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On the non-radial oscillations of a star II. Further amplifications

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The two algorisms that one uses for determining the complex characteristic frequencies, $\sigma = \sigma_0 + \mathrm{i}\sigma_\mathrm{i}$, belonging to the quasi-normal modes of oscillations of a star, are shown to be equivalent. In the first algorism, one searches directly in the complex σ-plane to satisfy the requirement that at infinity there are only outgoing waves (and no incoming waves). In the second algorism, one restricts oneself to real σs and selects the solution with the minimum flux of gravitational radiation at infinity.

1. Introduction

The present paper is of the nature of a supplement to two recent papers (Chandrasekhar & Ferrari 1991*a*, *b*; these papers will be referred to here after as Papers I and II) on the non-radial oscillations of a star: it is addressed to proving the equivalence of the two algorisms (to be described presently) by which one may obtain the complex characteristic values belonging to the quasi-normal modes (and to justify in particular the use of the 'Breit–Wigner' formula in the context of one of them).

2. The two algorisms for determining the characteristic frequencies belonging to the quasi-normal modes

The problem of determining the complex frequencies belonging to quasi-normal modes is strictly one in characteristic values of a system of linear differential equations together with boundary conditions. The equations are

(*a*) equations I, (72)–(75) governing the four metric scalars L, N, V, and W for the interior of the star, $r \leqslant r_1$; and

(*b*) the Zerilli equation, I, (93) for the vacuum exterior to the star, $r \geqslant r_1$.

The boundary conditions are the following.

(*a*) At the centre, $r = 0$: (i) freedom from singularity in all the physical variables.

(*b*) At the surface of the star, $r = r_1$: (ii) $W = W_{,r} = 0$ (equations I, (87) and (88)), and (iii) the continuity of Z and $Z_{,r}$ (equations I, (95)–(97)).

(*c*) At infinity, $r \to \infty$: (iv) we have *only* outgoing waves and *no* incoming waves.

(It may be noted that the boundary conditions (ii) ensure that the perturbation, δp, in the pressure vanishes at $r = r_1$ and a requirement of Einstein's vacuum equation is satisfied.)

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It is this last boundary condition (iv), which, by its complex nature, requires the characteristic values to be complex.

Of the two algorisms (or methods) for determining the characteristic values, one is *direct* and the other is *indirect*. In the direct method one simply searches in the complex σ-plane with the guide that the real part of σ must be in the neighbourhood of the value given by the Newtonian theory and that the imaginary part of σ must be small compared with its real part (see (12) below). In the indirect method adopted in Papers I and II one restricts oneself to real σs and satisfies the boundary conditions (i)–(iii) for all σ. The asymptotic behaviour of the Zerilli function, Z, for $r \to \infty$ will then be given by (I, eqn (98) and II, eqn (78))

$$Z \to \alpha(\sigma) \cos \sigma r_* - \beta(\sigma) \sin \sigma r_*, \tag{1}$$

where $\alpha(\sigma)$ and $\beta(\sigma)$ are two functions of σ determined (as described in I, §8) by the integration of the Zerilli equation from the surface of the star at $r = r_1$ to a sufficiently large value of r. The flux of gravitational radiation at infinity determined by the average value of Z^2 and proportional to $\alpha^2 + \beta^2$ exhibits (as one finds) a deep minimum at some value of σ ($= \sigma_0$ say) and in its neighbourhood the behaviour (I, equation (99)),

$$\alpha^2 + \beta^2 = \text{const.} \times [(\sigma - \sigma_0)^2 + \sigma_{\text{i}}^2]. \tag{2}$$

One then interprets σ_0 and σ_{i} as the real and the imaginary parts of the required complex characteristic-value by appealing to the 'Breit–Wigner' formula. But the Breit–Wigner formula as commonly derived and stated (as for example in Landau & Lifshitz (1977, pp. 603–611)) is not directly applicable to the circumstances described: what one shows in fact is that, consistently with the rules of quantum mechanics (such as the unitarity of the S-matrix, etc.) and in the neighbourhood of a *posited* resonance (at an energy level E_0) the scattering cross section must have the behaviour,

$$\text{cross section} = \frac{\text{const.}}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}, \tag{3}$$

where Γ is the half-width of the resonance. As thus formulated, there is no direct connection between equation (3), for the cross section, and equation (2), for the variation of the flux of radiation at infinity. Instead of justifying on physical grounds (as one has) the interpretation of σ_0 and σ_{i} in an empirically established relation (2), we shall provide an explicit demonstration of the equivalence of the two algorisms that we have described for determining the complex characteristic frequency in the context of the particular problem we have considered.

3. The equivalence of the two algorisms

In the context of the problem considered in Paper I, it is on the solution of the Zerilli equation (I, eqn (93)) that we must impose the complex boundary condition that there are no incoming waves from infinity.

In the indirect algorism for determining $\sigma_0 + \text{i}\sigma_{\text{i}}$, we restrict ourselves to real solutions for real σ of the equation,

$$\frac{\text{d}^2 Z}{\text{d}r_*^2} + (\sigma^2 - V)\, Z = 0, \tag{4}$$

for the potential barrier V defined in I, eqn (94). As a basis for the solutions of equation (4), we shall take the two linearly independent solutions, Z_1 and Z_2, which, at infinity, have the asymptotic behaviours,

$$Z_1 \to \cos \sigma r_* + O(r_*^{-1}) \quad \text{and} \quad Z_2 \to \sin \sigma r_* + O(r_*^{-1}). \tag{5}$$

To emphasize the dependence of Z_1 and Z_2 on σ (as a parameter) we shall, on occasions, write

$$Z_1(\sigma\,; r_*) \quad \text{and} \quad Z_2(\sigma\,; r_*). \tag{6}$$

The Wronskian of these two solutions is

$$[Z_1, Z_2]_{r_*} = Z_{1,r_*} Z_2 - Z_{2,r_*} Z_1 = \text{const.}(-\sigma). \tag{7}$$

The solution for Z one obtains by integrating equation (4) forward from the surface of the star at $r = r_1$ is of the form (cf. I, eqn (98))

$$Z = \alpha(\sigma)\, Z_1(\sigma\,; r_*) - \beta(\sigma)\, Z_2(\sigma\,; r_*), \tag{8}$$

where $\alpha(\sigma)$ and $\beta(\sigma)$ are (as described earlier), functions of σ determined by the integrations for different initially assigned real values of σ. One then identifies σ at the minimum of $(\alpha^2+\beta^2)$ as σ_0 and determines σ_{i} by the curvature of the $(\alpha^2+\beta^2, \sigma)$-curve at σ_0.

In the direct algorism for solving the characteristic-value problem, one seeks a complex solution of equation (4) for a complex σ. Letting

$$\sigma_{\mathrm{c}} = \sigma + \mathrm{i}\sigma_{\mathrm{i}} \quad \text{and} \quad Z_{\mathrm{c}} = Z + \mathrm{i}Z_{\mathrm{i}}, \tag{9}$$

in equation (4) and separating the real and the imaginary parts of the equation, we obtain the pair of equations,

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}r_*^2} - VZ + (\sigma^2 - \sigma_{\mathrm{i}}^2)\, Z - 2\sigma\sigma_{\mathrm{i}}\, Z_{\mathrm{i}} = 0, \tag{10}$$

$$\frac{\mathrm{d}^2 Z_{\mathrm{i}}}{\mathrm{d}r_*^2} - VZ_{\mathrm{i}} + (\sigma^2 - \sigma_{\mathrm{i}}^2)\, Z_{\mathrm{i}} + 2\sigma\sigma_{\mathrm{i}}\, Z = 0. \tag{11}$$

The case of interest in our present context is when

$$\sigma_{\mathrm{i}} \ll \sigma. \tag{12}$$

For the quadrupole oscillations of the polytropic model considered in Paper I, for example, $\sigma_{\mathrm{i}}/\sigma \approx 10^{-3}$. (It may also be noted here, that in the standard derivations of the Breit–Wigner formula, an assumption equivalent to (12) is implicit.) One readily infers from equations (10) and (11), that Z_{i} is $O(\sigma_{\mathrm{i}})$ when the condition (12) is satisfied. Therefore, writing

$$Z_{\mathrm{i}} = \sigma_{\mathrm{i}}\, Y \tag{13}$$

and neglecting all quantities of $O(\sigma_{\mathrm{i}}^2)$, we find that equations (10) and (11) reduce to

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}r_*^2} + (\sigma^2 - V)\, Z = 0 \tag{14}$$

and

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}r_*^2} + (\sigma^2 - V)\, Y + 2\sigma Z = 0. \tag{15}$$

If $Z(\sigma;r_*)$ is one of a family of real solutions of equation (14), continuous in the parameter σ, then (as one can readily verify),

$$Y(\sigma;r_*)=\frac{\partial}{\partial\sigma}Z(\sigma;r_*) \tag{16}$$

provides a corresponding family of solutions of equation (15). The solutions for $Y(\sigma;r_*)$ given by equation (16) have the *unique* property that they satisfy *all* the boundary conditions that one may have imposed on $Z(\sigma;r_*)$: the solutions have none of the arbitrariness that the general solution,

$$Y=2Z_1\int^{r_*}ZZ_2\,dr_*-2Z_2\int^{r_*}ZZ_1\,dr_*, \tag{17}$$

of the inhomogeneous equation (15), has.

Since the real part of Z_c satisfies the same equation (4) (for the real part of σ_c) and satisfies, also, the necessary boundary conditions (i)–(iii), we may consistently choose for it the same family of solutions (8). The corresponding family of solutions for Y is given by

$$Y=\alpha'(\sigma)\,Z_1(\sigma;r_*)-\beta'(\sigma)\,Z_2(\sigma;r_*)+\alpha(\sigma)\frac{\partial}{\partial\sigma}Z(\sigma;r_*)-\beta(\sigma)\frac{\partial}{\partial\sigma}Z(\sigma;r_*), \tag{18}$$

where primes denote differentiations with respect to σ. It is important to note in this connection that $Z_1(\sigma;r_*)_{,\sigma}$ and $Z_2(\sigma;r_*)_{,\sigma}$ have, at infinity, the asymptotic behaviours:

$$\left[\frac{\partial}{\partial\sigma}Z_1(\sigma;r_*)\right]_{r_*\to\infty}=-\left\{\lim_{\Delta\sigma\to0}\left[\frac{\sin\left(\frac{1}{2}\Delta\sigma r_*\right)}{\frac{1}{2}\Delta\sigma}\right]\sin\sigma r_*\right\}_{r_*\to\infty} \tag{19}$$

and

$$\left[\frac{\partial}{\partial\sigma}Z_2(\sigma;r_*)\right]_{r_*\to\infty}=+\left\{\lim_{\Delta\sigma\to0}\left[\frac{\sin\left(\frac{1}{2}\Delta\sigma r_*\right)}{\frac{1}{2}\Delta\sigma}\right]\cos\sigma r_*\right\}_{r_*\to\infty}. \tag{20}$$

The required solution for Z_c belonging to $\sigma+i\sigma_i$ is, therefore,

$$Z_c=\alpha(\sigma)\,Z_1-\beta(\sigma)\,Z_2+i\sigma_i\,[\alpha'(\sigma)\,Z_1-\beta'(\sigma)\,Z_2+\alpha(\sigma)\,Z_{1,\sigma}-\beta(\sigma)\,Z_{2,\sigma}], \tag{21}$$

where we have suppressed the distinguishing parentheses, $(\sigma;r_*)$, as no longer necessary. Under the circumstances we are presently considering the problem, namely $\sigma\gg\sigma_i$, we can ignore the terms in $Z_{1,\sigma}$ and $Z_{2,\sigma}$ (with the factors α and β) in comparison with the terms in Z_1 and Z_2 (with the factors α' and β'): in the case considered below for example (cf. equation (29) and the entries in table 1) while $\alpha'\approx27$ and $\beta'\approx-94$, α and β are of order 10^{-2}. (Some additional remarks concerning the neglect of the forms in $Z_{1,\sigma}$ and $Z_{2,\sigma}$ are made at the end of this section.)

Ignoring then the terms in $Z_{1,\sigma}$ and $Z_{2,\sigma}$ we have for the asymptotic behaviour of the solution (21) at infinity:

$$\begin{aligned}z_c&\to(\alpha+i\sigma_i\,\alpha')\cos\sigma r_*-(\beta+i\sigma_i\,\beta')\sin\sigma r_*\\&=\tfrac{1}{2}[(\alpha-\sigma_i\,\beta')+i(\beta+\sigma_i\,\alpha')\,e^{i\sigma r_*}+\tfrac{1}{2}[(\alpha+\sigma_i\,\beta')-i(\beta-\sigma_i\,\alpha')]\,e^{-i\sigma r_*}.\end{aligned} \tag{22}$$

The condition that there be no incoming waves from infinity requires that the coefficient of $e^{i\sigma r_*}$ in the foregoing expansion vanishes, i.e.

$$\alpha-\sigma_i\,\beta'=0\quad\text{and}\quad\beta+\sigma_i\,\alpha'=0. \tag{23}$$

Table 1. *The values of* α, β, *and* $(\alpha^2+\beta^2)$ *for some assigned values of* σ *for the polytropic model considered in Paper I*

σ	α	β	$\alpha^2+\beta^2$
0.324825	-0.9198×10^{-2}	0.4658×10^{-2}	0.1063×10^{-3}
0.324845	-0.9734×10^{-2}	0.2750×10^{-2}	0.1023×10^{-3}
0.324865	-0.1026×10^{-1}	0.8620×10^{-3}	0.1061×10^{-3}

It follows that

$$\sigma_i = \alpha/\beta' = -\beta/\alpha', \tag{24}$$

and

$$\alpha\alpha' + \beta\beta' = 0. \tag{25}$$

Therefore, *the real part* (σ_0) *of the complex characteristic frequency belonging to the quasi-normal mode occurs where* $(\alpha^2+\beta^2)$, *as a function of* σ, *attains its minimum: and the imaginary part* (σ_i) *is given by* $\alpha/\beta' = -\beta/\alpha'$ *at the minimum*. The first part of this statement is in accord with the prescription that was adopted: the second part provides an alternative formula (new in this connection) for the imaginary part.

Letting $(\alpha^2+\beta^2)_0$ denote the minimum value of $\alpha^2+\beta^2$ at σ_0, then in its neighbourhood, we have the Taylor expansion,

$$\begin{aligned}\alpha^2+\beta^2 &= (\alpha^2+\beta^2)_0 + \tfrac{1}{2}(\alpha^2+\beta^2)''_{\sigma=\sigma_0}(\sigma-\sigma_0)^2 \\ &= \tfrac{1}{2}(\alpha^2+\beta^2)''_{\sigma=\sigma_0}\left[(\sigma-\sigma_0)^2 + 2\frac{(\alpha^2+\beta^2)_0}{(\alpha^2+\beta^2)''_{\sigma=\sigma_0}}\right].\end{aligned} \tag{26}$$

In the algorism adopted the second term in the square brackets is interpreted as σ_i^2:

$$\sigma_i^2 = 2\frac{(\alpha^2+\beta^2)_0}{(\alpha^2+\beta^2)''_{\sigma=\sigma_0}}. \tag{27}$$

The alternative formulae for σ_i given in equation (24), requires:

$$2\frac{(\alpha^2+\beta^2)_0}{(\alpha^2+\beta^2)''_{\sigma=\sigma_0}} = \left(\frac{\alpha}{\beta'}\right)^2 = \left(\frac{\beta}{\alpha'}\right)^2 = -\frac{\alpha\beta}{\alpha'\beta'}. \tag{28}$$

The origin of this last relation is not clear. But we have verified it numerically, with sufficient precision, in particular cases. With the numerical data provided in table 1, we find:

$$\left.\begin{aligned}&\sigma_0 = 0.324845, \quad \alpha' = -26.6556, \quad \beta' = -94.9041, \\ &\sigma_i = 1.026\times10^{-4}\,(=\alpha/\beta') \quad \text{and} \quad \sigma_i = 1.032\times10^{-4}\,(=-\beta/\alpha'),\end{aligned}\right\} \tag{29}$$

while by a parabolic fitting to the $(\alpha^2+\beta^2, \sigma)$-curve in Paper I, we found (I, eqn (102))

$$\sigma_0 = 0.3248 \quad \text{and} \quad \sigma_i = 1.026\times10^{-4}. \tag{29$'$}$$

It will be noted that a very high precision in the numerical integrations is needed before we can attain even a few percent reliability in the derived values of σ_i.

It is perhaps worth emphasizing that the relations (24) and (25) are valid only in an asymptotic sense in the limit $|\sigma_i|/|\sigma| \to 0$. The limit is to be understood in the sense that while the terms in $Z_{1,\sigma}$ and $Z_{2,\sigma}$ (with the coefficients α and β) in equation (21) must eventually dominate over the terms in Z_1 and Z_2 (with the coefficients α' and β'), the asymptotic behaviour (22) will be established long before the neglected terms begin to dominate; and this will increasingly be the case as $|\sigma_i|/|\sigma| \to 0$. Or, as Bernard Schutz has expressed more clearly (in a personal communication), the assumed behaviour will be valid for 'the field far away at a finite value of r_* which is in the far zone but not so far that the exponential growth has taken over: for $|\sigma_i| \ll |\sigma|$ such a radius should exist'.

3. Concluding remark

We may first draw attention to the fact that the analysis of §3 provides an entirely general method for determining the characteristic values, $E_0 - \frac{1}{2}\mathrm{i}\Gamma$, of any quantal system that scatters radiation such as, for example, a radioactive nucleus that decays with the emission of an α-particle. But the analysis leaves open the question whether equation (2), together with the relations (24) and (25), and the Breit–Wigner formula (3) represent alternative formulations of the same underlying physical problem. In the appendix to this paper, Roland Winston addresses this question.

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Appendix A. The Breit–Wigner formula

In this appendix we establish a connection between the calculational methods of this paper and conventional discussions of scattering. In particular, we make explicit the relationship between equation (2) and the so-called Breit–Wigner resonances derived from quasi-stationary states. In describing scattering by a centrally symmetric field, the wave function and the scattering amplitude for some fixed value of angular momentum, l, are treated as analytic functions of the complex energy E (in the notation of this paper $\sigma = \text{const.}\sqrt{E}$, and the constant of proportionality is real and positive); and one makes this function single valued by cutting the complex plane along the positive real E-axis. This choice of cut makes $\mathrm{Re}\sqrt{(-E)}$ everywhere positive. By writing the asymptotic solution for large values of r_* in the form,

$$Z(E) \approx A(E)\,\mathrm{e}^{-\mathrm{i}\sigma r_*} + B(E)\,\mathrm{e}^{+\mathrm{i}\sigma r_*}, \tag{A 1}$$

one readily verifies that $A(E) = B^*(E)$ for real $E > 0$. In other words, the solution $Z(E)$ is real for real E. The scattering amplitude follows in the usual way:

$$S_l = \mathrm{e}^{2\mathrm{i}\delta_l} = (-1)^{l+1}(B^*/B), \tag{A 2}$$

where δ_l is the phase shift.

One knows now that the scattering amplitude may have poles which from the relation (A 2) corresponds to zeros of the function $B(E)$. From equation (A 1) the solution at a pole possesses only outgoing waves. Suppose we postulate that such a pole in fact exists and lies close to the positive real σ-axis at some 'complex' energy $E = E_0 - \frac{1}{2}\mathrm{i}\Gamma$; then we may expand about the zero:

$$B(E) \approx b(E_0)(E - E_0 + \tfrac{1}{2}\mathrm{i}\Gamma). \tag{A 3}$$

From this expansion, the Breit–Wigner resonance formula for the scattering cross-section and phase shift readily follow. Equation (25) (in the text) corresponds to

$|B(E)|^2$ attaining its minimum value along the real axis at $E = E_0$, while the 'width' (at which twice the minimum value is attained) determines Γ. Further, the complex 'energy' $E = E_0 - \frac{1}{2}\mathrm{i}\Gamma$ is interpreted as the eigenvalue of a quasi-stationary state or resonance which decays exponentially with a lifetime $\tau = \hbar/\Gamma$ since the probability $|\psi|^2$ of the initial state decays with time as $\mathrm{e}^{-\Gamma t/\hbar}$.

Conversely, if the characteristic value problem is solved for a range of real σ and the function $|B(E)|^2$ is found to have a minimum at E_0 in the manner described in the text, then at E_0,

$$BB^{*\prime} + B^*B' = 0 \quad \text{or} \quad B'/B = -(B^{*\prime}/B^*) \tag{A 4}$$

(where B' denotes $\mathrm{d}B/\mathrm{d}E$). Setting aside the trivial case $B' = 0$, we conclude that B'/B is imaginary at E_0, say, $-2\mathrm{i}/\Gamma$. *The logarithmic derivative is purely imaginary at* E_0. Then we may analytically continue the function B in the complex plane and expand in the vicinity of E_0:

$$B(E) \approx B(E_0)\,[1 + (B'/B)_{E_0}(E - E_0)] \approx B(E_0)\,[1 + 2\mathrm{i}(E - E_0)/\Gamma]. \tag{A 5}$$

Therefore, a zero of the function $B(E)$ will lie just below E_0 at $E = E_0 - \frac{1}{2}\mathrm{i}\Gamma$ provided Γ is positive and small compared with E_0. All the results leading to the foregoing discussion of the Breit–Wigner formula follow from this property.

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