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Border collision bifurcations in a soft impact system

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Abstract

The present work deals with the dynamics of a mechanical switching system in which the state variables are continuous at the switching events, but the first derivative of the vector field changes discontinuously across the switching boundary. Earlier works have shown that hard impacting systems yield discrete maps with a term of power $1/2$, and stick-slip systems yield discrete maps with a term of power $3/2$. Maps of the first kind exhibit square-root singularity while those of the second kind are smooth, and therefore no border collision bifurcation occur in them. In this Letter we consider an impacting system with a wall cushioned with spring-damper support. The spring is constrained such that the force on the mass changes discontinuously at a grazing contact. We focus our attention on the change in the Jacobian matrix of a fixed point caused by grazing. We show that a typical property of border collision in such systems is that the determinant remains invariant and the trace shows a singularity at the grazing point. We also explain the observed bifurcations based on the available theory of border collision bifurcations.

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Keywords: Bifurcation; Switching dynamical system; Border collision; Soft impact

1. Introduction

This Letter studies the dynamical behavior of a mass undergoing soft impacts. Soft impacts occur in many practical mechanical systems [1,2] where there is some "cushioning" at the impacting surfaces—meant for reducing the noise and chatter. It can be visualized as a mass impacting not with a hard wall, but with a spring-damper support in front of a wall. The existence of the spring-damper type cushion introduces some special features in the system dynamics—which is the focus of the investigation in the present Letter.

This dynamical system can be expressed, mathematically, as two sets of first order ordinary differential equations, and the orbit switches from one set to the other depending on some condition on the state variables. Such switching dynamical systems are of profound practical importance. All power electronic circuits are of this type [3]. Many mechanical systems like impacting systems [4,5] and those involving stick slip motion [6,7], and hydraulic systems with valves (including the human heart [8]) are also examples of switching dynamical systems.

The theory of bifurcations in smooth dynamical systems is well developed. The bifurcations in piecewise smooth systems demand a different theory. It has been shown that switching systems yield discrete-time maps with two or more compartments in the state space, and the system description in each compartment is given by different functional forms [9,10]. In such systems, as a parameter is varied, a fixed point may collide with the borderline between compartments. This may cause a discrete change in the eigenvalues, resulting in a sudden change in the dynamical behavior. Such bifurcations are called *border collision bifurcations* [11,12].

It follows that border collision bifurcations can occur only if the derivative (or the Jacobian) of the map changes discontin-

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uously across the borderline between compartments. A comprehensive study of the form of discrete maps for piecewise smooth mechanical systems has been done earlier. Nordmark [6] showed that an impact oscillator with hard impacts yields a map with square root singularity. Dankowicz and Nordmark [13] analysed the stick-slip oscillations, and showed that such a system yields a map with 3/2-type singularity. Di Bernardo et al. [14] have analyzed normal forms maps using the method of zero-time discontinuity mapping (ZDM) for a class of hybrid systems. Of particular interest is the classification on the basis of continuity of flows. They show that the zero-time discontinuity mapping has (i) a square root singularity at the grazing point if the flow on the right-hand side is not equal to the flow on left-hand side, and (ii) a 3/2-type singularity at the grazing point in case where the flows are equal on both sides but there is a discontinuity in the first or the second derivative of the flow.

To illustrate the character of the two functional forms, consider the following maps.

Power-of-1/2 map:

$$x_{n+1} = \begin{cases} ax_n + \mu & \text{if } x \leqslant 0, \\ bx_n^{1/2} + \mu & \text{if } x \geqslant 0. \end{cases} \tag{1}$$

Power-of-3/2 map:

$$x_{n+1} = \begin{cases} ax_n + \mu & \text{if } x_n \leqslant 0, \\ ax_n - bx_n^{3/2} + \mu & \text{if } x_n \geqslant 0. \end{cases} \tag{2}$$

The map (1) exhibits a singularity in slope—the so-called square-root singularity—as $x_n \to 0^+$. On the other hand, the map (2) is continuous at the critical point $x_n = 0$, and the derivative has the same value at the two sides of the critical point. This implies that the map is smooth, and hence border collision bifurcations are not expected to occur in such a system.

At the point of grazing, if the cushioning surface does not apply any force on the mass, there will be no local change in the vector field, and therefore one would not expect any abrupt change in the Jacobian of the fixed point. In order to study the nature of the abrupt change in the Jacobian, we consider a system with precompressed spring at the cushion—which causes a discontinuous change in the force applied on the mass at a grazing point. Therefore in such a system there is a discontinuous change in the first derivative of the vector field. To our knowledge, the nature of local bifurcations in such a system has not been investigated yet. In this Letter we show that border collisions do occur in such a system, and offer explanation of the bifurcation phenomena based on the available theory of border collision bifurcations [15]. We also investigate the specific character of the two-dimensional normal form of the Poincaré map, and show that in such a system, at all border collision events the determinant of the Jacobian matrix remains invariant while the magnitude of the trace approaches infinity at one side of the border.

2. System description

The system considered in this Letter consists of a mass M supported by a spring k_1 and a damper R_1 attached to a rigid wall (see Fig. 1). There is a sinusoidally varying force

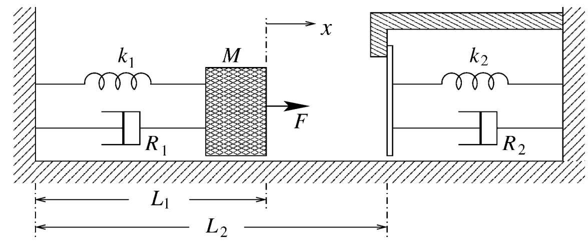

Fig. 1. Schematic diagram of the system.

$$F = F_m \cos \omega t$$

acting on the mass. This part is a simple spring-mass-damper system. When the spring is relaxed, the right end of the mass is at a distance L_1 from the wall, and x is the elongation from the unstretched position.

On the other side there is a wall with spring k_2 and damper R_2 (and no mass) to cushion the impact. For the sake of simplicity of the model, the equilibrium positions of both springs are taken as L_1. The spring of the cushion is constrained so that it is compressed by a displacement $L_2 - L_1$ from its relaxed position. Thus the impacting surface is at a distance L_2.

If $x + L_1 < L_2$ then it is a simple harmonic oscillator given by the equation

$$M\ddot{x} + R_1\dot{x} + k_1 x = F_m \cos \omega t.$$

We call it system-1.

If $x + L_1 \geqslant L_2$ then an impact occurs. Since there is no mass at the impacting surface, following the impact simply the spring and damper constants change, to give the equation

$$M\ddot{x} + (R_1 + R_2)\dot{x} + (k_1 + k_2)x = F_m \cos \omega t.$$

We call it system-2.

Here we are interested in the bifurcations occurring when the trajectory grazes the boundary between system-1 and system-2. The discrete observations are done in synchronism with the external periodic input, to obtain a "stroboscopic sampling". It is known that such switching dynamical systems yield maps that are piecewise smooth. We are interested in the character of the discrete map at the two sides of the borderline.

3. Observed bifurcation phenomena

In this section we study the bifurcation phenomena in this system, with the amplitude F_m of the forcing function as the bifurcation parameter and the other parameters fixed at $M = 1$ kg, $k_1 = k_2 = 1$ N/m, $R_1 = R_2 = 0.1$ Ns/m, $L_2 - L_1 = 0.5$ m, and $\omega = 0.8$ rad/s. A bifurcation diagram, with F_m varied from 0.175 N to 0.3 N is shown in Fig. 2(a). This diagram has been obtained by using a number of initial conditions for each parameter value, so that the coexisting attractors are visible. A close-up of the diagram in the parameter range $F_m = 0.176$ to $F_m = 0.186$ is shown in Fig. 2(b).

In the following discussion, specific periodic orbits are identified by a symbol like PiTj, where i implies the periodicity of

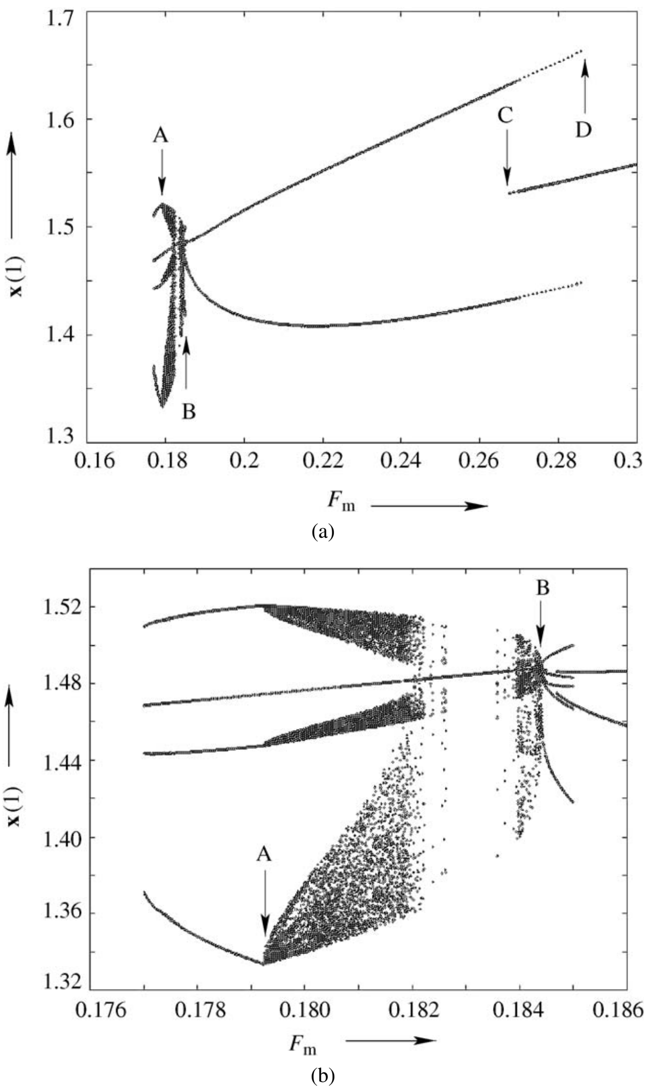

Fig. 2. (a) The bifurcation diagram of the system. (b) Close-up view of the parameter range from 0.176 to 0.186 to show the details.

the orbit and j implies the number of transitions from system-1 to system-2, i.e., the number of impacts in that orbit.

At low values of F_m, the period-1 orbit (P1T0 type, without any impact) exists and is stable. At $F_m \approx 0.1768$, another stable period-3 orbit comes into existence through a saddle-node bifurcation. At $F_m \approx 0.179245$ this period-3 orbit undergoes a bifurcation as the continuous-time trajectory grazes the cushioning surface, and the orbit abruptly turns chaotic. We call it bifurcation point A.

As the parameter is increased further, at $F_m \approx 0.1843909$ the P1T0 type period-1 orbit grazes the boundary and loses stability. We call it bifurcation point B. Following this bifurcation, the orbit moves to a coexisting chaotic orbit. Subsequently a period-2 orbit becomes stable and persists for a large range of the parameter.

As the parameter is increased further, the P1T1 orbit again becomes visible at $F_m \approx 0.261$ and the stable P2T1 orbit suddenly disappears at $F_m \approx 0.287$. To probe these events, we plot the bifurcation diagram by following the fixed points (not by

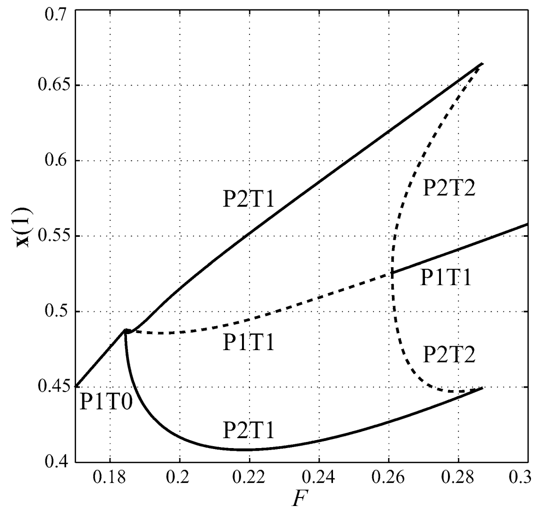

Fig. 3. The bifurcation diagram obtained by following the fixed points.

obtaining the asymptotically stable orbits as is the common practice). For this, the fixed points, irrespective of their stability, are obtained by solving the algebraic conditions numerically [16]. The resulting diagram is presented in Fig. 3.

It shows that the unstable P1T1 orbit undergoes a subcritical period doubling bifurcation at $F \approx 0.261$ and becomes stable. We call this the bifurcation point C. The P2T2 type orbit originating at this bifurcation exists for $F_m > 0.261$, i.e., at the same side as the stable P1T1 orbit. This orbit collides with the stable P2T1 orbit at $F_m \approx 0.28692$ and both the orbits disappear (bifurcation point D).

To investigate the nature of these bifurcations, we show in Fig. 4 the continuous-time orbits which undergo the bifurcations A, B, C, and D. It shows that at bifurcations A, B, and D, the orbit grazes the line representing the switching condition between system-1 and system-2, and therefore these are related to the grazing condition. The orbit in Fig. 4(c) does not show grazing, and hence point C must be related to a smooth bifurcation. In case of bifurcation D, both the stable P2T1 orbit and the unstable P2T2 orbit undergo grazing, and at the bifurcation point they become identical. This orbit is shown in Fig. 4(d).

The question now is: Do these grazing conditions lead to border collision bifurcations? The answer hinges on whether the eigenvalues of a fixed point change abruptly as an orbit grazes the state space boundary.

4. Calculation of the eigenvalues

To investigate this point, we need to obtain the eigenvalues of the fixed point that collides with the border (corresponding to the continuous-time orbit that undergoes grazing). We have calculated the eigenvalues following a procedure particularly suitable for switching dynamical systems, presented in a separate paper [16].

For bifurcation point A, the eigenvalues of the stable period-3 orbit just before the bifurcation are 0.142531 and 0.635898, and those of the unstable period-3 orbit just after the bifurcation

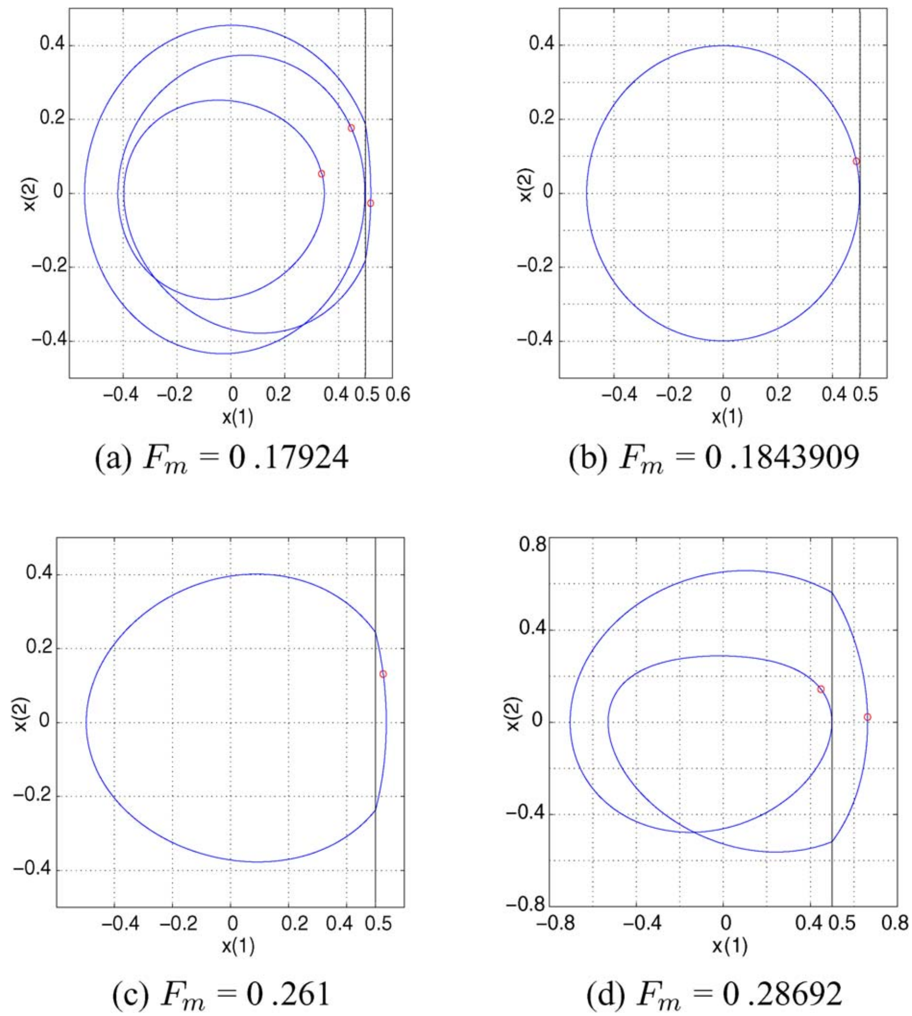

Fig. 4. The continuous-time orbits close to bifurcation points A, B, C, and D. The points of discrete-time observation for obtaining the stroboscopic map are also shown.

are -0.0001316 and -688.514. For the bifurcation point B, the eigenvalues of the P1T0 fixed point at $F_m = 0.1843909$ (before grazing) are $6.633115 \times 10^{-3} \pm i0.6751993$, and that of the P1T1 fixed point at $F_m = 0.1844$ (after grazing) are $-6.102994462061591 \times 10^3, -7.470650757568364 \times 10^{-5}$. At the bifurcation point C, for the parameter value $F_m = 0.2609$ the unstable P1T1 orbit's eigenvalues are -1.00047867 and -4.29169277, while for $F_m = 0.261$, the stable P1T1 orbit's eigenvalues are -0.99941506 and -4.29601072. Just before the bifurcation point D, i.e., just before the disappearance of the orbits at $F_m = 0.28692$, the stable P2T1 orbit has the eigenvalues $-0.16072169 \pm i0.39923669$ and the unstable P2T2 orbit has the eigenvalues 11235.37065 and $1.64855405 \times 10^{-5}$.

It is therefore clear that at bifurcation points A and B, there is an abrupt change in the eigenvalues of the fixed point at the grazing incident. For the bifurcation point C, one eigenvalue assumes the value of -1, which implies that the event is a smooth period-doubling bifurcation. At bifurcation point D, a stable and an unstable fixed point collide with the borderline simultaneously, and their eigenvalues are widely different. Bifurcations A, B, and D are therefore clear cases of border collision bifurcation.

The eigenvalues of the fixed point seem to be jumping discontinuously over a large distance on the complex plane when an orbit grazes the switching boundary. The question is: Is there an underlying pattern in the seemingly arbitrary jump of the eigenvalues?

5. Character of the normal form

As shown in [15,17], the local character of border collision bifurcations can be analysed by obtaining the piecewise linear "normal form" map in the neighborhood of the borderline. The normal form is given by

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} \underbrace{\begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}}_{\mathbf{J}_L} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \leqslant 0 \\ \underbrace{\begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}}_{\mathbf{J}_R} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \geqslant 0 \end{cases} \tag{3}$$

where τ_L is the trace and δ_L is the determinant of the Jacobian matrix $\mathbf{J}_L$ of the system at a fixed point in $R_A := \{(x, y) \in$

Table 1
The parameters of the normal form corresponding to the bifurcation points A, B, and D

Bifurcation	τ_L	δ_L	τ_R	δ_R
A	0.7784	0.0906	−688.5141	0.0906
B	0.0126623	0.45593	-6.1029944×10^3	0.45593
D	−0.32144	0.18522	1.123537×10^4	0.18522

$\Re^2$: $x \leqslant 0\}$ and close to the border and τ_R is the trace and δ_R is the determinant of the Jacobian matrix $\mathbf{J}_R$ of the system evaluated at a fixed point in $R_B := \{(x, y) \in \Re^2\text{: } x \geqslant 0\}$ near the border.

The parameters of the normal form, namely, the trace and determinant of the Jacobian matrix at the two sides of the border, depends on the system description and the parameters chosen. But it is not possible to establish a closed form functional relationship between the parameters of the normal form and the actual system parameters. For each border collision bifurcation, the parameters of the normal form have to be obtained by computing the eigenvalues of the fixed point before and after the border collision event. These parameters, computed from the eigenvalues presented in the last section are tabulated in Table 1.

For the bifurcation point A, the parameters satisfy the condition

$$2\sqrt{\delta_L} < \tau_L < 1 + \delta_L \quad \text{and} \quad \tau_R < -(1 + \delta_R),$$

and the theory of border collision bifurcation [15,17] predicts that there should be a direct transition from a stable fixed point to a chaotic orbit. In the actual system the fixed point under consideration is period-3. Therefore we observe a transition from a period-3 orbit to a 3-piece chaotic orbit.

For the bifurcation point B the orbit becomes locally unstable following the border collision bifurcation. The available theory of border collision bifurcations predicts that following this bifurcation a period-1 and a period-2 orbit should exist, but both should be unstable. The bifurcation diagram obtained by following the period-1 and period-2 orbits close to this bifurcation point (Fig. 5) show the evolution of these orbits. It is found that the P2T1-type period-2 orbit, which was unstable immediately following the border collision bifurcation, attains stability at $F_m = 0.184403$ (at $F_m = 0.184402$, the eigenvalues are -1.059081004 and -0.19611911 while at $F_m = 0.184403$, the eigenvalues are -0.98396256 and -0.21108037). But the system does not collapse in the parameter range where both the orbits are unstable, and moves to a coexisting chaotic orbit.

As for case D, the theory of border collision bifurcation [15,17] predicts that the merging and disappearance of a pair of fixed points would occur if

$$\tau_R > 1 + \delta_R \quad \text{and} \quad -(1 + \delta_L) < \tau_L < (1 + \delta_L).$$

In the present case this condition is satisfied, and so this is a "border collision fold bifurcation".

However, the most remarkable results emerging from Table 1 are that

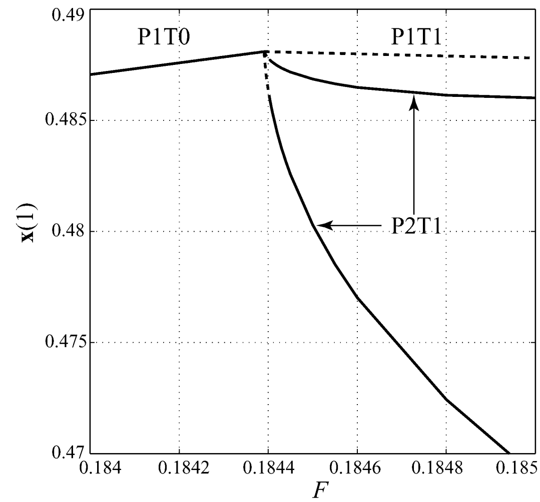

Fig. 5. The bifurcation diagram obtained by following the period-1 and period-2 fixed points.

- at all border collision events for the soft impact system, the determinant at the two sides of the borderline were the same, and
- the trace at one side of the borderline assumes a very high value.

The eigenvalues were obtained at the fixed point which is dependent on the parameter value chosen. Our investigations indicated that the trace changes very fast as one chooses a parameter value closer and closer to the bifurcation point, but the determinant does not change appreciably. This raises further questions:

(1) Does the magnitude of τ_R tend to infinity as one approaches the bifurcation point?
(2) Are the determinants exactly equal at the two sides of the normal form map?
(3) How does the variation of the trace and determinant depend on the character of the cushioning surface?

To settle these issues, we plot the variation of the determinant of the Jacobian matrix with the amplitude of the forcing function F_m for various values of k_2 and R_2 (Fig. 6) for the bifurcation event B, where the P1T0 orbit collides with the switching surface. It shows that the determinants at the two sides of the border collision event are indeed equal, and with increasing values of the parameter, the determinant falls linearly. It also shows that the slope of the curve obtained for the P1T1 orbit depends on the stiffness of the spring k_2 and the coefficient of the damper R_2. For very hard and frictionless impact ($k_2 \to \infty$, $R_2 \to 0$), the determinant is expected to be essentially constant over a large parameter range.

In Fig. 7 we plot the variation of the trace as a function of the parameter F_m. Since the value changes over several orders of magnitude and since it has negative value, we plot the logarithm of the negative of the trace. The graph clearly shows that the trace changes discontinuously at the border, and that the value of the trace approaches $-\infty$ as the bifurcation para-

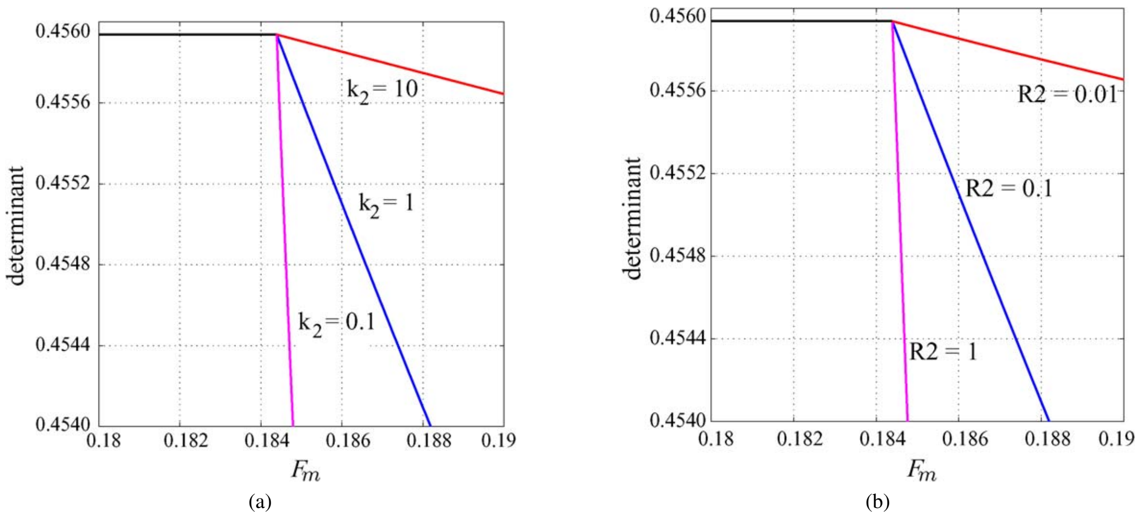

Fig. 6. The variation of the determinant of the Jacobian matrix as a function of the parameter F_m (a) for various values of k_2 with R_2 fixed at 0.1 N s/m, and (b) for various values of R_2, with K_2 fixed at 1.0 N/m.

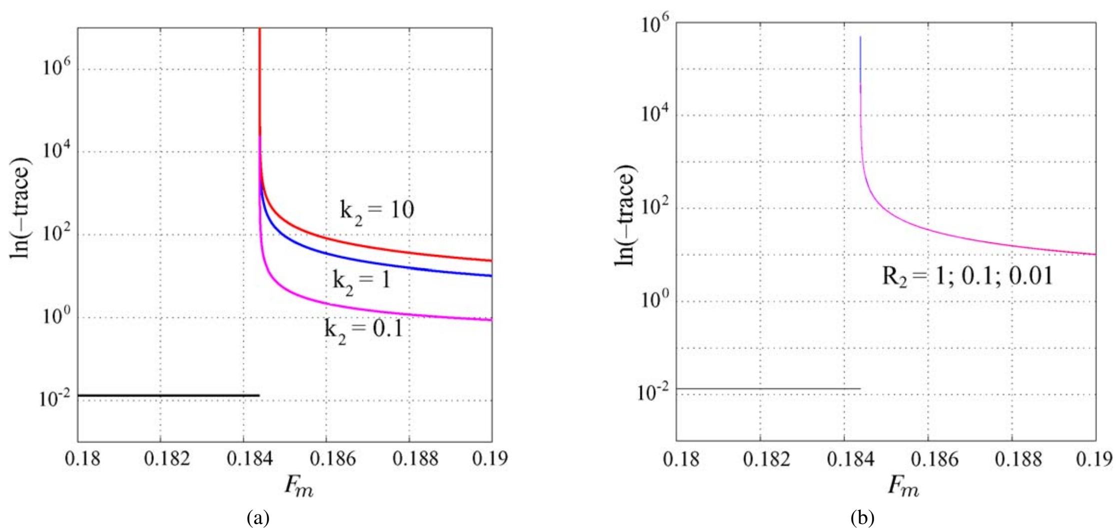

Fig. 7. The variation of the trace of the Jacobian matrix as a function of the parameter F_m, (a) for various values of k_2 with R_2 fixed at 0.1 N s/m, and (b) for various values of R_2, with K_2 fixed at 1.0 N/m.

meter is approached from the right side. The rate of approach depends on the softness of the impacting surface (given by the stiffness of the spring k_2) but is independent of the coefficient of the damper R_2.

6. Conclusions

In this Letter we have considered an impact oscillator with a constrained spring-damper cushion in front of the impacting wall. In this system there is no discontinuity in the state variables, and only the first derivative of the flow changes discontinuously across the switching surface. Our numerical investigations show that the Jacobian matrix of a fixed point undergoes abrupt change as it hits the border—or in continuous-time, when an orbit grazes the switching manifold. There is an underlying pattern in the seemingly arbitrary jump of the eigenvalues, which becomes clear only when one looks at the normal form.

For an orbit with impact, the magnitude of the trace approaches infinity as the parameter approaches the critical value corresponding to the grazing condition, implying a square-root like singularity in the trace of the Jacobian matrix. The rate of approach is dependent on the spring constant of the cushion, but is independent of the damping factor. Another remarkable observation is that at every occurrence of border collision, the even though the Jacobian elements are widely different, the determinants at the two sides of the border are the same. The determinant varies linearly as the fixed point moves away from the border with the change of a parameter, with different rates of change for various values of k_2 and R_2.

Though the theory of border collision bifurcations for 2D normal form map should not logically be applicable to systems with derivative singularity, we find that the predictions of the theory, when extended to very large values of the trace, do satisfy the bifurcations observed in this system. However, further investigation is necessary to ascertain the limits of applicability of the available theory to systems with square-root singularity.

We believe that the observations reported in this Letter will pave way for a proper understanding of the local bifurcations that occur in practical systems with components undergoing soft impact. These observations also call for further development of the theory of border collision bifurcations for two-dimensional maps with singularity in the trace.

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