

DYNAMICAL FRICTION

I. GENERAL CONSIDERATIONS: THE COEFFICIENT OF DYNAMICAL FRICTION

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ABSTRACT

In this paper it is shown that a star must experience *dynamical friction*, i.e., it must suffer from a systematic tendency to be decelerated in the direction of its motion. This dynamical friction which stars experience is one of the direct consequences of the fluctuating force acting on a star due to the varying complexion of the near neighbors. From considerations of a very general nature it is concluded that the *coefficient of dynamical friction*, η , must be of the order of the reciprocal of the time of relaxation of the system. Further, an independent discussion based on the two-body approximation for stellar encounters leads to the following explicit formula for the coefficient of dynamical friction:

$$\eta = 4\pi m_1 (m_1 + m_2) \frac{G^2}{v^3} \log_e \left[\frac{D_0 \overline{u^2}}{G(m_1 + m_2)} \right] \int_0^v N(v_1) dv_1,$$

where m_1 and m_2 denote the masses of the field star and the star under consideration, respectively; G , the constant of gravitation; D_0 , the average distance between the stars; $\overline{u^2}$, the mean square velocity of the stars; $N(v_1) dv_1$, the number of field stars with velocities between v_1 and $v_1 + dv_1$; and, finally, v , the velocity of the star under consideration. It is shown that the foregoing formula for η is in agreement with the conclusions reached on the basis of the general considerations. Finally, some remarks are made concerning the further development of these ideas on the basis of a proper statistical theory.

1. *General considerations.*—In a first approximative discussion¹ of the fluctuating part of the gravitational field acting on a star we may conveniently describe it in terms of two functions: a function $W(\mathbf{F})$, which governs the probability of occurrence of a force $\mathbf{F}$ per unit mass acting on a star, and a function $T(|\mathbf{F}|)$, which gives the average time during which such a force acts. On this assumption we can properly visualize the motion of the representative point in the velocity space as follows: The representative point suffers random displacements in a manner that can be described in terms of the theory of random flights.² More specifically, the star may be assumed to suffer a large number of discrete increments in velocity of amounts $|\mathbf{F}| T(|\mathbf{F}|)$ occurring in random directions. The mean square increase in velocity which the star may be expected to suffer in a time t (large compared to the mean periods of the elementary fluctuations in $\mathbf{F}$) is then given by

$$\overline{|\Delta \mathbf{u}|^2} = \overline{|\mathbf{F}|^2 T(|\mathbf{F}|)} t. \quad (1)$$

Equivalently, we may describe the same situation by asserting that the probability function $W(\mathbf{u}, t)$, governing the occurrence of the velocity $\mathbf{u}$ at time t , satisfies the *diffusion equation*

$$\frac{\partial W}{\partial t} = q \nabla_{\mathbf{u}}^2 W, \quad (2)$$

¹ S. Chandrasekhar, *Ap. J.*, **94**, 511, 1941.

² For a general discussion of this and related theories see a forthcoming article by the writer in the *Reviews of Modern Physics*.

where the *diffusion coefficient* q has the value

$$q = \frac{1}{6} \overline{|\mathbf{F}|^2 T}. \tag{3}$$

If the star has a velocity $\mathbf{u}_0$ at time $t = 0$, then the solution of the diffusion equation (2) which will be appropriate for describing the distribution of $\mathbf{u}$ at later times is clearly

$$W(\mathbf{u}, t; \mathbf{u}_0) = \frac{1}{(4\pi q t)^{3/2}} e^{-|\mathbf{u} - \mathbf{u}_0|^2 / 4qt}. \tag{4}$$

It is now seen that formula (1) is an immediate consequence of the foregoing solution for W .

We shall now indicate why the considerations of the preceding paragraph can be valid only for times which are short compared to $\overline{|\mathbf{u}|^2} / \overline{|\mathbf{F}|^2 T}$, where $\overline{|\mathbf{u}|^2}$ denotes the mean square velocity of the stars in an appropriately chosen local standard of rest. For, if $W(\mathbf{u}, t; \mathbf{u}_0)$ according to equation (4), described the stochastic variations of $\mathbf{u}$ for all times, then the probability for a star to suffer any assigned arbitrarily large acceleration can be made as close to unity as we may choose by allowing t to be sufficiently large. This conclusion is, however, contrary to what we should expect on quite general grounds, namely, that $W(\mathbf{u}, t; \mathbf{u}_0)$ tends to a Maxwellian distribution, independently of $\mathbf{u}_0$ as $t \rightarrow \infty$. Expressed somewhat differently, we should strictly suppose that the stochastic variations in the velocity which a star suffers must be such as to leave an initial Maxwellian distribution of the velocities invariant. Defining, now, a stochastic process as *conservative* if it leaves a Maxwellian distribution unchanged, it is clear that the process described by equation (2) is *nonconservative*. Consequently, equation (2) is suitable for describing the underlying physical situation only for times t which satisfy the inequality

$$t \ll \frac{\overline{|\mathbf{u}|^2}}{\overline{|\mathbf{F}|^2 T}}. \tag{5}$$

The question now arises as to how our earlier approximate considerations can be modified so as to make the underlying stochastic process conservative. Now, as has been made familiar in the physical theories of Brownian motion by Ornstein, Uhlenbeck, and others,³ this can be achieved by the introduction of *dynamical friction*. More particularly, we suppose that the acceleration, $\Delta \mathbf{u}$, which a star suffers in a time Δt , which is short compared to the time intervals during which $\mathbf{u}$ may change appreciably but long compared to the periods of the elementary fluctuations in $\mathbf{F}$, can be expressed as the sum of two terms in the form

$$\Delta \mathbf{u} = \delta \mathbf{u}(\Delta t) - \eta \mathbf{u} \Delta t, \tag{6}$$

where the first term on the right-hand side is governed by the probability distribution (cf. eq. [4])

$$\psi(\delta \mathbf{u}[\Delta t]) = \frac{1}{(4\pi q \Delta t)^{3/2}} e^{-|\delta \mathbf{u} - \text{grad}_{\mathbf{u}} q \Delta t|^2 / 4q \Delta t} \tag{7}$$

and where the second term represents a *deceleration* of the star in the direction of its motion by an amount proportional to $|\mathbf{u}|$. The constant of proportionality, η , can therefore be properly defined as the *coefficient of dynamical friction*.

With the underlying stochastic process defined as in equation (6) the distribution

³ See the article quoted in n. 2 for further amplifications of what follows in the text.

function $W(\mathbf{u}, t + \Delta t)$ at time $t + \Delta t$ can be derived from the distribution $W(\mathbf{u}, t)$ at the earlier time t by means of the integral equation

$$W(\mathbf{u}, t + \Delta t) = \int_{-\infty}^{+\infty} W(\mathbf{u} - \Delta \mathbf{u}, t) \psi(\mathbf{u} - \Delta \mathbf{u}; \Delta \mathbf{u}) d(\Delta \mathbf{u}), \quad (8)$$

where $\psi(\mathbf{u}; \Delta \mathbf{u})$ denotes the *transition probability* (cf. eqs. [6] and [7])

$$\psi(\mathbf{u}; \Delta \mathbf{u}) = \frac{1}{(4\pi q \Delta t)^{3/2}} e^{-|\Delta \mathbf{u} - \text{grad}_{\mathbf{u}} q \Delta t + \eta \mathbf{u} \Delta t|^2 / 4q \Delta t}. \quad (9)$$

Expanding $W(\mathbf{u}, t + \Delta t)$, $W(\mathbf{u} - \Delta \mathbf{u}, t)$, and $\psi(\mathbf{u} - \Delta \mathbf{u}; \Delta \mathbf{u})$, which occur in equation (8) in the form of Taylor series, evaluating the various moments of $\Delta \mathbf{u}$ according to the distribution (9), and passing finally to the limit $\Delta t = 0$, we obtain the following equation, which is of the Fokker-Planck type:

$$\frac{\partial W}{\partial t} = \text{div}_{\mathbf{u}} (q \text{grad}_{\mathbf{u}} W) + \text{div}_{\mathbf{u}} (\eta W \mathbf{u}). \quad (10)$$

At this point we may explicitly draw attention to the fact that the foregoing equation is valid also when q and η are functions of $\mathbf{u}$.

Finally, the condition that the Maxwellian distribution

$$\left(\frac{3}{2\pi \overline{|\mathbf{u}|^2}} \right)^{3/2} e^{-3|\mathbf{u}|^2 / 2\overline{|\mathbf{u}|^2}} \quad (11)$$

satisfy equation (10) *identically* requires that q and η be related according to

$$\frac{q}{\eta} = \frac{1}{3} \overline{|\mathbf{u}|^2} = \text{constant}. \quad (12)$$

Now the solution of equation (10) appropriate for describing the distribution of the velocities at time t , given that $\mathbf{u} = \mathbf{u}_0$ at time $t = 0$, is

$$W(\mathbf{u}, t; \mathbf{u}_0) = \left[\frac{3}{2\pi \overline{|\mathbf{u}|^2} (1 - e^{-2\eta t})} \right]^{3/2} e^{-3|\mathbf{u} - \mathbf{u}_0 e^{-\eta t}|^2 / 2\overline{|\mathbf{u}|^2} (1 - e^{-2\eta t})}. \quad (13)$$

In writing down the foregoing solution we have assumed that q and η are constants. We readily verify that $W(\mathbf{u}, t; \mathbf{u}_0)$, according to equation (13), tends to our earlier solution (4) for $t \ll \eta^{-1}$ in virtue of the relation (12); moreover, it tends to the Maxwellian distribution (11) as $t \rightarrow \infty$. Accordingly, η^{-1} can be taken as a measure of the *time of relaxation* of the system. Combining equations (3) and (12), we have

$$\frac{1}{\eta} = 2 \frac{\overline{|\mathbf{u}|^2}}{\overline{|\mathbf{F}|^2 T}}, \quad (14)$$

which agrees with the customary definition of the time of relaxation except for a factor 2.⁴

Summarizing the conclusions reached, we may say that *general considerations such as the invariance of the Maxwellian distribution to the underlying stochastic process require that stars experience dynamical friction during their motion and that the coefficient of dynamical friction be of the order of the reciprocal of the time of relaxation of the system.*

2. An elementary derivation of the coefficient of dynamical friction on the two-body approximation for stellar encounters.—In the preceding section we have seen how the exist-

⁴ Cf. Chandrasekhar, *Ap. J.*, **94**, 511, 1941 (see particularly §§ 7, 8, and 9).

ence of dynamical friction can be inferred on quite general grounds. We shall now show how the operation of such a force can also be derived from a direct analysis of the fluctuating force acting on a star. It is perhaps simplest and most instructive to examine the problem on an approximation in which the fluctuations in $\mathbf{F}$ are analyzed in terms of single stellar encounters each idealized as a two-body problem. On this approximation the increments in velocity, $\Delta v_{||}$ and $\Delta v_{\perp}$, which a star with velocity $\mathbf{v}_2 = |\mathbf{v}_2|$ and mass m_2 suffers as the result of an encounter in directions which are respectively parallel to and perpendicular to the direction of motion are⁵

$$\Delta v_{||} = -\frac{2m_1}{m_1 + m_2} [(v_2 - v_1 \cos \theta) \cos \psi + v_1 \sin \theta \cos \Theta \sin \psi] \cos \psi \quad (15)$$

and

$$\Delta v_{\perp} = \pm \frac{2m_1}{m_1 + m_2} [v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta - \{(v_2 - v_1 \cos \theta) \cos \psi + v_1 \sin \theta \cos \Theta \sin \psi\}^2]^{1/2} \cos \psi, \quad (16)$$

where m_1 and v_1 denote the mass and the velocity of a typical field star and the rest of the symbols have the same meanings as in *Stellar Dynamics*, chapter ii (see, particularly, pp. 51-64).

According to equation (16), and as can, indeed, be expected on general symmetry grounds, $\Delta v_{\perp}$, when summed over a large number of encounters, vanishes identically. But this is not the case with $\Delta v_{||}$, for the net increase in the velocity which the star suffers in the direction of its motion during a time Δt (long compared to the periods of the elementary fluctuations but short compared to the time intervals during which $\mathbf{v}_2$ may be expected to change appreciably) is given by

$$\Sigma \Delta v_{||} = \Delta t \int_0^{\infty} dv_1 \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{D_0} dD \int_0^{2\pi} \frac{d\Theta}{2\pi} \{2\pi N(v_1, \theta, \varphi) V D \Delta v_{||}\}, \quad (17)$$

where the various integrations are, with respect to the different parameters, defining the single encounters. The integration over Θ , the inclination of the orbital plane to the fundamental plane containing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, is readily effected, and we are left with

$$\Sigma \Delta v_{||} = -4\pi \frac{m_1}{m_1 + m_2} \Delta t \int_0^{\infty} dv_1 \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{D_0} dD N(v_1, \theta, \varphi) \times V(v_2 - v_1 \cos \theta) \frac{D}{1 + \frac{D^2 V^4}{G^2 (m_1 + m_2)^2}}, \quad (18)$$

where we have substituted for $\cos^2 \psi$ from *Stellar Dynamics*, equation (2.301). The integral over the impact parameter D when extended from 0 to ∞ diverges; but for reasons explained in *Stellar Dynamics*, page 56, we allow for D only a finite range of integration, namely, from 0 to D_0 , where D_0 is of the order of the average distance between the stars. Performing, now, the integration over D , we obtain

$$\Sigma \Delta v_{||} = -2\pi m_1 (m_1 + m_2) G^2 \Delta t \int_0^{\infty} dv_1 \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi N(v_1, \theta, \varphi) \frac{1}{V^3} \times (v_2 - v_1 \cos \theta) \log(1 + \vartheta^2 V^4), \quad (19)$$

⁵ Cf. S. Chandrasekhar, *Principles of Stellar Dynamics*, p. 229 (eq. [5.721]), University of Chicago Press, 1942. This monograph will be referred to hereafter as "*Stellar Dynamics*."

where we have written

$$\vartheta = \frac{D_0}{G(m_1 + m_2)}. \quad (20)$$

If we now assume that the distribution of the velocities v_1 is spherical, then $N(v_1, \theta, \varphi)$ has the form (cf. *Stellar Dynamics*, eq. [2.336])

$$N(v_1, \theta, \varphi) = N(v_1) \frac{1}{4\pi} \sin \theta. \quad (21)$$

Substituting the foregoing form for $N(v_1, \theta, \varphi)$ in equation (19) and performing the integration over φ , we obtain

$$\Sigma \Delta v_{||} = -\pi m_1 (m_1 + m_2) G^2 \Delta t \int_0^\infty dv_1 N(v_1) \int_0^\pi d\theta \frac{\sin \theta}{V^3} \times (v_2 - v_1 \cos \theta) \log(1 + \vartheta^2 V^4) \quad (22)$$

To effect the integration over θ , we shall use the relative velocity V as the variable of integration instead of θ . Since

$$V^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos \theta, \quad (23)$$

we have

$$\left. \begin{aligned} V dV &= v_1 v_2 \sin \theta d\theta, \\ v_2 - v_1 \cos \theta &= \frac{1}{2v_2} (V^2 + v_2^2 - v_1^2). \end{aligned} \right\} \quad (24)$$

Using relations (24), we find that equation (22) can be reduced to the form

$$\Sigma \Delta v_{||} = -\frac{1}{2} \pi m_1 (m_1 + m_2) \frac{G^2}{v_2^2} \Delta t \int_0^\infty \frac{1}{v_1} N(v_1) J dv_1, \quad (25)$$

where we have used J to denote

$$J = \int_{|v_1 - v_2|}^{(v_1 + v_2)} \left(1 + \frac{v_2^2 - v_1^2}{V^2} \right) \log(1 + \vartheta^2 V^4) dV. \quad (26)$$

After an integration by parts the expression for J becomes

$$\left. \begin{aligned} J &= \left(V - \frac{v_2^2 - v_1^2}{V} \right) \log(1 + \vartheta^2 V^4) \Big|_{|v_1 - v_2|}^{(v_1 + v_2)} \\ &\quad - 4 \int_{|v_1 - v_2|}^{(v_1 + v_2)} \left(1 - \frac{v_2^2 - v_1^2}{V^2} \right) \frac{\vartheta^2 V^4}{1 + \vartheta^2 V^4} dV. \end{aligned} \right\} \quad (27)$$

Now, under most conditions of practical interest $\vartheta^2 V^4$ is generally very large compared to unity (cf. *Stellar Dynamics*, eqs. [2.323] and [2.347]; also eq. [5.215]). Hence, to a sufficient accuracy we have

$$J = \left[\left(V - \frac{v_2^2 - v_1^2}{V} \right) \log(1 + \vartheta^2 V^4) - 4 \left(V + \frac{v_2^2 - v_1^2}{V} \right) \right]_{|v_1 - v_2|}^{(v_1 + v_2)}. \quad (28)$$

After some further reductions we find that the foregoing equation becomes

$$J = \begin{cases} 2 v_1 \log (1 + \vartheta^2 [v_1 + v_2]^4) (1 + \vartheta^2 [v_1 - v_2]^4) & (v_1 < v_2), \\ 2 v_1 \log (1 + 16 \vartheta^2 v_1^4) - 8 v_1 & (v_1 = v_2), \\ 2 v_1 \log \frac{1 + \vartheta^2 (v_1 + v_2)^4}{1 + \vartheta^2 (v_1 - v_2)^4} - 16 v_2 & (v_1 > v_2). \end{cases} \quad (29)$$

Again, since $\vartheta^2(v_1 + v_2)^4$ and $\vartheta^2(v_1 - v_2)^4$ are also generally very large compared to unity, we can further simplify equation (29) to

$$J = \begin{cases} 8 v_1 \log \vartheta (v_2^2 - v_1^2) & (v_1 < v_2), \\ 4 v_1 \log 4 \vartheta v_1^2 - 8 v_1 & (v_1 = v_2), \\ 8 v_1 \log \frac{v_1 + v_2}{v_1 - v_2} - 16 v_2 & (v_1 > v_2). \end{cases} \quad (30)$$

The foregoing formula for J shows that in an approximation in which we retain only the "dominant term" (cf. *Stellar Dynamics*, pp. 62-64) we have

$$J = \begin{cases} 8 v_1 \log \vartheta |\overline{\mathbf{u}}|^2 & (v_1 < v_2), \\ 0 & (v_1 > v_2), \end{cases} \quad (31)$$

where $|\overline{\mathbf{u}}|^2$ may be taken to denote the mean square velocity of the stars in the system. According to equations (30) and (31), we have the remarkable result that *to a sufficient accuracy only stars with velocities less than the one under consideration contribute to $\Sigma \Delta v_{||}$* . As we shall see presently, it is precisely on this account that dynamical friction appears on our present analysis.

Combining equations (25) and (31), we have

$$\Sigma \Delta v_{||} = -4\pi m_1 (m_1 + m_2) \frac{G^2}{v_2^2} \log (\vartheta |\overline{\mathbf{u}}|^2) \Delta t \int_0^{v_2} N(v_1) dv_1. \quad (32)$$

Finally, if we assume that the velocities v_1 are distributed according to Maxwell's law, then

$$\int_0^{v_2} N(v_1) dv_1 = \frac{4j^3}{\pi^{1/2}} N \int_0^{v_2} e^{-j^2 v_1^2} v_1^2 dv_1, \quad (33)$$

where N denotes the number of stars per unit volume and j is a parameter which measures the dispersion of the velocities in the system. Expressing the integral on the right-hand side of equation (33) in terms of the error integral

$$\Phi(x) = \frac{2}{\pi^{1/2}} \int_0^x e^{-x^2} dx, \quad (34)$$

and substituting the result in equation (32), we find that

$$\Sigma \Delta v_{||} = -4\pi N m_1 (m_1 + m_2) \frac{G^2}{v_2^2} \log (\vartheta |\overline{\mathbf{u}}|^2) \Delta t [\Phi(x_0) - x_0 \Phi'(x_0)], \quad (35)$$

where we have written $x_0 = jv_2$.

Equation (35) shows that the star does, in fact, experience dynamical friction and that the coefficient of dynamical friction has the value

$$\eta = 4\pi N m_1 (m_1 + m_2) \frac{G^2}{v_2^3} \log (\vartheta |\mathbf{u}|^2) [\Phi(x_0) - x_0 \Phi'(x_0)] . \quad (36)$$

It is now of interest to see that with the coefficient of dynamical friction defined as in equation (36) we can directly verify the existence of a relation of the form (12). For, according to equations (2.356) and (5.724) in *Stellar Dynamics*, we have

$$\Sigma \Delta v_{||}^2 = \frac{8}{3} \pi N m_1^2 \frac{G^2}{v_2^3} |\mathbf{u}|^2 \log (\vartheta |\mathbf{u}|^2) \Delta t [\Phi(x_0) - x_0 \Phi'(x_0)] . \quad (37)$$

Hence,

$$\frac{\Sigma \Delta v_{||}^2}{\eta \Delta t} = \frac{2}{3} \frac{m_1}{m_1 + m_2} |\mathbf{u}|^2 , \quad (38)$$

which is to be compared with equation (12). It is thus seen that a detailed analysis of the fluctuating field of the near-by stars in terms of individual stellar encounters idealized as two-body problems fully confirms the conclusions reached in § 1 on the basis of certain general principles.

3. Dynamical friction as a consequence of the statistical properties of the fluctuating gravitational field of a random distribution of stars.—The discussion of dynamical friction in § 1, while sufficiently general for a first orientation in the subject, suffers, nevertheless, from certain drawbacks. For example, in writing down the probability distribution for $\delta \mathbf{u}(\Delta t)$ (eq. [7]) we have assumed that it has spherical symmetry. However, to be entirely general we should rather suppose that $\psi(\delta \mathbf{u}[\Delta t])$ has the form

$$\psi(\delta \mathbf{u}[\Delta t]) = \frac{1}{\pi^{3/2}} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} e^{-(\alpha_{11} \delta u_1^2 + \alpha_{22} \delta u_2^2 + \alpha_{33} \delta u_3^2 + 2\alpha_{12} \delta u_1 \delta u_2 + 2\alpha_{23} \delta u_2 \delta u_3 + 2\alpha_{31} \delta u_3 \delta u_1) / \Delta t} , \quad (39)$$

where $\delta \mathbf{u} = (\delta u_1, \delta u_2, \delta u_3)$ and $(\alpha_{\mu\nu})$ is a symmetric tensor of the second rank. The components of $(\alpha_{\mu\nu})$ can very well depend on $\mathbf{u}$. While it would not be difficult to write down for the correspondingly more general form of the transition probability the appropriate generalization of equation (10), we should not be able to make much practical use of such an equation without some direct knowledge concerning $(\alpha_{\mu\nu})$. In other words, a detailed statistical analysis of the fluctuating part of the gravitational field acting on a star must precede a discussion of the necessary generalization of equation (10). A start in this direction has recently been made by Chandrasekhar and von Neumann in two papers.⁶ Particularly in their second paper, where all the first and the second moments of $\dot{\mathbf{F}}$ for given $\mathbf{F}$ and $\mathbf{v}$ have been evaluated, a direct indication for the existence of dynamical friction on the statistical theory has indeed been found. However, a complete solution of the problem will require a more far reaching discussion than has yet been undertaken. But the general outlines of such a theory are not difficult to foresee. For, the essential information which is needed is, of course, the average force, $\overline{\mathbf{F}_t}$, per unit mass acting on a star at time t when a force $\mathbf{F}_0$ acted at time $t = 0$. The statistical problem is thus merely one of finding the joint distribution $W(\mathbf{F}_0, \mathbf{F}_t)$ of $\mathbf{F}_0$ and $\mathbf{F}_t$, where

$$\mathbf{F}_0 = G \sum_i M_i \frac{\mathbf{r}_i}{|\mathbf{r}_i|^3} \quad (40)$$

⁶ *Ap. J.*, **95**, 489, 1942, and **97**, 1, 1943.

and

$$\mathbf{F}_t = G \sum_i M_i \frac{\mathbf{r}_i + \mathbf{V}_i t}{|\mathbf{r}_i + \mathbf{V}_i t|^3}. \quad (41)$$

In equations (40) and (41) $\mathbf{r}_i$ and $\mathbf{V}_i$ denote, respectively, the position and the velocity of a typical field star *relative* to the one under consideration. By an application of Markoff's method (cf. the papers of Chandrasekhar and von Neumann) we readily find that the required distribution is formally given by

$$W(\mathbf{F}_0, \mathbf{F}_t) = \frac{1}{64\pi^6} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\boldsymbol{\rho} \cdot \mathbf{F}_0 + \boldsymbol{\sigma} \cdot \mathbf{F}_t)} A(\boldsymbol{\rho}, \boldsymbol{\sigma}) d\boldsymbol{\rho} d\boldsymbol{\sigma}, \quad (42)$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are two auxiliary vectors and

$$A(\boldsymbol{\rho}, \boldsymbol{\sigma}) = e^{-N C(\boldsymbol{\rho}, \boldsymbol{\sigma})} \quad (43)$$

and where

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \left[1 - e^{iGM \left\{ \frac{\mathbf{r} \cdot \boldsymbol{\rho}}{r^3} + \frac{(\mathbf{r} + \mathbf{V}t) \cdot \boldsymbol{\sigma}}{|\mathbf{r} + \mathbf{V}t|^3} \right\}} \right] \tau(V, M) dM dr dV. \quad (44)$$

In equation (44) $\tau(V, M)$ governs the probability of occurrence of a star with a relative velocity V and with a mass M .

For our purposes it would, however, be sufficient to know the first moment of $\mathbf{F}_t$ for given $\mathbf{F}$ and $\mathbf{v}$, in which case we shall need only the behavior of $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}| \rightarrow 0$. It is not difficult to push the formal theory a little further, but without going into these developments here it is clear that in terms of $\overline{\mathbf{F}_t}(\mathbf{F}_0, \mathbf{v})$ we shall be able to solve the entire problem of the stochastic variation of $\mathbf{F}$ acting on a star. More particularly the consideration of the integral

$$\int_0^{\infty} \overline{\mathbf{F}_t}(\mathbf{F}_0, \mathbf{v}) dt \quad (45)$$

will not only provide us with the means of giving a precise meaning to the notion of the mean life of $\mathbf{F}$ but will also disclose in a direct manner the existence of dynamical friction on the statistical theory. We shall return to the development of the theory along these lines on a later occasion.

4. General remarks.—To avoid misunderstandings we shall make some remarks (which are otherwise obvious) concerning the reasons for introducing the new notion of dynamical friction and avoiding the usage of the term “viscosity.” First, the physical ideas underlying the concepts of dynamical friction and viscosity are quite distinct: thus, while the “coefficient of dynamical friction” refers to the systematic deceleration which *individual* stars experience during their motion, “viscosity,” as commonly understood, refers to the sheering force exerted by one *element of gas* on another. Second, dynamical friction is an exact notion expressing the systematic decelerating effect of the fluctuating field of force acting on a star *in motion*, in contrast to viscosity, which, as a concept, is valid only when averaged over times which are long compared to the time of relaxation of the system and over spatial dimensions which are large compared to the mean free paths of the individual molecules. Thus, while the introduction of dynamical friction in stellar dynamics presents no difficulty, the circumstances are very different for a rational introduction of “viscosity” in the subject (cf. *Stellar Dynamics*, pp. 76–78 and 184).