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On the non-radial oscillations of slowly rotating stars induced by the Lense–Thirring effect

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In a star that is rotating so slowly that the distortion of its figure may be ignored, the axial modes of non-radial oscillation exhibiting resonance can be excited by the polar modes of perturbation by the coupling derived from the dragging of the inertial frame by the rotation of the star (i.e. by the Lense–Thirring effect). The coupling of these two modes of opposite parity is subject to the standard selection rule, $\Delta l = \pm 1$. Also, the excitation of $(l+1)$-axial mode by the l-polar mode is favoured relative to the excitation of the $(l-1)$-axial mode, in conformity with the 'propensity' rule. As an illustrative example, the excitation of the sextupole axial modes of oscillation by the quadrupole polar perturbations is considered in some detail; and it is shown that both the real and the imaginary parts of the characteristic frequency of the quasi-normal modes decrease dramatically with the amplitude of the coupling. The relatively very long damping times of these rotationally induced oscillations may be a decisive factor in their eventual detection in neutron stars following the glitches.

1. Introduction

The non-radial oscillations of a star may be considered as manifestations attendant to the scattering of incident gravitational waves by the curvature of the space-time and the matter content of the star. The incident waves can be of two kinds: polar and axial; and the manner of their scattering by a static spherical star is very different. The scattering of polar gravitational waves exhibits resonances with determinate half-widths. In contrast, the scattering of axial gravitational waves resembles scattering by a soft-core Coulomb potential and exhibits no resonances. The origin of this difference is that while the incidence of polar gravitational waves excites fluid motions in the star, the incidence of axial gravitational waves does not. A mathematical theory of the non-radial oscillations of a star based on this point of view has recently been developed by Chandrasekhar & Ferrari (1991 *a*; this paper will be referred to hereafter as Paper I; the earlier paper (Chandrasekhar & Ferrari 1990), in which the basic idea germinated, will be referred to hereafter as Paper II).

In this paper, we shall show that the incidence of axial gravitational waves on a star in slow rotation, with an angular velocity Ω so small that the distortion of its figure of order Ω^2 can be ignored, will, by virtue of the coupling of the axial and the polar modes of perturbations by the *Lense–Thirring effect* (i.e. by the 'dragging' of the inertial frame by the rotation of the star) excite fluid motions inside the star with the result that the scattering of these waves will exhibit resonance. It will appear that the coupling of these two modes of perturbation, of opposite parity, resulting in the resonant scattering of the axial gravitational waves, is in conformity with the

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standard requirements of symmetry: the *Laporte rule* – that even modes can couple only with the odd modes and conversely – and the *selection rule*, $\Delta l = \pm 1$ (for a more complete statement of the selection rule see §5).

The resonant oscillations of a star induced by the Lense–Thirring effect have no counterparts in the Newtonian theory since they derive from the purely general relativistic effect of the dragging of the inertial frame by rotation. These axial modes of oscillation should, in principle, occur in neutron stars (known to be rotating) following 'glitches', even as the normal non-radial modes of oscillations of the Earth are excited following earthquakes. The question whether such oscillations can be observed in neutron stars is problematic. In this paper, we shall, however, be concerned only with establishing the theoretical result.

2. The equations governing the stationary space-time of a slowly rotating star

The equations governing the space-time of a slowly rotating star were first written down by Hartle (1967; for a synopsis of the basic equations see Chandrasekhar & Miller (1974)). The metric of the space-time is of the standard form,

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu}(\mathrm{d}t)^2 - \mathrm{e}^{2\psi}(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2 - \mathrm{e}^{2\mu_2}(\mathrm{d}x^2)^2 - \mathrm{e}^{2\mu_3}(\mathrm{d}x^3)^2, \tag{1}$$

where ν, ψ, μ_2, and μ_3 differ from those for a spherical non-rotating star by quantities of order Ω^2 and ω (zero for a non-rotating star) is of order Ω. Thus the metric functions are of the form

$$\nu + \nu^{(\Omega)},\ \psi + \psi^{(\Omega)},\ \mu_2 + \mu_2^{(\Omega)},\ \mu_3 + \mu_3^{(\Omega)}, \tag{2}$$

where ν, ψ, μ_2, and μ_3 denote the functions appropriate to a static spherical star (given in I, §3) and $\nu^{(\Omega)}$, $\psi^{(\Omega)}$, $\mu_2^{(\Omega)}$, and $\mu_3^{(\Omega)}$ are 'corrections' of $O(\Omega^2)$. The equations determining these corrections are written out in detail in Chandrasekhar & Miller (1974, §2), we shall not need them in our present context. But the equation determining ω is central for our present purposes. It is given by

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\mu_2+\nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{3}$$

where
$$\varpi = \Omega - \omega, \tag{4}$$

and (according to I, eq. (8))

$$(\mu_2+\nu)_{,r} = \mathrm{e}^{2\mu_2}(\epsilon + p)\,r. \tag{5}$$

In the vacuum outside the star, $\mu_2 + \nu = 0$, and equation (3) reduces to

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} = 0. \tag{6}$$

The solution of this equation, appropriate to the problem on hand, is

$$\varpi = \Omega - 2Jr^{-3}, \tag{7}$$

where J denotes the angular momentum of the star. The continuity of ϖ at the boundary, $r = r_1$, of the star requires

$$(\varpi_{,r})_{r=r_1} = 6Jr_1^{-4}. \tag{8}$$

Returning to equation (3), we shall find it convenient to measure ϖ in units of its value ϖ_c at the centre and define a *standard solution* ϖ with the behaviour,

$$\varpi = 1+\varpi_2 r^2+\varpi_4 r^4+\dots, \tag{9}$$

at $r=0$, where (as one may readily verify)

$$\varpi_2 = \tfrac{2}{5}(\epsilon_0+p_0), \quad \varpi_4 = \tfrac{1}{7}[\tfrac{3}{5}(\epsilon_0+p_0)^2+a_2+b_2^{(2)}] \tag{10}$$

and ϵ_0, p_0, a_2, and $b_2^{(2)}$ are defined in I, eq. (52). In terms of the standard solution for ϖ,

$$J = +\tfrac{1}{6}r_1^4\varpi_c(\varpi_{,r})_{r=r_1} \quad \text{and} \quad \Omega = \varpi_c(\varpi+\tfrac{1}{3}r_1\varpi_{,r})_{r=r_1}. \tag{11}$$

3. The basic equations of the problem

The complete set of equations governing a perfect fluid configuration in rotation, for an underlying spacetime that is time-dependent and axisymmetric, were written down by Chandrasekhar & Friedman (1972; this paper will be referred to hereafter as Paper III). The metric of the space-time considered was the standard one, namely,

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi-\omega\,dt-q_2\,dx^2-q_3\,dx^3)^2-e^{2\mu_2}(dx^2)^2-e^{2\mu_3}(dx^3)^2, \tag{12}$$

where ν, ψ, μ_2, μ_3, ω, q_2, and q_3 are all functions of t, $x^2(=r)$, and $x^3(=\theta)$. In Paper III, the components of the Riemann, the Ricci, and the Einstein tensors were obtained in a tetrad formalism (as in *The mathematical theory of black holes* (*M.T.*), Chandrasekhar 1983); but the equations of hydrodynamics were not. In the Appendix this omission is rectified; they follow readily from the divergence of the energy–momentum tensor

$$T_{(a)(b)} = (\epsilon+p)\,u_{(a)}\,u_{(b)} - p\eta_{(a)(b)}, \tag{13}$$

where we have enclosed the indices in parentheses to denote that they are tetrad components.

With the definitions,

$$\varphi_{,t} = \Omega \quad \text{and} \quad x^\alpha_{,t} = v^\alpha \quad (\alpha = 2,3), \tag{14}$$

the tetrad components of the four-velocity are

$$u^{(0)} = \frac{1}{\surd(1-V^2)}, \quad u^{(\alpha)} = \frac{v^{(\alpha)}}{\surd(1-V^2)}, \quad u^{(1)} = \frac{v^{(1)}}{\surd(1-V^2)}, \tag{15}$$

$$v^{(\alpha)} = v^\alpha e^{\mu_\alpha-\nu}; \quad v^{(1)} = e^{\psi-\nu}(\Omega-\omega-q_2 v^2-q_3 v^3), \tag{16}$$

and

$$V^2 = [v^{(1)}]^2+[v^{(2)}]^2+[v^{(3)}]^2. \tag{17}$$

In the *stationary state*,

$$v^{(\alpha)} = 0, \quad v^{(1)} = V = e^{\psi-\nu}(\Omega-\omega) = e^{\psi-\nu}\varpi; \tag{18}$$

and in the *perturbed state*,

$$v^{(\alpha)}, u^{(\alpha)} \text{ and } q_\alpha \text{ are quantities of the first order of smallness,} \tag{19}$$

while

$$v^{(1)} = V = e^{\psi-\nu}(\Omega-\omega) = e^{\psi-\nu}\varpi. \tag{20}$$

continues to be valid (cf. equations (16) and (17)).

As in Paper I (eq. (13)) and Paper II (eq. (97)) we shall express the tetrad components of the four velocity, $u_{(2)}$ and $u_{(3)}$, (in the perturbed state) in terms of the Lagrangian displacements, ξ_2 and ξ_3 in the manner:

$$u_{(\alpha)} = \delta u_\alpha = \xi_{\alpha,0} = \mathrm{i}\sigma\xi_\alpha, \tag{21}$$

where it will be noted that we have dispensed with the parentheses distinguishing the tetrad components.

The relevant linearized equations

The linearized equations that govern the perturbations of a uniformly rotating star have all been written down in Paper III. Since in some of the equations of that paper the tensor components of the four-velocity are used and the Lagrangian displacements are also differently defined, we shall transcribe the relevant equations in accord with our present conventions and notations. However, for a comparison with the equations of Paper III, it may be useful to note that

$$\xi_\alpha \text{ as defined in Paper III is } -\xi^a\,\mathrm{e}^{\mu_\alpha-\nu} \text{ as defined in this paper;} \tag{22}$$

and the combination

$$u_0\,u^1\,\mathrm{e}^{-2\psi+2\nu+2\mu_\alpha}\xi_\alpha \text{ that occurs in Paper III}$$

$$\text{is } -\frac{\varpi}{1-V^2}\mathrm{e}^{\mu_a+\nu}\xi_\alpha \text{ with our present definitions.} \tag{23}$$

The basic equations for our present purposes are eq. (143), (145), (151), (152), and (155) of Paper III. Transcribed in our present notation, they are

$$-2\frac{\epsilon+p}{1-V^2}\xi_2\,\mathrm{e}^{\nu+\mu_2} = [\delta(\psi+\mu_3)_{,2}+(\psi-\nu)_{,2}\,\delta\psi+(\mu_3-\nu)_{,2}\,\delta\mu_3-(\psi+\mu_3)_{,2}\,\delta\mu_2] + \tfrac{1}{2}\mathrm{e}^{-\psi-\nu+\mu_2-\mu_3}X\omega_{,3}, \tag{24}$$

$$-2\frac{\epsilon+p}{1-V^2}\xi_3\,\mathrm{e}^{\nu+\mu_3} = [\delta(\psi+\mu_2)_{,3}+(\psi-\nu)_{,3}\,\delta\psi+(\mu_2-\nu)_{,3}\,\delta\mu_2-(\psi+\mu_2)_{,3}\,\delta\mu_3] - \tfrac{1}{2}\mathrm{e}^{-\psi-\nu+\mu_3-\mu_2}X\omega_{,2}, \tag{25}$$

$$\delta\omega_{,2}-q_{2,0,0} = -4\frac{\epsilon+p}{1-V^2}\mathrm{e}^{\nu+\mu_2}\xi_2\varpi+\varpi_{,2}(3\,\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3)-\mathrm{e}^{-3\psi+\nu+\mu_2-\mu_3}X_{,3}, \tag{26}$$

$$\delta\omega_{,3}-q_{3,0,0} = -4\frac{\epsilon+p}{1-V^2}\mathrm{e}^{\nu+\mu_3}\xi_3\varpi+\varpi_{,3}(3\delta\psi-\delta\nu-\delta\mu_3+\delta\mu_2)+\mathrm{e}^{-3\psi+\nu+\mu_3-\mu_2}X_{,2}, \tag{27}$$

and

$$(\mathrm{e}^{-3\psi+\nu-\mu_2+\mu_3}X_{,2})_{,2}+(\mathrm{e}^{-3\psi+\nu+\mu_2-\mu_3}X_{,3})_{,3}+\sigma^2\mathrm{e}^{-3\psi-\nu+\mu_2+\mu_3}X$$
$$= [\varpi_{,2}(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3)]_{,3}-[\varpi_{,3}(3\delta\psi-\delta\nu-\delta\mu_3+\delta\mu_2)]_{,2}$$
$$-4\left(\frac{\epsilon+p}{1-V^2}\mathrm{e}^{\nu+\mu_2}\xi_2\varpi\right)_{,3}+4\left(\frac{\epsilon+p}{1-V^2}\mathrm{e}^{\nu+\mu_3}\xi_3\varpi\right)_{,2}, \tag{28}$$

where (cf. Paper I, eq. (142))

$$X = \mathrm{e}^{3\psi+\nu-\mu_2-\mu_3}Q_{23} \tag{29}$$

and is what is denoted by Q in Paper III (eq. (142)).

The problem which we wish to consider in the context of equation (28) is the following. But first a clarification of the assumptions underlying the derivation of this equation.

Equation (28) is one of the *exact* equations governing the axisymmetric perturbations of a fluid configuration, rotating uniformly with an angular velocity Ω, and a space-time with a metric of the standard form (12) that describes it consistently with Einstein's equations. In obtaining the perturbation equations, the assumption has been made that the amplitudes of all the quantities describing the perturbation have the *same* time-dependence $e^{i\sigma t}$. It is clear that in the eventual solution for the desired quasi-normal modes, the angular velocity Ω will appear as a parameter.

As stated in the Introduction (§1), we wish to consider the case when the angular velocity Ω is sufficiently small that the distortion of the figure of order Ω^2 can be neglected. Then to order Ω, the diagonal metric functions ν, ψ, μ_2, and μ_3 of the initial stationary space-time are the same as those for the spherical star with no rotation; but there are 'corrections' of order Ω^2 derived from the terms $\nu^{(\Omega)}$, $\psi^{(\Omega)}$, $\mu_2^{(\Omega)}$, and $\mu_3^{(\Omega)}$ (cf. equation (2)). The only effect of order Ω on the unperturbed metric is the non-diagonal term $\omega(=\Omega-\varpi)$ whose presence leads to the Lense–Thirring effect.

Returning to equation (28), we observe that the terms on the right-hand side, involving ξ_2, ξ_3, $\delta\nu$, $\delta\psi$, $\delta\mu_2$, and $\delta\mu_3$ (describing the polar perturbations) occur with ϖ or $\varpi_{,i}$ $(i = 2, 3)$ as factors. Accordingly, it will suffice to consider for these perturbations their zero-order values for a spherical star: for whatever changes these perturbations may experience (besides those of order Ω^2 derived from $\nu^{(\Omega)}$, $\psi^{(\Omega)}$, etc.) can only be of order Ω. At the same time, the small rotation Ω will affect X (describing the axial mode) *directly* since $\varpi = \Omega - \omega$ is the source for Q_{02} and Q_{03} (as is manifest from *M.T.*, equation 4(*d*), p. 141). Remembering further that

$$\omega \text{ is a function only of } r, \tag{30}$$

we may conclude that, under the circumstances envisaged, equation (28), inclusive of terms of order Ω, can be replaced by

$$(e^{-3\psi+\nu-\mu_2+\mu_3}X_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3}X_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3}X$$
$$= \varpi_{,r}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)_{,\theta} - 4[(\epsilon+p)\, e^{\nu+\mu_2}\xi_2\varpi]_{,\theta} + 4[(\epsilon+p)\, e^{\nu+\mu_3}\xi_3\varpi]_{,r}, \tag{31}$$

where ξ_2, ξ_3, $\delta\psi$, $\delta\nu$, $\delta\mu_2$, and $\delta\mu_3$ are given by the solutions obtained for them in Paper I.

The mathematical problem we have formulated leaves open the question, how the polar modes of a spherical star may be affected to order Ω by their coupling with the axial modes. We consider this question briefly in the concluding section, §8.

4. The reduction of equation (31)

It will be observed that the left-hand side of equation (31) is the same as I, eq. (141). It can accordingly be reduced to the form (cf. I, eq. (143)).

$$\frac{1}{\sin^3\theta}\left(\frac{e^{\nu-\mu_2}}{r^2}X_{,r}\right)_{,r} + \frac{e^{\nu+\mu_2}}{r^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial X}{\partial\theta}\right)_{,\theta} + \sigma^2\frac{e^{-\nu+\mu_2}}{r^4\sin^3\theta}X, \tag{32}$$

Expressing X in the form,

$$X = \sum_{l=2}^{\infty} X_l(r)\, C_{l+2}^{-\frac{3}{2}}(\theta), \tag{33}$$

as a series in the Gegenbauer polynomials, $C_{l+2}^{-\frac{3}{2}}(\theta)$, we obtain

$$\sum_{l=2}^{\infty}\left\{\left(\frac{e^{\nu-\mu_2}}{r^2}X_{l,r}\right)_{,r}-(l-1)(l+2)\frac{e^{\nu+\mu_2}}{r^4}X_l+\sigma^2\frac{e^{-\nu+\mu_2}}{r^4}X_l\right\}\frac{C_{l+2}^{-\frac{3}{2}}(\theta)}{\sin^3\theta}$$
$$=\frac{e^{\nu-\mu_2}}{r^2\sin^3\theta}\sum_{l=2}^{\infty}C_{l+2}^{-\frac{3}{2}}(\theta)\,\mathscr{D}_l X_l(r), \quad (34)$$

where

$$\mathscr{D}_l=\partial_{rr}-\frac{e^{2\mu_2}}{r}[2+r^2(\epsilon-p)-6M(r)/r]\,\partial_r-\frac{(l-1)(l+2)}{r^2}e^{2\mu_2}+\sigma^2 e^{2(\mu_2-\nu)}. \quad (35)$$

The 'source terms' on the right-hand side of equation (31) is expressible as a sum over the different polar modes belonging to the different Legendre polynomials $P_l(\cos\theta)$. Considering a particular l-mode, we find from the solutions listed in I, equations (23), that

$$(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3)_{,\theta}=(4T-N-L-\kappa V)P_{l,\theta}+2V(P_{l,\theta}\cot\theta)_{,\theta}. \quad (36)$$

Eliminating T in favour of W with the aid of the equation (I, eq. (29)),

$$T-V+L=-W, \quad (37)$$

we find after some further reductions that the contribution to the source derived from the metric perturbations belonging to a particular l-polar mode is given by

$$\varpi_{,r}[(4W+N+5L+2nV)P_{l,\mu}+2V\mu P_{l,\mu,\mu}]\surd(1-\mu^2), \quad 38)$$

where
$$\mu=\cos\theta. \quad (39)$$

Similarly, we find, by making use successively of I, eq. (25) and (46), that the corresponding contribution derived from the fluid motions is given by

$$\begin{aligned}-4[(\epsilon+p)e^{\nu+\mu_2}\xi_2\varpi]_{,\theta}&+4[(\epsilon+p)e^{\nu+\mu_3}\xi_3\varpi]_{,r}\\&=2[-U\varpi+(W\varpi)_{,r}]P_{l,\theta}\\&=-2\{[W_{,r}+(Q-1)\nu_{,r}W]\varpi-W_{,r}\varpi-W\varpi_{,r}\}P_{l,\theta}\\&=-2W[\varpi_{,r}-\varpi(Q-1)\nu_{,r}]P_{l,\mu}\surd(1-\mu^2). \quad (40)\end{aligned}$$

Finally, combining equations (34), (38), and (40), we obtain

$$\sum_{l=2}^{\infty}[\mathscr{D}_l X_l(r)]C_{l+2}^{-\frac{3}{2}}(\mu)=e^{\mu_2-\nu}r^2(1-\mu^2)^2\sum_{l=2}^{\infty}S_l(r,\mu), \quad (41)$$

where

$$S_l=\varpi_{,r}[(2W_l+N_l+5L_l+2nV_l)P_{l,\mu}+2V_l\mu P_{l,\mu,\mu}]+2\varpi W_l(Q-1)\nu_{,r}P_{l,\mu}, \quad (42)$$

and the subscripts l to the functions W, N, L, and V distinguish that they belong to a particular polar l-mode.

Letting

$$r_*=\int_0^r e^{-\nu+\mu_2}\,dr \quad (43)$$

and
$$X_l=rZ_l, \quad (44)$$

(as in I, eq. (146) and (147)), we find

$$\sum_{l=2}^{\infty}\left\{\frac{d^2Z_l}{dr_*^2}+\sigma^2Z_l-\frac{e^{2\nu}}{r^3}[l(l+1)r+r^3(\epsilon-p)-6M(r)]Z_l\right\}C_{l+2}^{-\frac{3}{2}}(\mu)$$

$$= e^{\nu-\mu_2}r(1-\mu^2)^2\sum_{l=2}^{\infty}S_l(r,\mu). \quad (45)$$

Outside the boundary of the star ($r > r_1$),

$$\epsilon = p = 0, \quad M(r) = M, \quad \mu_2+\nu = 0, \quad W = 0, \quad (46)$$

and

$$\varpi_{,r} = 6Jr^{-4}; \quad (47)$$

and equation (45) becomes

$$\sum_{l=2}^{\infty}\left\{\frac{d^2Z_l}{dr_*^2}+\sigma^2Z_l-\frac{e^{2\nu}}{r^3}[l(l+1)r-6M]Z_l\right\}C_{l+2}^{-\frac{3}{2}}(\mu)$$

$$= 6\frac{e^{2\nu}}{r^3}J(1-\mu^2)^2\sum_{l=2}^{\infty}[(N_l+5L_l+2nV_l)P_{l,\mu}+2\mu V_lP_{l,\mu,\mu}]. \quad (48)$$

5. The Laporte, the selection, and the propensity rules

In view of the known orthogonality relation,

$$\int_{-1}^{+1}C_{m+2}^{-\frac{3}{2}}C_{l+2}^{-\frac{3}{2}}\frac{d\mu}{(1-\mu^2)^2} = \frac{18(m-2)!}{(2m+1)(m+2)!}\delta_{ml}, \quad (49)$$

among the Gegenbauer polynomials $C_{l+2}^{-\frac{3}{2}}$, we obtain, on multiplying equation (41) by $C_{m+2}^{-\frac{3}{2}}/(1-\mu^2)^2$ and integrating over the range of μ, the equation

$$\frac{18(m-2)!}{(2m+1)(m+2)!}\mathscr{D}_{m+2}X_{m+2} = r^2 e^{\mu_2-\nu}\sum_{l+2}^{\infty}\int_{-1}^{+1}C_{m+2}^{-\frac{3}{2}}S_l(r,\mu)\,d\mu. \quad (50)$$

The further reduction of the right-hand side of this equation requires the evaluation of the integrals

$$\int_{-1}^{+1}\mu P_{l,\mu,\mu}C_{m+2}^{-\frac{3}{2}}(\mu)\,d\mu \quad (51)$$

and

$$\int_{-1}^{+1}P_{l,\mu}C_{m+2}^{-\frac{3}{2}}(\mu)\,d\mu. \quad (52)$$

Since $C_{l+2}^{-\frac{3}{2}}(\mu)$ and $P_{l,\mu}$ and $\mu P_{l,\mu,\mu}$ are of opposite parities, it follows directly that *polar modes belonging to even ls can couple only with axial modes belonging to odd ls and conversely*. This is Laporte's rule known in atomic theory and for much the same reason. And as in atomic theory more is true: a *selection rule*,

$$l = m+1 \quad \text{or} \quad l = m-1 \quad (53)$$

obtains by virtue of the integrals (51) and (52) vanishing if the rule is violated. The proof is elementary if use is made of the following relation between the Gegenbauer and the Legendre functions:

$$C_{l+2}^{-\frac{3}{2}} = \frac{3(1-\mu^2)^2}{(l+2)(l+1)l(l-1)}P_{l,\mu,\mu}. \quad (54)$$

Thus, by making use of this relation and a known recurrence relation among the Gegenbauer polynomials, the integral (51) can be transformed to the form,

$$\tfrac{1}{3}(l+2)(l+1)\,l(l-1)\int_{-1}^{+1}\mu C_{m+2}^{-\frac{3}{2}}C_{l+2}^{-\frac{3}{2}}\frac{\mathrm{d}\mu}{(1-\mu^2)^2}$$

$$=\frac{(l+2)(l+1)\,l(l-1)}{3(2m+1)}\int_{-1}^{+1}[(m+3)\,C_{m+3}^{-\frac{3}{2}}+(m-2)\,C_{m+1}^{-\frac{3}{2}}]\frac{C_{l+2}^{-\frac{3}{2}}}{(1-\mu^2)^2}\mathrm{d}\mu, \quad (55)$$

which by the orthogonality relation (49) is non-vanishing only if $l=m+1$ or $m-1$.

Considering next the integral (52) and rewriting it in the form,

$$\frac{3}{(m+2)(m+1)\,m(m-1)}\int_{-1}^{+1}(1-\mu^2)^2P_{m,\mu,\mu}P_{l,\mu}\,\mathrm{d}\mu, \quad (56)$$

we transform the integrand, successively, in the following manner:

$$(1-\mu^2)\,[2\mu P_{m,\mu}-m(m+1)\,P_m]\,P_{l,\mu}$$

$$=2P_{m,\mu}[\mu(1-\mu^2)\,P_{l,\mu}]-m(m+1)\,(1-\mu^2)\,P_mP_{l,\mu}; \quad (57)$$

and the first term in the second line, after an integration by parts, becomes

$$-2P_m[-l(l+1)\,\mu P_l+(1-\mu^2)\,P_{l,\mu}]. \quad (58)$$

After these transformations the integrand becomes

$$2l(l+1)\,\mu P_mP_l-[m(m+1)+2]\,(1-\mu^2)\,P_mP_{l,\mu}; \quad (59)$$

or by making use of the known recurrence relations among the Legendre polynomials we obtain,

$$\frac{l(l+1)}{2l+1}\{[m(m+1)+2(l+2)]\,P_{l+1}-[m(m+1)-2(l-1)]\,P_{l-1}\}\,P_m. \quad (60)$$

The vanishing of the integral if the selection rule (53) is violated is now manifest.

In atomic theory, besides the Laporte and the selection rules, one has, in recent times formulated a '*propensity rule*' which, among other things, states that 'in light absorption, the transitions $l\rightarrow l+1$ are strongly favoured over the transitions $l\rightarrow l-1$' (Fano 1985); and the basis for this propensity essentially derives from the sum rules. Propensity operates also in the coupling between the polar and the axial modes we are presently considering. But it derives from a different cause: in the manner in which the behaviour of X at the origin is affected by the polar source-terms.

It is known that the functions N_l, L_l, V_l, and W_l which describe the polar perturbations at the origin have the behaviour r^l (I, eq. (82)). Therefore the inhomogeneous terms on the right-hand side of the equation for a particular l have the behaviour

$$r^3S_l\sim r^{l+3}; \quad (61)$$

while X_{l+1} and X_{l-1} have the behaviours (cf. I, eq. (152)):

$$X_{l+1}=X^{(0)}r^{l+3}+X^{(2)}r^{l+5}+\dots, \quad (62)$$

$$X_{l-1}=X^{(0)}r^{l+1}+X^{(2)}r^{l+3}+X^{(3)}r^{l+5}+\dots. \quad (63)$$

It follows that the source-terms contribute directly to the coefficient $X^{(2)}$ in the expansion for X_{l+1} while it contributes only to $X^{(3)}$ in the expansion for X_{l-1}: the coupling of the polar perturbation belonging to l affects the function X_{l+1} more strongly than it does X_{l-1}; in other words, a propensity in favour of the transition $l \rightarrow l+1$.

6. The equations governing sextupole axial oscillations induced by quadrupole polar oscillations

In this case,

$$C_5^{-\frac{3}{2}} = \tfrac{3}{8}\mu(1-\mu^2)^2; \quad P_2 = \tfrac{1}{2}(3\mu^2-1), \tag{64}$$

and the evaluation of the angular integrals (51) and (52) is immediate. And we find for the equation coupling the sextupole axial oscillations with the quadrupole polar oscillations:

$$\mathscr{D}_3 X_3 = 8r^2\,\mathrm{e}^{\mu_2-\nu}[\varpi_{,r}(2W_2+N_2+5L_2+6V_2)+2\varpi(Q-1)\,\nu_{,r}\,W_2], \tag{65}$$

where

$$\mathscr{D}_3 = \partial_{rr} - \frac{\mathrm{e}^{2\mu_2}}{r}[2+r^2(\epsilon-p)-6M(r)/r]\,\partial_r - \frac{10}{r^2}\mathrm{e}^{2\mu_2} + \sigma^2\,\mathrm{e}^{2(\mu_2-\nu)}. \tag{66}$$

The solutions for W_2, N_2, etc., have at the origin the behaviour (cf. I, eq. (82))

$$(W_2, N_2, L_2, V_2) = (W_2^{(0)}, N_2^{(0)}, L_2^{(0)}, V_2^{(0)})\,r^2 + O(r^4). \tag{67}$$

The solutions obtained in the manner described in Paper I, §7 are 'normalized' so that for all assigned σ^2,

$$L_2^{(0)} = 1. \tag{68}$$

In using these particular solutions for W_2, etc., it will be convenient to use for ϖ the standard solution (9) having at the origin the behaviour,

$$\varpi = 1 + \varpi_2 r^2 + \ldots; \tag{69}$$

and if ϖ_c denotes the value of ϖ at the origin, we can rewrite equation (65) in the form,

$$\mathscr{D}_3 X_3 = 8L_2^{(0)}\,\varpi_c\,\mathrm{e}^{\mu_2-\nu}r^2\,[\varpi_{,r}(2W_2+N_2+5L_2+6V_2)+2\varpi\nu_{,r}(Q-1)\,W_2], \tag{70}$$

with the understanding that the functions W_2, etc., are normalized with the choice $L_2^{(0)} = 1$ and that ϖ denotes the standard solution. By 'absorbing' $L_2^{(0)}$ in the amplitude of X, by writing $L_2^{(0)}X_3$ in place of X_3, we shall obtain the equation

$$\mathscr{D}_3 X_3 = 8\varpi_c\,\mathrm{e}^{\mu_2-\nu}r^2[\varpi_{,r}(2W_2+N_2+5L_2+6V_2)+2\varpi\nu_{,r}(Q-1)\,W_2]. \tag{71}$$

At $r=0$, X_3 has the series expansion

$$X_3 = X_3^{(0)}r^5 + X_3^{(2)}r^7 + O(r^9), \tag{72}$$

where (cf. I, eq. (152))

$$X_3^{(2)} = \tfrac{1}{18}\{X_3^{(0)}[5(\tfrac{5}{3}\epsilon_0 - p_0) - \sigma_0^2]\} + 8\varpi_c\,\mathrm{e}^{-\nu_0}[2\varpi_2(2W_2^{(0)}+N_2^{(0)}+5+6V_2^{(0)}) + 2a(Q_0-1)\,W_2^{(0)}]\}, \tag{73}$$

and $a = p_0 + \frac{1}{3}\epsilon_0$ (cf. I, eq. (52)).

With the aid of the expansion (72) (with $X_3^{(2)}$ given by equation (73) and $X_3^{(0)}$ set equal to one) equation (70) can be integrated forward to the boundary at $r = r_1$. The integration can then be continued into the vacuum outside the star with the equation (cf. equation (48)),

$$\frac{\mathrm{d}^2 Z_3}{\mathrm{d}r_*^2} + \sigma^2 Z_3 - \frac{6\mathrm{e}^{2\nu}}{r^3}(2r-M)\,Z_3 = \frac{48J}{r^3}\varpi_c\,\mathrm{e}^{2\nu}(N_2+5L_2+6V_2), \tag{74}$$

where in accordance with the choice of the standard solution for ϖ, J has the value (cf. equation (11)),

$$J = \tfrac{1}{6}r_1^4(\varpi_{,r})_{r=r_1}. \tag{75}$$

And for the simultaneous integration of the functions N_2, L_2, and V_2 we may use the equations (52)–(54) given in *M.T.* on p. 148.

The integration of equation (70) beyond $r = r_1$ must be started with the values (cf. I, eq (153), (154)),

$$Z_3(r = r_1) = \lim_{r \to r_1 - 0} (X_3/r) \tag{76}$$

and

$$Z_{3,r_*}(r = r_1) = \left(1 - \frac{2M}{r_1}\right) \lim_{r \to r_1 - 0} \left[\frac{1}{r^2}(rX_{3,r} - X_3)\right]. \tag{77}$$

It follows from the solutions for N_l, L_l, and V_l given in *M.T.* (p. 152) that at infinity they are $O(1/r)$ so that the source-term is $O(1/r^4)$. Therefore to $O(1/r^2)$ the source terms have no effect on the asymptotic behaviour of Z_3 that follows from I, eq. (155), namely,

$$\begin{aligned} Z_3 \to &+\left\{\alpha_0 - \frac{6\beta_0}{r\sigma} - \frac{3}{2r^2\sigma^2}(10\alpha_0 - M\sigma\beta_0) + \ldots\right\}\cos\sigma r_* \\ &-\left\{\beta_0 + \frac{6\alpha_0}{r\sigma} - \frac{3}{2r^2\sigma^2}(10\beta_0 + M\sigma\alpha_0) + \ldots\right\}\sin\sigma r_*. \end{aligned} \tag{78}$$

As in Paper 1, the integration of equation (74) must be continued to a sufficiently large r that by matching with the expansion (78) we can determine α_0 and β_0. And again as in Paper I, the characteristic frequencies of the quasi-normal modes can be determined by the behaviour of $\alpha_0^2 + \beta_0^2$ as a function of σ. If the axial modes we are considering are resonantly scattered, $\alpha_0^2 + \beta_0^2$ should exhibit a deep minimum at some determinate frequency σ_0 with the behaviour

$$\alpha_0^2 + \beta_0^2 = \text{const.}\,[(\sigma - \sigma_0)^2 + \sigma_{\mathrm{i}}^2]. \tag{79}$$

A comparison of the results of the numerical integrations with this formula will determine the real and the imaginary parts of the quasi-normal mode.

The propensity rule

As an example of the operation of the propensity rule, we shall consider the excitation of the quadrupole axial oscillations by the sextupole polar perturbations. In this case

$$C_4^{-\frac{3}{2}} = \tfrac{3}{8}(1 - \mu^2)^2 \quad \text{and} \quad P_3 = \tfrac{1}{2}\mu(5\mu^2 - 3); \tag{80}$$

and we find that in place of equation (70) we now have

$$\mathscr{D}_2 X_2 = -\tfrac{8}{7}\varpi_{\mathrm{c}}\, \mathrm{e}^{\mu_2 - \nu} r^2 [\varpi_{,r}(2W_3 + N_3 + 5L_3) + 2\varpi(Q-1)\,\nu_{,r}\, W_3], \tag{81}$$

where a factor $L_2^{(0)}$ has been absorbed in the amplitude X_2 and it is assumed that ϖ denotes the standard solution and that the solutions W_3, N_3, and L_3 have at the origin the behaviour

$$(W_3, N_3, L_3) = (W_3^{(0)}, N_3^{(0)}, 1)\, r^3 + O(r^5), \tag{82}$$

i.e. the solutions are so 'normalized' that $L_2^{(0)} = 1$.

From equation (81) it readily follows that at $r = 0$, we have the series expansion

$$X_2 = r^4 + \tfrac{1}{14}[4(\epsilon_0 - p_0) - \sigma_0^2]\, r^6 + X_3^{(0)} r^8 + \dots, \tag{83}$$

where $X_3^{(0)}$ includes the term

$$-\tfrac{8}{7}\varpi_c\, e^{-\nu_0}\, [2\varpi_2(2W_3^{(0)} + N_3^{(0)} + 5) + 2a(Q_0 - 1)\, W_3^{(0)}]. \tag{84}$$

Therefore, the sextupole polar perturbations affect the behaviour of X_2 at the origin only in $O(r^6)$, not $O(r^4)$ – an example of the operation of the propensity rule.

7. An illustrative example

Since we are concerned in this paper only with establishing that the scattering of axial gravitational waves by a slowly rotating star must exhibit resonance, we have considered, for the purposes of illustration, the excitation of the sextupole axial oscillations by the dominant quadrupole perturbations for the same polytropic model as in Paper I, §9. By the procedure described in §6 (and in greater detail in I, §§8 and 9), we have determined, by direct numerical integration of the relevant equations, the real and the imaginary parts of the characteristic frequency, $\sigma = \sigma_0 + i\sigma_i$, of the quasi-normal modes for some assigned values of $8\varpi_c$. The results are listed in table 1. For comparison it may be noted that for the quadrupole polar oscillations (I, eq. (103)),

$$\sigma_0 = 0.3248 \quad \text{and} \quad \sigma_i = 1.00 \times 10^{-4}. \tag{85}$$

With regard to the accuracy of the values of σ_0 and σ_i listed in table 1: the values of σ_0 are reliable to the last significant figure retained; with respect to σ_i, on account of the difficulty of attaining the necessary numerical accuracy, they are reliable only to within 20–30%.

We must address ourselves at this point to the questions that have been raised on the validity of determining σ_0 and σ_i by locating the vertex and ascertaining the curvature of the parabola that fits the numerically evaluated values of $\alpha_0^2 + \beta_0^2$ in the manner described in §6. It has also been argued that an appeal to 'Breit and Wigner' is not persuasive! We have, however, been able to show (Chandrasekhar & Ferrari 1991*b*) that σ_0 and σ_i determined *as afore described are the same, one will obtain by a direct determination of* $\sigma_0 + i\sigma_i$ *as the solution of a problem in complex characteristic values of the underlying system of differential equations with the boundary conditions at the centre and at the surface of the star* (as stated in I, §7) *together with the additional requirement that there are no incoming waves from infinity*. The proof of the identity of the values of σ_0 and σ_i (so long as $\sigma_i \ll \sigma_0$) determined by the two algorisms provides some additional identities which are useful as checks on the derived values of σ_i. The checks are fulfilled by the solution for the quasi-normal mode determined in Paper I; and we have also used them in testing the reliability of the values of σ_i listed in table 1. The details of this investigation will be published shortly.

We turn next to the range of $8\varpi_c$ that is consistent with the requirement that the rotation be slow. Quite generally the requirement is (J. Friedman, personal communication)

$$\Omega r_1 \ll \surd(M/r_1). \tag{86}$$

With Ω given by equation (11), the required condition is

$$\varpi_c(\varpi + \tfrac{1}{3} r \varpi_{,r})_{r=r_1} \ll \surd(M/r_1^3). \tag{87}$$

Table 1. *The real and the imaginary parts of the characteristic frequencies of the sextupole axial modes of oscillation induced by the quadrupole perturbations for various assigned amplitudes,* $L_2^{(0)}/X_3^{(0)}$ $(=g)$

$8\varpi_c q$	σ_0	σ_i
0.6	0.12417	0.16×10^{-5}
0.9	0.3064	0.11×10^{-5}
1.0	0.3438	0.15×10^{-5}

For the polytropic model considered in this paper

$$\left.\begin{aligned} M = 0.20541, \quad r_1 = 1.57641, \\ (\varpi)_{r=r_1} = 1.60806, \quad \text{and} \quad (\varpi_{,r})_{r=r_1} = 0.210367.\end{aligned}\right\} \tag{88}$$

Inserting these values in (87), we obtain

$$8\varpi_c \ll 1.064. \tag{89}$$

For values for $8\varpi_c q$ for which we have determined σ_0 and σ_i, with the exception of the last entry for $8\varpi_c q = 1$, are in the range of possibilities.

Quite generally, for more realistic neutron star models (than the one considered in this paper) $M \sim r_1$ and one has the requirement,

$$\Omega r_1 \ll 1 \tag{90}$$

and we obtain (in place in (89))

$$8\varpi_c \ll 2.95. \tag{91}$$

It is important to observe in this connection that the terms in ϖ affect the expansion of X_3 at the origin only in the second term leaving the amplitude $X_3^{(0)}$ unchanged (cf. equation (73)).

8. Concluding remarks

As we have stated at the outset, the excitation of resonant axial modes of oscillation in slowly rotating stars by their coupling with polar modes, derived from the dragging of the inertial frame, has no counterpart in the Newtonian theory: it is a purely general relativistic effect. But a related question requires clarification. If the coupling provided by ω affects the axial modes, should not the polar modes be similarly affected? The answer is in the affirmative. Thus, for X given by a solution for a non-rotating star (i.e. by a solution of I, eq. (145)) equation (25) shows how X will induce changes of order Ω in the various quantities that describe the polar perturbations of a spherical star. Together with equation (24) (in which the effect of ω is absent by virtue of its independence on θ) and the remaining field equations listed in Appendix A, we shall be led to a system of linear equations that will make determinate the changes of order Ω in $\delta\nu$, $\delta\psi$, etc. The analysis will be more complex than for the axial modes treated in this paper – as the analysis of polar perturbations always is! But we need anticipate no insuperable difficulty in obtaining the desired solution. The interest in the solution, however, is perhaps only marginal: the effect of ω on the polar modes will only be to augment the phenomenon of resonant oscillation already present. In contrast, the effect of ω on the axial modes is to excite a manner of oscillation that was not there.

The mathematical problem formulated in the context of equation (25) and solved in this paper applies to a physical situation that may be described as follows: In view of the decoupling of the polar and the axial modes of a spherical star, we may suppose that initially the star is oscillating, simultaneously, in *both* modes with arbitrarily assigned amplitudes, i.e. arbitrarily (and independently) assigned values for the amplitude $L^{(0)}$ of the polar mode (cf. equation (68)) and the coefficient $X^{(0)}$ of r^{l+2} in the expansion I, eq. (152) for the axial mode. Let a star so oscillating in both modes be set in slow rotation with an angular velocity Ω. The resulting coupling of the two modes, provided by ω, will effect changes of order Ω in them. The effect on the axial mode has been analysed in this paper: it results, as we have shown, in the excitation of resonant oscillations. The effect on the polar mode, in augmenting the resonant oscillation already present, has not been analysed.

One may perhaps infer that the resonant axial modes of oscillation will be excited in neutron stars following 'glitches' which some of them exhibit at irregular intervals. Glitches, which have sometimes been described as 'star-quakes', may set the star oscillating in its dominant quadrupole mode and possibly in some axial modes as well. Since neutron stars are known to be rotating, the accompanying $l = 2$ polar perturbations will excite, consistently with the selection rule $\Delta l = \pm 1$, the sextupole axial mode favoured by the propensity rule. It is for this reason that this case was treated in some detail in §7. And as it was pointed out, the very much longer damping times for these axial modes may well be a decisive factor in their eventual detection.

This paper suggests several further investigations. Reference to one particular investigation has been made in §7. In conjunction with the earlier Paper I a host of other problems suggests itself. We hope to consider some of them in the near future.

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Note added on 7 December 1990

A referee has pointed out to us that, since both polar and axial gravitational waves are present as $r \to \infty$, the minimization of $\alpha_0^2 + \beta_0^2$, as carried out in the text (§6, 7) may not correspond to establishing a quasi-normal mode in the strict sense. It seems to us, however, that since the amplitudes at infinity are of order $O(1)$ for the polar waves and of $O(\Omega)$ for the axial waves, the minimization with respect to the two contributions to the flux of radiation at infinity may be carried out independently. The matter, however, clearly requires a more careful investigation that is beyond the scope of this paper.

Appendix A. The equations of hydrodynamics and the remaining field equations for the general time-dependent axisymmetric space-time

In Paper II §10 the equations of hydrodynamics for the standard time-dependent axisymmetric space-time were written down for the case when the only non-vanishing components of the four-velocity are $u_{(0)}$, $u_{(2)}$, and $u_{(3)}$. It will be useful to have the equations written down when $u_{(1)}$ is also non-zero. They are needed in the general context of this paper; and they will supplement and complete Paper III.

By making use of the Ricci rotation coefficients listed in *M.T.* (p. 82, eq. (91)), as in Paper II, we now find (when $u_{(1)} \neq 0$) that II, equations (94) are enlarged to the set:

$$\begin{aligned}(\epsilon+p)^{-1}\{\mathrm{e}^{-\nu}(1-u_{(0)}^2)\,p_{,0}+\mathrm{e}^{-\mu_2}u_{(0)}\,u_{(2)}\,p_{,2}+\mathrm{e}^{-\mu_3}u_{(0)}\,u_{(3)}\,p_{,3}\}\\ = +\mathrm{e}^{-\nu}[u_{(0)}\,u_{(0),0}+\mu_{2,0}\,u_{(2)}^2+\mu_{3,0}\,u_{(3)}^2]\\ -\mathrm{e}^{-\mu_2}u_{(2)}\,[\nu_{,2}\,u_{(0)}+u_{(0),2}]-\mathrm{e}^{-\mu_3}u_{(3)}\,[\nu_{,3}\,u_{(0)}+u_{(0),3}]\\ +\mathrm{e}^{-\nu}u_{(1)}\,[\psi_{,0}\,u_{(1)}-\mathrm{e}^{\psi-\mu_2}Q_{20}\,u_{(2)}-\mathrm{e}^{\psi-\mu_3}Q_{30}\,u_{(3)}],\end{aligned} \tag{A 1}$$

$$\begin{aligned}(\epsilon+p)^{-1}\{\mathrm{e}^{-\mu_2}(1+u_{(2)}^2)\,p_{,2}-\mathrm{e}^{-\nu}u_{(2)}\,u_{(0)}\,p_{,0}+\mathrm{e}^{-\mu_3}u_{(2)}\,u_{(3)}\,p_{,3}\}\\ = -\mathrm{e}^{-\mu_2}\,[u_{(2)}\,u_{(2),2}+\nu_{,2}\,u_{(0)}^2-\mu_{3,2}\,u_{(3)}^2]\\ +\mathrm{e}^{-\nu}u_{(0)}\,[\mu_{2,0}\,u_{(2)}+u_{(2),0}]-\mathrm{e}^{-\mu_3}u_{(3)}\,[\mu_{2,3}\,u_{(2)}+u_{(2),3}]\\ +\mathrm{e}^{-\mu_2}u_{(1)}\,[\psi_{,2}\,u_{(1)}-\mathrm{e}^{\psi-\nu}Q_{20}\,u_{(0)}+\mathrm{e}^{\psi-\mu_2}Q_{23}\,u_{(3)}],\end{aligned} \tag{A 2}$$

$$[2 \rightleftharpoons 3], \tag{A 3}$$

$$\begin{aligned}(\epsilon+p)^{-1}u_{(1)}\,[\mathrm{e}^{-\nu}u_{(0)}\,p_{,0}-\mathrm{e}^{-\mu_2}u_{(2)}\,p_{,2}-\mathrm{e}^{-\mu_3}u_{(3)}\,p_{,3}]\\ = -\mathrm{e}^{-\nu}u_{(0)}\,[u_{(1),0}+\psi_{,0}\,u_{(1)}]+\mathrm{e}^{-\mu_2}u_{(2)}\,[u_{(1),2}+\psi_{,2}\,u_{(1)}]\\ +\mathrm{e}^{-\mu_3}u_{(3)}\,[u_{(1),3}+\psi_{,3}\,u_{(3)}].\end{aligned} \tag{A 4}$$

First, we observe that the zero-order terms in equation (A 2), describing the stationary state, give:

$$(\epsilon+p)^{-1}p_{,2} = -\nu_{,2}\,u_{(0)}^2+u_{(1)}\,[\psi_{,2}\,u_{(1)}+\mathrm{e}^{\psi-\nu}\omega_{,2}\,u_{(0)}]. \tag{A 5}$$

By making use of the relations (cf. equation (15))

$$u_{(0)}^2-u_{(1)}^2 = 1 \quad \text{and} \quad u_{(1)} = \mathrm{e}^{\psi-\nu}\varpi, \tag{A 6}$$

we can reduce equation (A 5) to the form

$$p_{,2} = (\epsilon+p)\left(-\nu_{,2}+\frac{VV_{,2}}{1-V^2}\right) = (\epsilon+p)\,(\ln u^0)_{,2}, \tag{A 7}$$

where

$$u^0 = \mathrm{e}^{-\nu}/\sqrt{(1-V^2)}, \tag{A 8}$$

denotes the *tensor* component. Similarly, from equation (A 3) (obtained from equation (A 2) by interchanging the indices (2) and (3)) we shall obtain,

$$p_{,3} = (\epsilon+p)\left(-\nu_{,3}+\frac{VV_{,3}}{1-V^2}\right) = (\epsilon+p)\,(\ln u^0)_{,3}. \tag{A 9}$$

Equations (A 7) and (A 9) are the standard equations of hydrostatic equilibrium of uniformly rotating stars (cf. Paper III, eq. (81)).

Next, by ignoring all terms of the second and higher orders in equation (A 2), we obtain:

$$(\epsilon+p)^{-1}p_{,2}+\nu_{,2}\,u_{(0)}^2-\psi_{,2}\,u_{(1)}^2-\mathrm{e}^{\psi-\nu}\varpi_{,2}\,u_{(1)}\,u_{(0)} = \mathrm{e}^{\mu_2-\nu}u_{(0)}\,u_{(2),0}-\mathrm{e}^{\psi-\nu}u_{(0)}\,u_{(1)}\,q_{2,0,0}. \tag{A 10}$$

The linearized version of this equation, appropriate for a linear perturbation theory, is

$$\delta\{(\epsilon+p)^{-1}p_{,2}+\nu_{,2}\,u_{(0)}^2-\psi_{,2}\,u_{(1)}^2-\mathrm{e}^{\psi-\nu}\varpi_{,2}\,u_{(1)}\,u_{(0)}\} = \mathrm{e}^{\mu_2-\nu}u_{(0)}\,u_{(2),0}-\mathrm{e}^{\psi-\nu}u_{(1)}\,u_{(0)}\,q_{2,0,0}. \tag{A 11}$$

Considering next equation (A 1) and ignoring terms of the second and higher orders, we directly obtain

$$(\epsilon+p)^{-1}[-\mathrm{e}^{-\nu}u_{(1)}^2 p_{,0}+\mathrm{e}^{-\mu_2}u_{(0)}u_{(2)}p_{,2}+\mathrm{e}^{-\mu_3}u_{(0)}u_{(3)}p_{,3}]$$
$$=\mathrm{e}^{-\nu}u_{(0)}u_{(0),0}-\mathrm{e}^{-\mu_2}u_{(2)}u_{(0),2}-\mathrm{e}^{-\mu_3}u_{(3)}u_{(0),3}-\mathrm{e}^{-\mu_2}\nu_{,2}u_{(0)}u_{(2)}-\mathrm{e}^{-\mu_3}\nu_{,3}u_{(0)}u_{(3)}$$
$$+\mathrm{e}^{-\nu}\psi_{,0}u_{(1)}^2-(\mathrm{e}^{\psi-\nu-\mu_2}\varpi_{,2}u_{(2)}+\mathrm{e}^{\psi-\nu-\mu_3}\varpi_{,3}u_{(3)})u_{(1)}. \qquad \text{(A 12)}$$

With the aid of the equilibrium equations (A 8) and (A 9), equation (A 12) can be reduced to the form

$$-\mathrm{e}^{-\nu}u_{(1)}^2\left(\frac{p_{,0}}{\epsilon+p}+\psi_{,0}\right)-\mathrm{e}^{-\nu}u_{(0)}u_{(0),0}+\mathrm{e}^{-\mu_2}u_{(2)}u_{(0),2}+\mathrm{e}^{-\mu_3}u_{(3)}u_{(0),3}$$
$$+\tfrac{1}{2}u_{(0)}V^2\left[\mathrm{e}^{-\mu_2}u_{(2)}\ln\left(\frac{u_{(1)}^2}{\varpi}\right)_{,2}+\mathrm{e}^{-\mu_3}u_{(3)}\ln\left(\frac{u_{(1)}^2}{\varpi}\right)_{,3}\right]=0. \qquad \text{(A 13)}$$

Similarly, from equation (A 4) we obtain

$$-\mathrm{e}^{-\nu}u_{(0)}\left[\frac{p_{,0}}{\epsilon+p}+\psi_{,0}+(\ln u_{(1)})_{,0}\right]+\left[\mathrm{e}^{-\mu_2}u_{(2)}\left(\ln\frac{u_{(1)}^2}{\varpi}\right)_{,2}+\mathrm{e}^{-\mu_3}u_{(3)}\left(\ln\frac{\mu_{(1)}^2}{\varpi}\right)_{,3}\right]=0. \qquad \text{(A 14)}$$

Eliminating $[(\epsilon+p)^{-1}p_{,0}+\psi_{,0}]$ from equations (A 13) and (A 14) and remembering that $u_{(0)}V^2=u_{(1)}^2/u_{(0)}$, we obtain

$$-u^{(a)}u_{(0),(a)}+\mathrm{e}^{-\nu}u_{(1)}^2(\ln u_{(1)})_{,0}-\frac{u_{(1)}^2}{2u_{(0)}}\left[\mathrm{e}^{-\mu_2}u_{(2)}\left(\ln\frac{u_{(1)}^2}{\varpi}\right)_{,2}+\mathrm{e}^{-\mu_3}u_{(3)}\left(\ln\frac{u_{(1)}^2}{\varpi}\right)_{,3}\right]=0. \qquad \text{(A 15)}$$

Expanding

$$\ln(u_{(1)}^2/\varpi)=2\ln u_{(1)}-\ln\varpi, \qquad \text{(A 16)}$$

and simplifying, we find after some further elementary reductions:

$$u^{(a)}(u_{(0)}^2-u_{(1)}^2)_{,(a)}+u_{(1)}^2[\mathrm{e}^{-\mu_2}u_{(2)}(\ln\varpi)_{,2}+\mathrm{e}^{-\mu_3}u_{(3)}(\ln\varpi)_{,3}]=0. \qquad \text{(A 17)}$$

The first term in this equation clearly vanishes and we are left with

$$\mathrm{e}^{-\mu_2}u_{(2)}(\ln\varpi)_{,2}+\mathrm{e}^{-\mu_3}u_{(3)}(\ln\varpi)_{,3}=0, \qquad \text{(A 18)}$$

or

$$\mathrm{e}^{-\nu}u_{(0)}(\ln\varpi)_{,0}-u^{(a)}(\ln\varpi)_{,(a)}=0. \qquad \text{(A 19)}$$

This last equation is equivalent to

$$\frac{\delta\varpi}{\varpi}=\frac{\delta V}{V}-(\psi-\nu), \qquad \text{(A 20)}$$

which is an expression of the continued validity of the relation,

$$\varpi=V\mathrm{e}^{-(\psi-\nu)}. \qquad \text{(A 21)}$$

Finally, we may note that since the equation,

$$u^{(a)}{}_{|(a)}=\mathrm{e}^{-\nu}[u_{(0),0}+(\psi+\mu_2+\mu_3)_{,0}u_{(0)}]$$
$$-\mathrm{e}^{-\mu_2}[u_{(2),2}+(\psi+\nu+\mu_3)_{,2}u_{(2)}]-\mathrm{e}^{-\mu_3}[u_{(3),3}+(\psi+\nu+\mu_2)_{,3}u_{(3)}], \qquad \text{(A 22)}$$

is the same whether or not $u_{(1)}=0$, the equations (cf. II, eqs (93) and (104))

$$\epsilon_{,(a)}+(\epsilon+p)u^{(a)}{}_{|(a)}=0, \qquad \text{(A 23)}$$

and
$$[Nu^{(a)}]_{|(a)} = N_{,(a)}\, u^{(a)} + Nu^{(a)}{}_{|(a)} = 0, \tag{A 24}$$
are unaffected and eqs I, (95), (105), and (108)–(110) continue to be valid under the present more general circumstances.

The remaining field equations

We have already written down in §3 (equations (24)–(28)) the linearized version of five of the necessary field equations. The remaining two equations derived from the linearization of the R_{23} and the G_{22} equations are:

$$\begin{aligned}\delta[(\nu+\psi)_{,2,3}+\nu_{,2}\nu_{,3}+\psi_{,2}\psi_{,3}-(\psi+\nu)_{,2}\mu_{2,3}-(\psi+\nu)_{,3}\mu_{3,2})]\\ = -2\mathrm{e}^{2\psi-2\nu}\omega_{,2}\,\omega_{,3}\,\delta\psi+\tfrac{1}{2}\mathrm{e}^{-\psi-\nu}(\mathrm{e}^{\mu_3-\mu_2}X_{,2}\,\omega_{,2}-\mathrm{e}^{\mu_2-\mu_3}X_{,3}\,\omega_{,3})\\ -2\frac{\epsilon+p}{1-V^2}\varpi\,(\mathrm{e}^{2\psi+\mu_2-\nu}\xi_2\,\omega_{,3}+\mathrm{e}^{2\psi+\mu_3-\nu}\xi_3\,\varpi_{,2});\end{aligned} \tag{A 25}$$

$$\begin{aligned}&\mathrm{e}^{-2\mu_2}\{\delta\nu_{,2}(\psi+\mu_3)_{,2}+\nu_{,2}\,\delta(\psi+\mu_3)_{,2}+\delta\psi_{,2}\mu_{3,2}+\psi_{,2}\,\delta\mu_{3.2}\\ &\quad -2\delta\mu_2\,[\nu_{,2}(\psi+\mu_3)_{,2}+\psi_{,2}\mu_{3,2}]\}\\ &\quad +\mathrm{e}^{-2\mu_3}\{\delta(\psi+\nu)_{,3,3}+\nu_{,3}\,\delta(\nu-\mu_3)_{,3}+(\nu-\mu_3)_{,3}\,\delta\nu_{,3}+\psi_{,3}\,\delta(\psi+\nu-\mu_3)_{,3}\\ &\quad +\delta\psi_{,3}(\psi+\nu-\mu_3)_{,3}-2\delta\mu_3[(\psi+\nu)_{,3,3}+\psi_{,3}\psi_{,3}+\nu_{,3}\nu_{,3}-\nu_{,3}\mu_{3,3}+\psi_{,3}(\nu-\mu_3)_{,3}]\}\\ &\quad +\sigma^2\,\mathrm{e}^{-2\nu}\,(\delta\psi+\delta\mu_3)-2\delta p = \mathrm{e}^{2\psi-2\nu}\,[\mathrm{e}^{-2\mu_2}(\omega_{,2})^2+\mathrm{e}^{-2\mu_3}(\omega_{,3})^2]\,\delta\psi\\ &\quad +\tfrac{1}{2}\mathrm{e}^{2\psi-2\nu}\,[\mathrm{e}^{-2\mu_2}(\omega_{,2})^2\,\delta\mu_3-\mathrm{e}^{-2\mu_3}(\omega_{,3})^2\,\delta\mu_2]+\frac{1}{2\sqrt{-g}}(X_{,3}\,\omega_{,2}+X_{,2}\,\omega_{,3})\\ &\quad +2\frac{\epsilon+p}{1-V^2}\varpi\,\mathrm{e}^{2\psi-2\nu}(\mathrm{e}^{\nu-\mu_2}\xi_2\,\omega_{,2}-\mathrm{e}^{\nu-\mu_3}\,\xi_3\,\omega_{,3}).\end{aligned} \tag{A 26}$$

Appendix B. Some details on the numerical procedures adopted

For the illustrative example considered in §7, the equations that had to be integrated are equations (70) and (74) for an assigned $8\varpi_{\mathrm{c}}$ and different values of σ. The source term on the right-hand sides of the equations is a superposition of the two linearly independent solutions satisfying the boundary conditions described in Paper I, §7.

As explained in Appendix D of Paper I, the range of integration inside the star, $r_0 < r < r_1$ (where r_0 is the initial starting point, chosen to be 0.01, and r_1 is the radius of the star), was divided into four intervals:

$$\left.\begin{array}{llll}(1) & r_0 \leqslant r \leqslant r_a; & r_a = 1.55, & \mathrm{d}r = 10^{-4};\\ (2) & r_a \leqslant r \leqslant r_b; & r_b = 1.5764, & \mathrm{d}r = 10^{-6};\\ (3) & r_b \leqslant r \leqslant r_c; & r_c = 1.5764123, & \mathrm{d}r = 10^{-8};\\ (4) & r_c \leqslant r \leqslant r_1; & r_1 = 1.5764124853, & \mathrm{d}r = 10^{-11}.\end{array}\right\} \tag{B 1}$$

These step sizes were chosen to guarantee that at the boundary of the star the functions and their derivatives were accurate up to the fifth significant figure. We therefore need to discuss only the accuracy of the axial functions Z and $Z_{,r}$ which provide the initial data for the integration outside the star. We shall consider two cases:

$$(\varpi_{\mathrm{c}} = 0.6,\ \sigma = 0.12417) \quad \text{and} \quad (\varpi_{\mathrm{c}} = 1.0, \sigma = 0.3438). \tag{B 2}$$

When the integrations were carried out with the step sizes specified in (B 1), the resulting values of Z and $Z_{,r}$ at $r = r_1$ were:

$$\left.\begin{aligned} 8\varpi_c = 0.6; \quad Z = 0.34008477, \quad Z_{,r} = -0.39928523; \\ 8\varpi_c = 1.0; \quad Z = 0.50468000, \quad Z_{,r} = -0.68065908. \end{aligned}\right\} \tag{B 3}$$

If the integrations are repeated with one-half of all the step sizes the result was:

$$\left.\begin{aligned} 8\varpi_c = 0.6; \quad Z = 0.34008407, \quad Z_{,r} = -0.39928641; \\ 8\varpi_c = 1.0; \quad Z = 0.50468656, \quad Z_{,r} = -0.68064822. \end{aligned}\right\} \tag{B 4}$$

The fractional changes in the derived values Z and $Z_{,r}$ are:

$$\left.\begin{aligned} 8\varpi_c = 0.6; \quad \Delta Z/Z \sim 2.1\times10^{-6}, \quad \Delta Z_{,r}/Z_{,r} \sim 3.0\times10^{-6}; \\ 8\varpi_c = 1.0; \quad \Delta Z/Z \sim 1.3\times10^{-5}, \quad \Delta Z_{,r}/Z_{,r} \sim 1.6\times10^{-5}. \end{aligned}\right\} \tag{B 5}$$

Therefore, by integrating the equations using the step sizes specified in (B 1), the functions Z and $Z_{,r}$ are accurate up to the fifth significant figure. The integration outside the star was carried out with a step $\mathrm{d}r_* = 0.05$. If the step is reduced to $\mathrm{d}r_* = 0.005$, the final values of $\alpha_0^2+\beta_0^2$ change only in the fourth significant figure; but this change does not affect the real and the imaginary parts of the characteristic frequencies listed in table 1.

Finally, it may be noted that if equation (3) for the standard solution for ϖ was integrated with the same step sizes (B 1), it was found that

$$\varpi = 1.60806609 \quad \text{and} \quad \varpi_{,r} = 0.21036682; \tag{B 6}$$

and when the step sizes were halved,

$$\varpi = 0.60806609 \quad \text{and} \quad \varpi_{,r} = 0.21036682. \tag{B 7}$$

The integration is therefore accurate to the eighth significant figure.

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Apropos of the *Note added on 7 December 1990*, the solution obtained in the text may first be clarified.

Since the polar perturbations of the spherical star that induces the resonant axial mode belongs to a frequency σ, that is different from its resonant frequency σ_0 ($= 0.324845$ for the polytropic model considered in §7) these polar waves are both ingoing and outgoing. In other words, by scattering polar waves off of a slowly rotating star with varying σ (keeping the ratio q of the amplitudes of the polar and axial perturbations fixed), we find that the emergent induced axial waves of order Ω reveal, as we have seen, the presence of a resonance state.

Since the problem presented by equation (70) is one in two-channel scattering (the two channels being the axial and the polar modes), it is clear that a whole range of problems with different initial conditions can be formulated. The problem with the greatest physical interest is perhaps the following: A spherical star initially oscillating in its quadrupole quasi-normal mode with its resonant frequency σ_0 is set in slow rotation. What is the amplitude of the induced emergent outgoing axial waves of the same frequency? The solution to this problem can be found as follows.

Absorbing $8L_2^{(0)}\varpi_c$ in the amplitude X_3, we first find the particular integral of the equation (cf. equation (70))

$$\mathscr{D}_3 X_3 = e^{\mu_2-\nu}\pi^2[\varpi_{,r}(2W_2+N_2+5L_2+6V_2)+2\varpi\nu_{,r}(Q-1)W_2], \tag{92}$$

which at $r = 0$ has the series expansion (cf. equation (73))

$$X_3 = e^{-\nu_0}[2\varpi_2(2W_2^{(0)}+N_2^{(0)}+5+6V_2^{(0)})+2a(Q_0-1)W_2^{(0)}]r^7+O(r^9). \tag{93}$$

The asymptotic behaviour at infinity of the solution so obtained, will have the form

$$\alpha_0\cos\sigma r_* - b_0\sin\sigma r_* = \tfrac{1}{2}[(\alpha_0+i\beta_0)e^{i\sigma r_*}+(\alpha_0-i\beta_0)e^{-i\sigma r_*}], \tag{94}$$

where α_0 and β_0 are determinate constants. For this solution we therefore have both incoming and outgoing axial waves. We can annul the incoming radiation by adding to the *particular integral* having the behaviour (94), a suitable multiple, $(X+iY)$ say, of the solution of the homogenous equation.

The solution of the homogenous equation as determined in I, §11 has at infinity the asymptotic behaviour (cf. I, eq. (155))

$$A_0\cos\sigma r_* - B_0\sin\sigma r_* = \tfrac{1}{2}[(A_0+iB_0)e^{i\sigma r_*}+(A_0-iB_0)e^{-i\sigma r_*}], \tag{95}$$

where we have written A_0 and B_0 in place of α_0 and β_0. The required condition for the vanishing of the incoming radiation is therefore,

$$(X+iY)(A_0+iB_0)+\alpha_0+i\beta_0 = 0, \tag{96}$$

or

$$X = -\frac{(\alpha_0 A_0+\beta_0 B_0)}{(A_0^2+B_0^2)} \quad \text{and} \quad Y = -\frac{(\beta_0 A_0-\alpha_0 B_0)}{(A_0^2+B_0^2)} \tag{97}$$

The resulting purely outgoing flux of gravitational radiation will be determined by

$$\tfrac{1}{2}[(X+iY)(A_0-iB_0)+\alpha_0-i\beta_0]e^{-i\sigma r_*} = iY(A_0+iB_0)e^{-i\sigma r_*}; \tag{98}$$

and the flux itself will be given by

$$[8\varpi_c L_2^{(0)}]^2 Y^2(A_0^2+B_0^2), \tag{99}$$

where we have restored the factor that had been suppressed.

For the polytropic model considered in §7 (and in Paper I) we find:

$$\left.\begin{aligned} \alpha_0 &= -0.1129\times10^5, \quad \beta_0 = +0.4504\times10^4,\\ A_0 &= +0.1506\times10^5, \quad B_0 = -0.6012\times10^4,\\ X &= 0.7496 \quad \text{and} \quad Y = 1.7\times10^{-4}; \end{aligned}\right\} \tag{100}$$

and

$$Y^2(A_0^2+B_0^2) \approx 8.0. \tag{101}$$