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Dangerous bifurcation at border collision: When does it occur?

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It has been shown recently that border collision bifurcation in a piecewise smooth map can lead to a situation where a fixed point remains stable at both sides of the bifurcation point, and yet the orbit becomes unbounded at the point of bifurcation because the basin of attraction of the stable fixed point shrinks to zero size. Such bifurcations have been named "dangerous bifurcations." In this paper we provide explanation of this phenomenon, and develop the analytical conditions on the parameters under which such dangerous bifurcations will occur.

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The bifurcations occurring in piecewise smooth maps have attracted significant research attention in recent years because of their applicability in a wide class of systems of practical interest [1] including switching circuits [2], impact oscillators [3] walking robots [4], and cardiac dynamics [5]. In such systems, as a parameter is varied, a fixed point can collide with the borderline between two smooth regions, resulting in an abrupt change in the Jacobian matrix. This leads to a new class of bifurcations, known as border collision bifurcations [6].

It has been shown that such bifurcations can lead to atypical transitions like a period-1 orbit directly bifurcating into a chaotic orbit, or a periodic orbit suddenly vanishing as it hits the border. Analysis of such bifurcations have been developed [6–8] depending on the eigenvalues of the Jacobian matrix at the two sides of the border. It has been found that under some conditions a stable fixed point occurs at both sides of a border collision event, and it was believed that this situation would result in no observable change in system behavior [7,8]. However, it has been recently shown [9] that border collision bifurcations can also lead to a peculiar situation where the system collapses at the point of border collision, even though the fixed point remains stable throughout the range of parameter variation. The basin of attraction of the stable fixed point shrinks as the parameter is varied toward the bifurcation value, and at the bifurcation point the basin of attraction has zero size. As a result, orbits starting from all points other than the fixed point become unbounded (at least from a local point of view in a neighborhood of the bifurcating point). This revelation is a matter of serious concern for practical systems that are modeled by piecewise smooth maps, because the eigenvalues of the fixed point do not give any signal of the impending collapse.

This possibility has been pointed out through numerical exploration but the mechanism causing its occurrence is not yet known. Moreover, in order to apply this knowledge in practical situations, it is necessary to know the conditions under which such bifurcations are expected to occur. The purpose of this paper is to explain the mechanism of such "dangerous bifurcations" and to obtain explicit conditions on the system parameters which lead to such behavior.

It has been shown [6] that dynamical phenomena related to border collision bifurcations (BCB) can be probed using the piecewise linear approximation in the neighborhood of the border crossing fixed point, expressed in the convenient normal form:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} \underbrace{\begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}}_{\mathrm{J}_L} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu, & x_k \leqslant 0, \\ \underbrace{\begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}}_{\mathrm{J}_R} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu, & x_k \geqslant 0, \end{cases} \tag{1}$$

where τ_L is the trace and δ_L is the determinant of the Jacobian matrix $\mathbf{J}_L$ of the system at a fixed point in $R_A := \{(x,y) \in \mathbb{R}^2 : x \leqslant 0\}$ and close to the border and τ_R is the trace and δ_R is the determinant of the Jacobian matrix $\mathbf{J}_R$ of the system evaluated at a fixed point in $R_B := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0\}$ near the border.

We assume that the determinants of the Jacobian matrices at the two sides of the border satisfy the conditions $1 > \delta_L > 0$ and $1 > \delta_R > 0$. For a specific combination of δ_L and δ_R, the type of BCB depends on the two parameters τ_L and τ_R. We restrict our attention to the parameter space region bounded by

$$-(1+\delta_L) < \tau_L < (1+\delta_L), \tag{2a}$$

$$-(1+\delta_R) < \tau_R < (1+\delta_R), \tag{2b}$$

because, the map has a fixed point attractor for $\mu < 0$ if (2a) is true, and for $\mu > 0$ if (2b) is true.

In order for an "attractor at infinity" to occur in addition to the stable fixed point, it is necessary that there must be an invariant subspace forming the boundary between the basins of attraction of the two attractors. It is known that the stable manifold of a saddle fixed point can form such a basin boundary. Therefore we probe which fixed point can serve this purpose.

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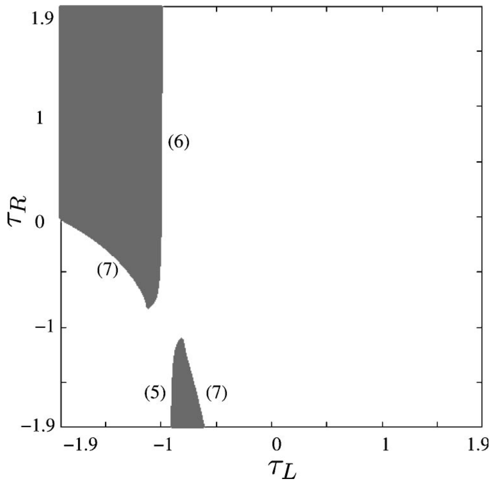

FIG. 1. Region of existence of the LRR orbit, given by the intersection of (3) and (4) for $\mu=-0.3$ and $\delta_L=\delta_R=0.9$. The contours are marked by the corresponding equations.

Since an unstable period-1 orbit does not occur in the region of parameter space given by (2), a period-1 orbit cannot form the basin boundary. A period-2 orbit exists for $\mu>0$ if $\tau_R<-(1+\delta_R)$ and for $\mu<0$ if $\tau_L<-(1+\delta_L)$, and therefore it cannot coexist with a stable period-1 orbit.

Next, we analyze the existence and the stability of high periodic orbits (HPO) starting from period-3. In the following discussion we will name particular types of orbits depending on the partitions (L or R) in which the points fall. For example, an LRR orbit implies a period-3 orbit with one point on the left-hand side and two points on the right-hand side. Suppose this orbit has the points (x_{L1},y_{L1}), (x_{R1},y_{R1}), and (x_{R2},y_{R2}). The conditions of existence of the LRR orbit are given by $x_{L1}<0$, $x_{R1}>0$, and $x_{R2}>0$. Out of these, the first one is always satisfied, since there cannot be a high-periodic orbit with all the points in one linear side. From the other two conditions we get the inequalities

$$\frac{(1+\tau_L-\delta_R+\tau_R\tau_L+\delta_L\delta_R+\delta_L\tau_R)\mu}{1+\delta_R^2\delta_L+\tau_L\delta_R+\delta_L\tau_R+\delta_R\tau_R-\tau_L\tau_R^2}>0, \quad (3)$$

$$\frac{(1+\tau_R-\delta_L+\tau_R\tau_L+\delta_L\delta_R+\tau_L\delta_R)\mu}{1+\delta_R^2\delta_L+\tau_L\delta_R+\delta_L\tau_R+\delta_R\tau_R-\tau_L\tau_R^2}>0. \quad (4)$$

Notice that the denominator of both these inequalities is the same. The region of existence of this orbit in the parameter space will be determined by the intersection of the regions given by the above inequalities. This region is plotted in Fig. 1. Since the LHS of the inequalities are expressed as ratios of two functions, the ratio will be positive if both the numerator and the denominator are positive, or if both are negative. This leads to the conclusion that the contours of the region of existence will be formed by

$$1+\tau_L-\delta_R+\tau_R\tau_L+\delta_L\delta_R+\delta_L\tau_R=0, \quad (5)$$

$$1+\tau_R-\delta_L+\tau_R\tau_L+\delta_L\delta_R+\tau_L\delta_R=0, \quad (6)$$

$$1+\delta_R^2\delta_L+\tau_L\delta_R+\delta_L\tau_R+\delta_R\tau_R-\tau_L\tau_R^2=0. \quad (7)$$

These lines are marked in Fig. 1.

Now, in order for this orbit to be stable, the eigenvalues of the composite matrix $\mathbf{J}_L\mathbf{J}_R^2$ must be inside the unit circle. Let τ be the trace and δ be the determinant of the above matrix. If the eigenvalues are complex conjugate, $|\lambda|^2=\delta<1$ since the system is assumed to be dissipative at both sides. Therefore the eigenvalues can go out of the unit circle only when they are real, giving the stability condition

$$-(1+\delta)<\tau<(1+\delta). \quad (8)$$

From this, the stability conditions for the LRR orbit are found to be

$$1+\delta_R^2\delta_L+\tau_L\delta_R+\delta_L\tau_R+\delta_R\tau_R-\tau_L\tau_R^2>0, \quad (9)$$

$$1+\delta_R^2\delta_L-\tau_L\delta_R-\delta_L\tau_R-\delta_R\tau_R+\tau_L\tau_R^2>0. \quad (10)$$

Notice that the left-hand side (LHS) in (9) is the same as the LHS in (7)—which corresponds to the condition where the eigenvalue becomes equal to +1.

If (9) is satisfied, then the denominator of the existence conditions becomes positive. Therefore in the parameter region shown in Fig. 1, the stable LRR orbit must coexist with the stable fixed point in L. These two orbits must have their own basins of attraction, separated by the stable manifold of a saddle-type periodic point. Which fixed point serves this purpose?

Notice that for the LLR type period-3 orbit, the stability conditions are

$$1+\delta_L^2\delta_R+\tau_R\delta_L+\delta_R\tau_L+\delta_L\tau_L-\tau_R\tau_L^2>0, \quad (11)$$

$$1+\delta_L^2\delta_R-\tau_R\delta_L-\delta_R\tau_L-\delta_L\tau_L+\tau_R\tau_L^2>0. \quad (12)$$

From these conditions we find that in the parameter region under consideration, the LLR orbit is unstable—a regular saddle. Indeed, this is the orbit whose stable manifold creates the basin boundary.

The condition of existence of the LLR orbit is given by $x_{L1}<0$, $x_{L2}<0$, and $x_{R1}>0$. Ignoring the last condition which is always satisfied, we get the inequalities

$$\frac{(1+\tau_R-\delta_L+\tau_L\tau_R+\delta_R\delta_L+\delta_R\tau_L)\mu}{1+\delta_L^2\delta_R+\tau_R\delta_L+\delta_R\tau_L+\delta_L\tau_L-\tau_R\tau_L^2}<0, \quad (13)$$

$$\frac{(1+\tau_L-\delta_R+\tau_L\tau_R+\delta_R\delta_L+\tau_R\delta_L)\mu}{1+\delta_L^2\delta_R+\tau_R\delta_L+\delta_R\tau_L+\delta_L\tau_L-\tau_R\tau_L^2}<0, \quad (14)$$

as the condition for its existence.

We now plot the intersection of these existence conditions in Fig. 2. In this case also the boundaries of the region are obtained from the numerators and denominators of (13) and (14), as

$$1+\tau_R-\delta_L+\tau_L\tau_R+\delta_R\delta_L+\delta_R\tau_L=0, \quad (15)$$

$$1+\tau_L-\delta_R+\tau_L\tau_R+\delta_R\delta_L+\tau_R\delta_L=0, \quad (16)$$

$$1+\delta_L^2\delta_R+\tau_R\delta_L+\delta_R\tau_L+\delta_L\tau_L-\tau_R\tau_L^2=0. \quad (17)$$

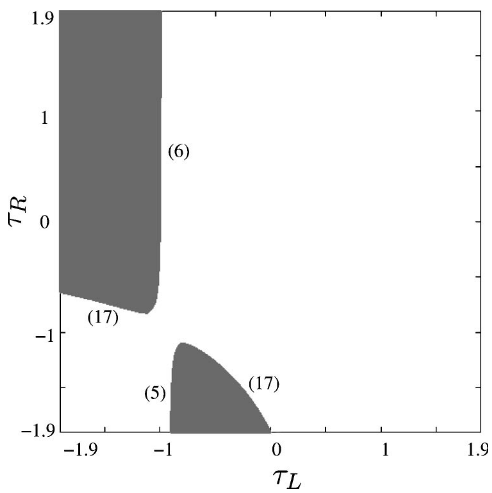

FIG. 2. The region of existence of the *LLR* orbit for $\mu=-0.3$ and $\delta_L=\delta_R=0.9$, obtained as the intersection of the parameter regions given by (13) and (14).

Now notice that (6) and (15) are identical and (5) and (16) are also identical. Therefore, the curves marked (5) in Fig. 1 and in Fig. 2, respectively, are identical, and the curves marked (6) in Fig. 1 and in Fig. 2 are also identical. But the curves marked (7) in Fig. 1 representing the first stability condition of the *LRR* orbit and that marked (17) in Fig. 2 representing the first stability condition of the *LLR* orbit are *not identical*.

This observation leads to the conclusion that whenever the stable *LRR* orbit exists, the unstable *LLR* orbit (we call it the *complementary* orbit) also exists. But the reverse is not true. In the parameter space there exists a region where the unstable *LLR* orbit exists but the stable *LRR* orbit does not. This region is obtained by subtracting the regions of existence of the *LRR* orbit from that of the *LLR* orbit, and is shown in Fig. 3. When τ_L is reduced from zero with τ_R fixed at a value lying between -1 and -1.9, if τ_L crosses the line marked (17), a single unstable period-3 orbit comes into existence. As it is reduced below the parameter value limited by curve (7), the stable *LRR* orbit begins to exist.

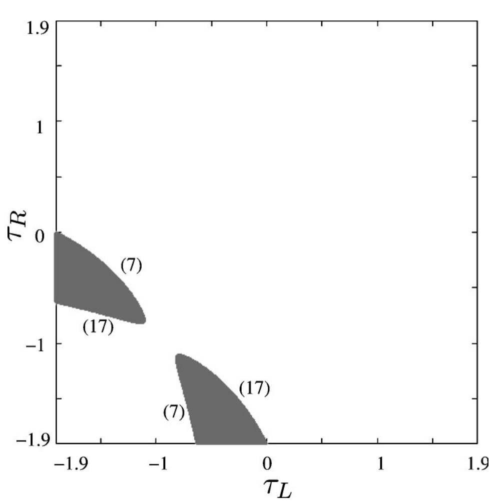

FIG. 3. The region where only the unstable *LRR* orbit exists along with the stable period-1 fixed point.

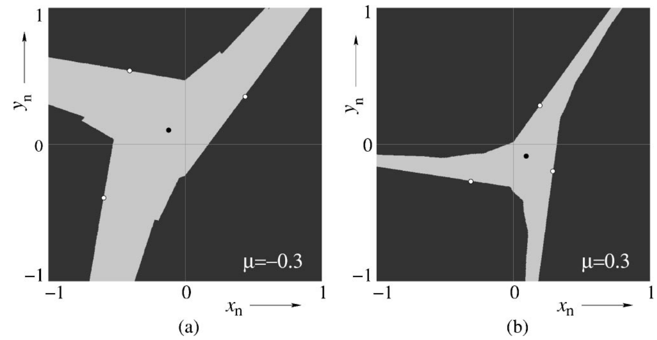

FIG. 4. The structure of the phase space for $\delta_L=\delta_R=0.9$, $\tau_L=-0.5$, $\tau_R=-1.5$. (a) For a negative value of the parameter μ and (b) for a positive value of μ. The stable fixed point is shown with a black dot and the unstable period-3 fixed point is shown with open circles. The basin of attraction of the stable fixed point reduces to zero size at $\mu=0$.

For a system with parameters placed in the shaded region in Fig. 3, a stable period-1 fixed point exists in L, and a regular saddle type period-3 fixed point also exists. Naturally, the stable manifold of the saddle fixed point will divide the phase space into two regions. One is the basin of attraction of the stable fixed point. In the absence of any other stable orbit, the other attractor will be at infinity.

Since every distance in this map scales with the parameter μ, as the parameter is varied toward zero, the maximum distance between the stable fixed point and the points of the unstable period-3 orbit reduces. At the point of border collision bifurcation, i.e., at $\mu=0$ the distance becomes zero. Therefore the area of the basin of attraction of the stable fixed point reduces to zero.

Because of the symmetry in the map, for positive values of μ, the same phenomenon will be observed. Only, the *LLR* orbit will now be stable and the *LRR* orbit will be unstable. In the equations, the terms with suffix L will have to be substituted by terms with suffix R and vice versa. For the same values of the determinants at the two sides, these regions obtained for $\mu>0$ will be identical with that obtained for $\mu<0$ (though the regions will be different if the determinants at the two sides are unequal). For a specific parameter combination placed in this region, the basins of attraction and the positions of the stable and unstable fixed points are shown in Fig. 4.

Now we take up the case of orbits of periodicity four and above. Obviously, there can be many different types of orbits of each periodicity. We define "regular orbits" as those orbits having the symbols L and R occurring consecutively at least in one cyclic permutation. We find that for $\mu>0$, two types of regular periodic orbits occur in the parameter region (2): (a) those with one point on the right-hand side and the rest on the left-hand side (the $L^{n-1}R$ orbits), and (b) those with two points on the left-hand side and the rest on the right-hand side (the L^2R^{n-2} orbits). For $\mu<0$, the regular orbits are of LR^{n-1} and $L^{n-2}R^2$ types. Based on numerical investigation, it is our conjecture that only these regular orbits are responsible for the dangerous BCBs.

For each stable high periodic orbit, there will be a *complementary orbit* with the following properties:

(1) It must have the same periodicity as the observed periodic orbit.

(2) Its symbol sequence differs from that of the stable periodic orbit by only one letter, with the exception of sequences consisting of only one symbol.

(3) Whenever a high periodic orbit coexists with a stable period-1 orbit, the stable manifold of the complementary orbit forms the basin boundary.

In the present case, the complementary of the $L^{n-1}R$ orbits are the $L^{n-2}R^2$ orbits, and the complementary of the LR^{n-1} orbits are the L^2R^{n-2} orbits. Note that the period-3 LLR orbit belongs to both $L^{n-1}R$ class and the L^2R^{n-2} class.

There will be regions of the parameter space where the above regular orbits of periods 4, 5, 6, etc., will occur. There will also be adjacent regions where these regular orbits do not occur, but their complementary orbits occur. These represent conditions for the occurrence of dangerous border collision bifurcations. To compute these, one must obtain the composite matrices of the regular orbits and their complementary orbits, compute their trace τ and determinant δ, and then substituting into the equation $\tau=1+\delta$ one gets the contours of the regions. These regions can be easily obtained using any symbolic computation program. For example, for $\mu<0$ there will be a region where the regular period-4 LR^3 orbit will not occur, but its complementary L^2R^2 orbit will occur. This region is delimited by two lines obtained by substituting the traces and determinants of the matrices $\mathbf{J}_L\mathbf{J}_R\mathbf{J}_R\mathbf{J}_R$ and $\mathbf{J}_L\mathbf{J}_L\mathbf{J}_R\mathbf{J}_R$ into the equation $\tau=1+\delta$.

Following this procedure, we obtain the parameter regions where the above phenomenon occurs with orbits of period 3, 4, 5, etc., i.e., where only the unstable complementary orbits occur, but the stable high-periodic orbits do not occur. These regions are shown in Fig. 5. In the dark areas of the region $\tau_L>\tau_R$, the complementary orbits belonging to the class $L^{n-1}R$ exist for $\mu<0$, and those belonging to the class $L^{n-2}R^2$ exist for $\mu>0$. By symmetry, in the region $\tau_L<\tau_R$ the complementary orbits belonging to the class L^2R^{n-2} exist for $\mu<0$, and those belonging to the class LR^{n-1} exist for $\mu>0$. Notice that the regions for the occurrence of dangerous bifurcation thus obtained analytically exactly match those obtained in Ref. [9] through numerical exploration.

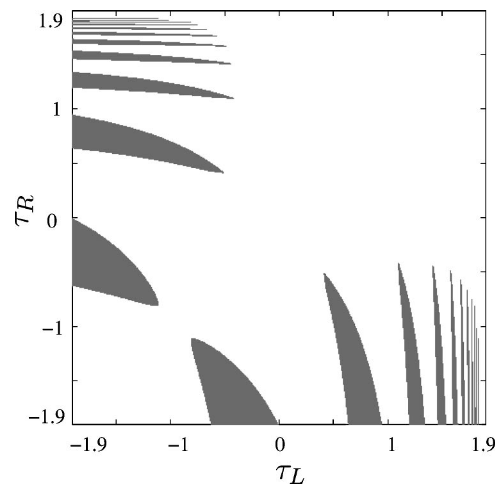

FIG. 5. The regions of dangerous border collision bifurcation obtained using the conditions of existence of complementary orbits for $\delta_L=0.9$, $\delta_R=0.9$.

In this paper we have shown that it is possible for a single unstable periodic orbit to come into existence in a piecewise linear continuous map, and that the dangerous border collision bifurcations in nonsmooth systems are caused by the occurrence of such singleton unstable orbits. We have described a method of obtaining the parameter space regions where such bifurcations occur. This knowledge will help avoid the occurrence of dangerous bifurcations in practical systems.

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