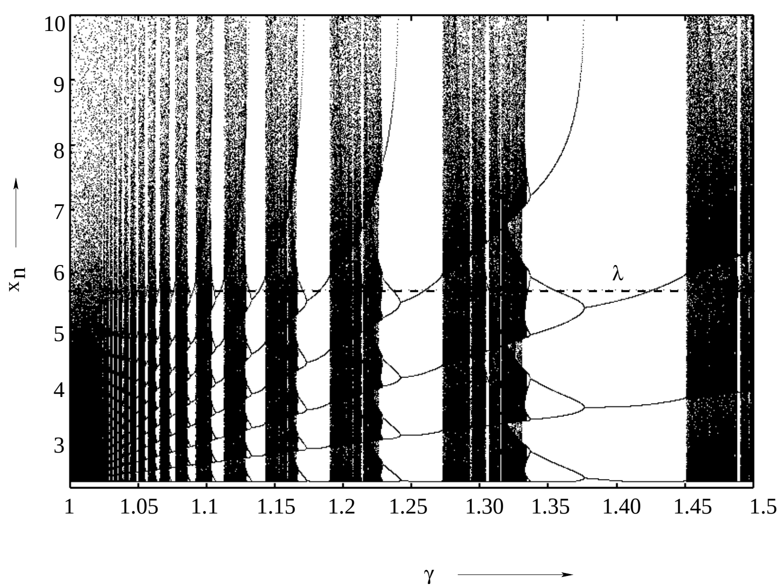

Fig. 2. Bifurcation diagram with $\alpha = 0.04$, $\beta = 2.8$, $\rho = 0.14$, $\lambda = 5.76$, and γ as the variable parameter. The diagram is truncated above $x_n = 10$.

A few features are noticeable in the bifurcation diagram. First, there is a succession of periodic windows with the following properties:

(1) Within each periodic window, as the bifurcation parameter γ is reduced, a period doubling cascade terminates in chaos.
(2) If n is the periodicity of the base orbit (lowest period orbit) at the highest parameter value in a perodic window and m is the periodicity of the base orbit at the highest parameter value in the next window, then $m = n + 1$, i.e., there is a period adding cascade as the parameter is reduced.
(3) The width of the periodic window reduces monotonically as the period adding cascade progresses.

In earlier studies, period adding cascades were observed in the study of piecewise smooth maps of finite magnitude and finite slope [2,4] and those with square root singularity of derivatives [5,6,1]. In maps of the former type it has been found that each periodic window originates at a border collsion. But for the system under the present study, the bifurcation diagram plotted for a larger range of x (Fig. 3) demonstrates that each periodic window does not emerge due to border collision. These occur at saddle-node bifurcations.

In systems with square root singularity it was observed that a period-n orbit always has $n - 1$ points on one side of the border and one point on the other side. Such orbits were called maximal. Though from Fig. 2 this may seem true also for the system presently under consideration, a closer scrutiny of Fig. 3 shows that the orbits are not maximal at the points of emergence of periodic windows. They become maximal as the periodic point crosses the borderline value of λ.

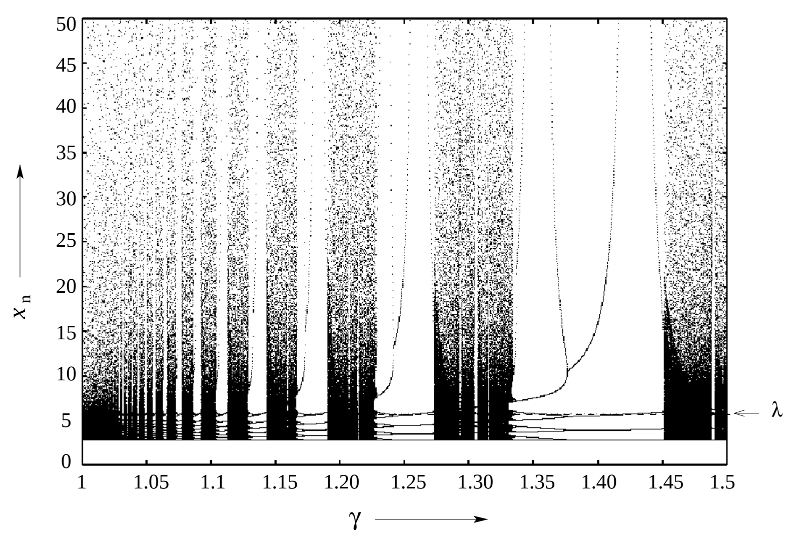

Fig. 3. Bifurcation diagram with horizontal lines drawn at $y = \beta$ and $y = \lambda$.

2 Border-collision bifurcations

In view of the differences, therefore, the theory developed for these other types of piecewise smooth maps cannot be applied for the map with magnitude singularity presently under consideration.

For this system, a few questions demand fresh answers. For which parameter values does the state reach infinite value? It is easy to see that in the chaotic windows, because of the ergodic nature of the orbit, iterates come arbitrarily close to the value of λ, and then the next iterate must assume infinite value. Therefore, even though a bifurcation diagram cannot be drawn for such high range of the variable, there must be points at infinity for all parameter values in all the chaotic windows. This is supported by the observation in [10] where such states were called "ejecting states."

This system reveals another interesting aspect: infinity is reached even within periodic windows — at the points of border collision.

The points of border-collision in the periodic windows can be determined theoretically. At the point of border collision, one point of the periodic orbit must have the value λ. Since $\lambda \mapsto \infty \mapsto \beta$, for a period three orbit the three points are β, λ, ∞. To find the value of the parameter γ at the crossover point we set

$$\gamma\beta + \frac{\alpha\beta}{(\beta-\lambda)^2} = \lambda \tag{4}$$

This yields a value $\gamma = 2.052577$ for which the period-3 orbit contains a point at infinity. Fig. 4 gives a closeup of the period-3 window showing this point.

At the point where the period-4 orbit touches infinity, three of the four points will be λ, ∞, and β. The fourth point obtained by substituting $x_n = \beta$ in (2) is

$$\gamma\beta + \frac{\alpha\beta}{(\beta-\lambda)^2}.$$

Since this point must map to λ, we obtain the parameter value γ from (2) as

$$\gamma\{\gamma\beta + \frac{\alpha\beta}{(\beta-\lambda)^2}\} + \frac{\alpha\{\gamma\beta + \frac{\alpha\beta}{(\beta-\lambda)^2}\}}{(\gamma\beta + \frac{\alpha\beta}{(\beta-\lambda)^2} - \lambda)^2} = \lambda \tag{5}$$

This yields the following 4-th degree equation in γ

$$a_0\gamma^4 + a_1\gamma^3 + a_2\gamma^2 + a_3\gamma + a_4 = 0 \tag{6}$$

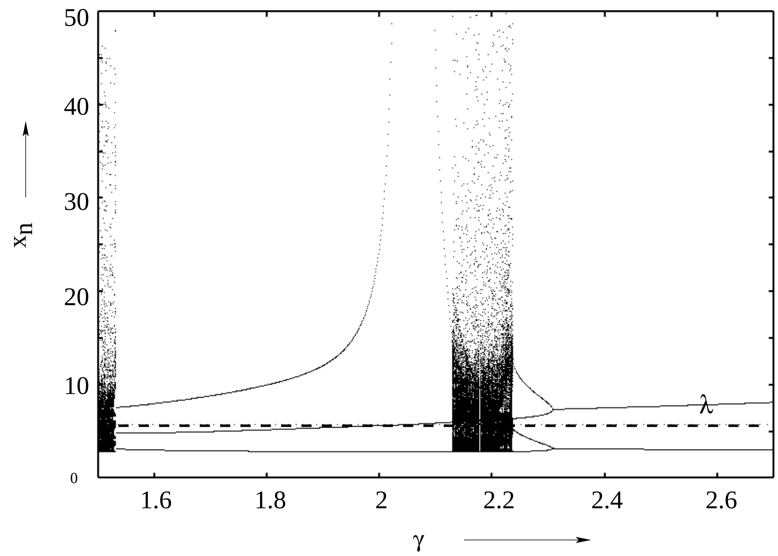

Fig. 4. Close up of the period-3 window.

where

$$a_0 = \beta^3$$

$$a_1 = \frac{3\alpha\beta^3}{(\beta-\lambda)^2} - 2\beta^2\lambda$$

$$a_2 = \frac{3\alpha^2\beta^3}{(\beta-\lambda)^4} - \frac{4\lambda\alpha\beta^2}{(\beta-\lambda)^2} + \beta\lambda^2 - \beta^2\lambda$$

$$a_3 = \frac{\alpha^3\beta^3}{(\beta-\lambda)^6} - \frac{2\alpha^2\beta^2\lambda}{(\beta-\lambda)^4} + \frac{\alpha\beta\lambda^2}{(\beta-\lambda)^2} + \alpha\beta - \frac{2\alpha\beta^2\lambda}{(\beta-\lambda)^2} + 2\beta\lambda^2$$

$$a_4 = \frac{\alpha^2\beta}{(\beta-\lambda)^2} - \frac{\alpha^2\beta^2\lambda}{(\beta-\lambda)^4} + \frac{2\lambda^2\alpha\beta}{(\beta-\lambda)^2} - \lambda^3.$$

Solving (6) using $\beta = 2.8, \lambda = 5.76, \alpha = 0.040$ yields $\gamma = 1.425510183648710$. Out of the other three solutions, one is negative and the other two complex conjugate — which are not possible values of the parameter.

In a similar manner, all the periodic orbits with a point at infinity and their corresponding parameter values can be obtained. The only requirement for such border-collision points to exist is $\beta < \lambda$.

3 Stability of periodic orbits

The rate of expansion/contraction at any periodic point is dependent on the derivative of the map function

$$\frac{dx_{n+1}}{dx_n} = \gamma + \frac{\alpha}{(x_n-\lambda)^2} - \frac{\alpha x_n}{(x_n-\lambda)^3} \qquad \text{for} \quad x_n < \lambda, \tag{7}$$

$$\frac{dx_{n+1}}{dx_n} = \frac{\rho}{(x_n-\lambda)^2} - \frac{2\rho x_n}{(x_n-\lambda)^3} \qquad \text{for} \quad x_n > \lambda. \tag{8}$$

Since border collision points are related to the singularity condition, and since from (7) and (8) we find that

$$\lim_{x_n\to\lambda^-} \frac{dx_{n+1}}{dx_n} \to \infty \quad \text{and} \quad \lim_{x_n\to\lambda^+} \frac{dx_{n+1}}{dx_n} \to -\infty,$$

one may tend to believe that periodic orbits containing a point at infinity cannot be stable. However, since

$$\lim_{x_n\to\infty} \frac{dx_{n+1}}{dx_n} \to 0,$$

this need not be true, and so it is interesting to work out the stability of the orbits containing infinity.

Such an orbit must have two points with values λ and ∞. The other points in the orbit have finite slope. Therefore the derivative of nth iterate map for that condition must be a finite number times the limiting values of

$$\left(\lim_{x_n\to\lambda^-} \gamma + \frac{\alpha}{(x_n-\lambda)^2} - \frac{\alpha x_n}{(x_n-\lambda)^3}\right) \times \left(\lim_{x_{n+1}\to\infty} \frac{\rho}{(x_{n+1}-\lambda)^2} - \frac{2\rho x_{n+1}}{(x_{n+1}-\lambda)^3}\right) \tag{9}$$

and

$$\left(\lim_{x_n\to\lambda^+} \frac{\rho}{(x_n-\lambda)^2} - \frac{2\rho x_n}{(x_n-\lambda)^3}\right) \times \left(\lim_{x_{n+1}\to\infty} \frac{\rho}{(x_{n+1}-\lambda)^2} - \frac{2\rho x_{n+1}}{(x_{n+1}-\lambda)^3}\right). \tag{10}$$

For obtaining the limiting values of (9) and (10), we transform these into a limit of a single expression. Using the symbolic computation facility of MATLAB, we substitute the expression for x_{n+1}, i.e., (2) or (3) depending on the position of x_n. For simplicity and compactness these expressions are denoted by F^- and F^+ as

$x_n \to \lambda^-$ or $x_n \to \lambda^+$. The limit expression becomes

$$\lim_{x_n \to \lambda^-} F^- \tag{11}$$

$$\lim_{x_n \to \lambda^+} F^+ \tag{12}$$

We find that in both cases limit exists and is zero. Therefore we conclude that the orbit including the point at infinity is superstable. It should be noted that the value of limit is independent of $\beta, \lambda, \alpha, \rho$ and hence the result is a general one.

The result was numerically confirmed by choosing a border collision point and then by calculating the product of the derivatives. It was observed that the product could be made arbitrarily small by choosing appropriate value of the variable x.

4 Bifurcation behavior for $\beta \geq \lambda$

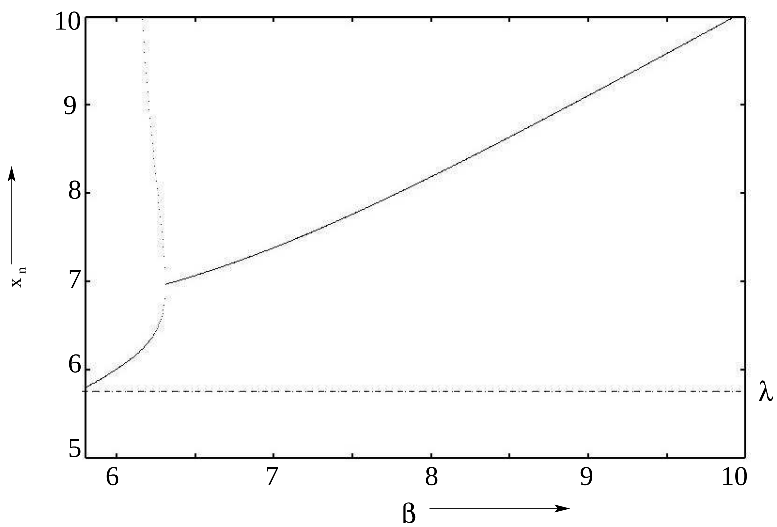

Fig. 5. Bifurcation diagram with respect to β for $\beta \geq \lambda$.

In the earlier sections the bifurcation behavior for the case $\beta < \lambda$ has been investigated. We now consider the situation where $\beta \geq \lambda$, and study the bifurcation phenomena as β is varied. The value of γ only changes the graph quantitatively and hence it is not considered to be the parameter. Moreover, ρ is assumed to be positive.

The behavior can be explained in the light of superstability of periodic orbits when border-collision occurs. When $\beta = \lambda$ the points λ, ∞ form a superstable 2-period orbit. As β is increased, the product of the derivatives approaches unity. At a point the period-2 orbit becomes unstable giving way to a stable period-1 orbit. This is a standard period-doubling bifurcation when β is reduced. The nature of the map

(Fig. 1) also suggests that period-1 orbit should be stable at very high values of β. This behavior is shown in the bifurcation diagram of Fig. 5.

However, if the initial iterate starts on the left-hand side of the map and remains forever on the left-hand side, then the behavior as shown in Fig. 5 is not exhibited.If $0 \leq \gamma + \frac{\alpha}{\lambda^2} \leq 1$ then the origin forms a stable fixed point and iterates starting on the left hand side of the map converge to the origin.

5 Scaling in the period-adding cascade

Figure 2 reveals that successive periodic windows have monotonically decreasing width, which suggests the existence of a Feigenbaum-type ratio. To check this, we obtained the parameter values at the points of saddle-node bifurcation where the periodic orbits come into existence. The ratio of the widths of successive periodic windows were taken. Fig. 6 shows the graph of the ratios of the widths versus the index.

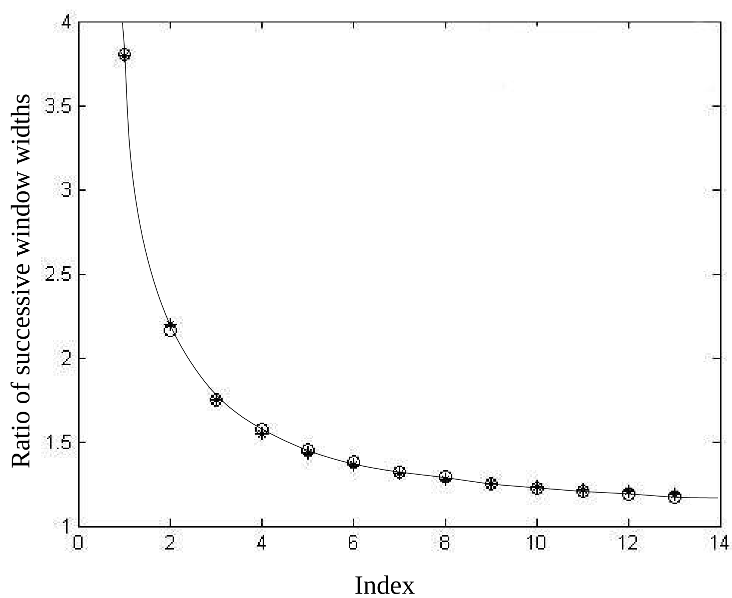

Fig. 6. The ratios of the widths between the successive appearance of periodic windows. $*$ indicates the numerically determined values of the parameter ratios, and $\circ$ indicates those obtained from (13).

Using standard curve-fitting technique, we fit the data into the curve

$$y = ax^b + c. \tag{13}$$

The obtained value of c was $1.091 \pm .033$. Hence the ratio of successive widths will tend to c.

6 Conclusions

In this work a piecewise smooth map was considered that has magnitude as well as slope singularity at a borderline value. It was found that the singularity condition, i.e., the state assuming infinte value, occurs both in chaotic mode as well as in periodic mode. We found that periodic orbits containing a point at infinity occur are the points of border collision, at which these orbits are superstable. The system exhibits period-adding cascade with deminishing width of successive periodic windows, and the ratio converges to a number $1.091 \pm .033$. There are several border-collisions within each periodic window.

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