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S. Chandrasekhar

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Hydromagnetic turbulence. II. An elementary theory

By S. Chandrasekhar, F.R.S.

University of Chicago

In this paper an elementary theory of hydromagnetic turbulence is developed along the lines of Heisenberg's theory of ordinary turbulence. The basic physical idea underlying this theory is to conceive the transformation of the kinetic energy at a particular wave number into kinetic energy and magnetic energy at higher wave numbers, and similarly, the transformation of the magnetic energy at a given wave number into kinetic energy and magnetic energy of higher wave numbers, as a cascade process which can be visualized in terms of suitably defined coefficients of eddy viscosity and eddy resistivity. The resulting equations for the cascade process have been solved under stationary conditions in the limiting case of zero viscosity and infinite electrical conductivity. It is shown that in this limiting case there exist two distinct modes of turbulence; these have been distinguished as the velocity mode and the magnetic mode respectively. In both modes equipartition between the two forms of energy prevail among the largest eddies present (i.e. as $k \to 0$); and the spectrum of both the kinetic energy and the magnetic energy have a Kolmogoroff behaviour for $k \to 0$. The two modes differ in their behaviour for $k \to \infty$. In the velocity mode the ratio of the magnetic energy to the kinetic energy tends to zero among the smallest eddies present (i.e. as $k \to \infty$), while in the magnetic mode the same ratio tends to about 2·6 as $k \to \infty$. The bearing of these results on the possible character of the interstellar magnetic fields is briefly discussed.

1. Introduction

In this paper an elementary theory of hydromagnetic turbulence will be developed which is similar in its scope and content to Heisenberg's (1948) theory of ordinary turbulence. The object in developing such a theory is to clarify in a simple scheme as to what is essentially taking place in hydromagnetic turbulence.

2. The cascade process in ordinary turbulence

The basic physical idea in current visualizations of the phenomenon of turbulence is that of the cascade of energy from the larger to the smaller eddies (cf. Onsager 1945). More precisely, one supposes that the energy is supplied to the large-scale eddies, i.e. to the Fourier components of the velocity field which are of the smallest wave numbers; and that this energy cascades to the higher wave numbers till it is finally dissipated as heat by viscosity in the region of the highest wave numbers. One thus envisages a transition probability Q, $(v, k'; v, k'')$ which will give the rate at which the kinetic energy, $\frac{1}{2}F(k')$, per unit volume and per unit wave number interval at the wave number k' will be transformed into kinetic energy (again, per unit wave number interval) at k'' $(>k')$. This transfer of energy from k' to k'' takes place as a result of the non-linear coupling between the different Fourier components; and this in turn is due to the inertial term in the equation of motion. In addition to this inertial exchange of energy, there will, of course, be the direct dissipation of energy by viscosity at each wave number; the amount of this latter dissipation is given by $\nu F(k)\,k^2$. Therefore, considering the rate of change of energy at any particular wave number k, we have

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \int_0^k Q(v, k'; v, k)\,\mathrm{d}k' - \int_k^\infty Q(v, k; v, k'')\,\mathrm{d}k'' - \nu F(k)\,k^2. \tag{1}$$

The first term on the right-hand side of this equation represents the gain in the energy density at k by the transformation of the energy at all lower wave numbers; the second term represents the loss due to the transformation of the energy at k into energy of higher wave numbers; and the last term represents the direct dissipation of the energy at k by viscosity.

In Heisenberg's theory the process underlying the inertial exchange of energy and the transition probability $Q(v, k'; v, k'')$ is described in terms of an eddy viscosity $\nu(k'')$ and one writes

$$Q(v, k'; v, k'') = F(k')\,k'^2\,\nu(k''). \tag{2}$$

From dimensional considerations one then writes for $\nu(k)$ the expression

$$\nu(k) = \kappa\sqrt{\frac{F(k)}{k^3}}, \tag{3}$$

where κ is a numerical constant. The expression

$$Q(v, k'; v, k'') = \kappa F(k')\,k'^2\sqrt{\frac{F(k'')}{k''^3}}, \tag{4}$$

which provides the basis for Heisenberg's theory, is thus obtained.

3. The cascade process in hydromagnetic turbulence

It would appear that in generalizing to hydromagnetic turbulence the considerations set out in § 2, we should introduce, in addition to $Q(v, k'; v, k'')$, the transition probabilities $Q(v, k'; h, k'')$ and $Q(h, k'; v, k'')$ to describe the transformation of the kinetic energy and the magnetic energy at the wave number k' into magnetic energy, respectively, kinetic energy at the wave number k'' ($>k'$). [Note that because of the linearity of Maxwell's equations, there can be no *direct* transformation of the magnetic energy at one wave number into magnetic energy at another wave number.]

Allowing for the transformation of the magnetic energy with $k' < k$ into kinetic energy in the wave number k and the transformation of the kinetic energy at k into magnetic energy in wave numbers exceeding k, we can now rewrite equation (1) in the form

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \int_0^k Q(v, k'; v, k)\,\mathrm{d}k' + \int_0^k Q(h, k'; v, k)\,\mathrm{d}k' - \int_k^\infty Q(v, k; v, k'')\,\mathrm{d}k'' - \int_k^\infty Q(v, k; h, k'')\,\mathrm{d}k'' - \nu F(k)\,k^2. \tag{5}$$

Similarly, if $G(k)$ denotes the spectrum of the turbulent magnetic field, we can by considering the processes which transform kinetic energy into magnetic energy and vice versa, write

$$\frac{1}{2}\frac{\partial G(k)}{\partial t} = \int_0^k Q(v, k'; h, k)\,\mathrm{d}k' - \int_k^\infty Q(h, k; v, k'')\,\mathrm{d}k'' - \lambda G(k)\,k^2. \tag{6}$$

The occurrence of the first two terms on the right-hand side of equation (6) requires no explanation; the last term represents the direct dissipation of the magnetic energy at k into Joule heat by electrical conductivity.

Some assumptions must now be made regarding the three transition probabilities we have introduced. We shall assume that $Q(v, k'; v, k'')$ will continue to be given by (4). The question then remains as to the manner of choice regarding the two remaining transition probabilities.

Considering first the transformation of the magnetic energy at a particular wave number into kinetic energy of higher wave numbers we may argue as follows: We know that by electrical conductivity, magnetic energy at any wave number can be dissipated as Joule heat and that the amount of this latter dissipation is given by $\lambda G(k)\,k^2$. And even as one pictures the inertial exchange of kinetic energy between two wave numbers k' and k'' ($>k'$) in terms of an eddy viscosity $\nu(k'')$, so we may now picture the transformation of magnetic energy at k' into kinetic energy at k'' in terms of an *eddy resistivity* $\lambda(k'')$. Thus (cf. equation (2))

$$Q(h, k'; v, k'') = G(k')\,k'^2\,\lambda(k''). \tag{7}$$

As to the choice of $\lambda(k)$ there is considerable arbitrariness. It is on this account that so far no serious attempt has been made to develop a theory along the lines of Heisenberg's for hydromagnetics. However, the theory presented in the preceding paper (Chandrasekhar 1955*b*) narrows this arbitrariness; indeed, as we shall see, it would seem to suggest a unique choice.

The equation (Chandrasekhar 1955*a*)

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)Q=-2Q\frac{\partial}{\partial r}D_5 Q, \tag{8}$$

for the defining scalar $Q(r,t)$ of the tensor $\overline{u_i(\mathbf{r}',t')\,u_j(\mathbf{r}'',t'')}$ in the framework of ordinary hydrodynamics is, in the framework of hydromagnetics, replaced by

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)Q=-2Q\frac{\partial}{\partial r}D_5 Q-2H\frac{\partial}{\partial r}D_5 H, \tag{9}$$

where H is the defining scalar of $\overline{h_i(\mathbf{r}',t'),\,h_j(\mathbf{r}'',t'')}$. The process visualized in Heisenberg's theory by $Q(v,k';v,k'')$ is exactly that represented by the term

$$-2Q\,\partial(D_5 Q)/\partial r$$

in the deductive theory leading to equation (8). The process underlying $Q(h,k';v,k'')$ which we are now trying to visualize is represented by the additional term

$$-2H\,\partial(D_5 H)/\partial r$$

in the deductive theory leading to equation (9). Accordingly, it would appear that $Q(h,k';v,k'')$ should be constructed out of $G(k)$ in the same manner as $Q(v,k';v.k'')$ was constructed out of $F(k)$. We shall, therefore, assume that (cf. equation (4))

$$Q(h,k';v,k'')=\kappa G(k')\,k'^2\sqrt{\frac{G(k'')}{k''^3}}, \tag{10}$$

where κ is the *same* numerical constant as in equation (4). By comparison with equation (7) it follows that our choice is

$$\lambda(k)=\kappa\sqrt{\frac{G(k)}{k^3}}. \tag{11}$$

With $Q(h,k';v,k'')$ chosen in the manner (7) and (10) we can 'deduce' an expression for the remaining transition probability $Q(v,k';h,k'')$ as follows:

Writing the equation of motion in the form (see Chandrasekhar (1955*b*) for the notation)

$$\frac{\partial u_i}{\partial t}=-\frac{\partial \varpi}{\partial x_i}-u_j\frac{\partial}{\partial x_j}u_i+h_j\frac{\partial}{\partial x_j}h_i+\nu\nabla^2 u_i, \tag{12}$$

we observe that the expressions (2) and (7) for $Q(v,k';v,k'')$ and $Q(h,k';v,k'')$ essentially interpret the operators

$$-u_j\frac{\partial}{\partial x_j}\quad\text{and}\quad +h_j\frac{\partial}{\partial x_j}, \tag{13}$$

as equivalent to the multiplication of k'^2 times the spectrum of the variable on which the differential operators act by $\nu(k'')$ and $\lambda(k'')$, respectively. By applying this rule to the equation

$$\frac{\partial h_i}{\partial t}=-u_j\frac{\partial}{\partial x_j}h_i+h_j\frac{\partial}{\partial x_j}u_i+\lambda\nabla^2 h_i, \tag{14}$$

we can write, the following expression for $Q(v,k';h,k'')$:

$$Q(v,k';h,k'') = G(k')\,k'^2\,\nu(k'') + F(k')\,k'^2\,\lambda(k''). \tag{15}$$

With $\nu(k)$ and $\lambda(k)$ given by equations (3) and (11) we have

$$Q(v,k';h,k'') = \kappa G(k')\,k'^2\sqrt{\frac{F(k'')}{k''^3}} + \kappa F(k')\,k'^2\sqrt{\frac{G(k'')}{k''^3}}. \tag{16}$$

Finally, substituting for the transition probabilities in accordance with equations (4), (10) and (16) in the cascade equations (5) and (6) we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\partial F(k)}{\partial t} = {} & \kappa\sqrt{\frac{F(k)}{k^3}}\int_0^k \mathrm{d}k'\,F(k')\,k'^2 + \kappa\sqrt{\frac{G(k)}{k^3}}\int_0^k \mathrm{d}k'\,G(k')\,k'^2 \\ & - \kappa F(k)\,k^2\int_k^\infty \mathrm{d}k''\sqrt{\frac{F(k'')}{k''^3}} - \kappa G(k)\,k^2\int_k^\infty \mathrm{d}k''\sqrt{\frac{F(k'')}{k''^3}} \\ & - \kappa F(k)\,k^2\int_k^\infty \mathrm{d}k''\sqrt{\frac{G(k'')}{k''^3}} - \nu F(k)\,k^2\end{aligned} \tag{17}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\partial G(k)}{\partial t} = {} & \kappa\sqrt{\frac{F(k)}{k^3}}\int_0^k \mathrm{d}k'\,G(k')\,k'^2 + \kappa\sqrt{\frac{G(k)}{k^3}}\int_0^k \mathrm{d}k'\,F(k')\,k'^2 \\ & - \kappa G(k)\,k^2\int_k^\infty \mathrm{d}k''\sqrt{\frac{G(k'')}{k''^3}} - \lambda G(k)\,k^2.\end{aligned} \tag{18}$$

Equations (17) and (18) are the basic equations of the present theory.

4. The energy integral in case of stationary turbulence

Adding equations (17) and (18) we obtain after some rearrangement of the terms that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}[F(k)+G(k)] = {} & \kappa\frac{\sqrt{F(k)}+\sqrt{G(k)}}{k^{\frac{3}{2}}}\int_0^k \mathrm{d}k'[F(k')+G(k')]\,k'^2 \\ & - \kappa[F(k)+G(k)]\,k^2\int_k^\infty \frac{\mathrm{d}k''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}] \\ & - \nu F(k)\,k^2 - \lambda G(k)\,k^2.\end{aligned} \tag{19}$$

An alternative form of this equation is

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}[F(k)+G(k)] = {} & -\nu F(k)\,k^2 - \lambda G(k)\,k^2 \\ & - \frac{\partial}{\partial k}\left\{\kappa\int_k^\infty \frac{\mathrm{d}k''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}]\int_0^k \mathrm{d}k'[F(k')+G(k')]\,k'^2.\right.\end{aligned} \tag{20}$$

Integrating this last equation from 0 to k, we obtain

$$\begin{aligned}-\frac{1}{2}\frac{\partial}{\partial t}\int_0^k \mathrm{d}k'[F(k')+G(k')] = {} & \int_0^k \mathrm{d}k'[\nu F(k')+\lambda G(k')]\,k'^2 \\ & + \kappa\int_k^\infty \frac{\mathrm{d}k''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}]\int_0^k \mathrm{d}k'[F(k')+G(k')]\,k'^2.\end{aligned} \tag{21}$$

The quantity on the left-hand side of this equation clearly represents the rate of net flow of energy (kinetic + magnetic) across the spectrum at the wave number k; it is the exact analogue of ϵ_k which Heisenberg introduces in his theory. In stationary turbulence this must be independent of k and we must require that

$$\kappa\int_k^\infty \frac{dk''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}]\int_0^k dk'[F(k')+G(k')]\,k'^2 + \int_0^k dk'[\nu F(k')+\lambda G(k')]\,k'^2 = \text{constant.} \tag{22}$$

This is the energy integral in stationary turbulence.

In the framework of ordinary hydrodynamics equation (22) reduces to

$$\left\{\kappa\int_k^\infty dk''\sqrt{\frac{F(k'')}{k''^3}}+\nu\right\}\int_0^k dk' k'^2 F(k') = \text{constant.} \tag{23}$$

This is the equation which is used for deriving the spectrum of stationary turbulence in Heisenberg's theory (cf. Chandrasekhar 1949). However, in the framework of hydromagnetics the integral (22) will not suffice to determine F and G under conditions of stationary turbulence. We must go back to equations (17) and (18) and consider them when the left-hand sides of both equations are set equal to zero; of the resulting equations, equation (22) is, of course, an integral.

5. Stationary turbulence in case $\lambda = \nu = 0$

The case of greatest interest is when $\lambda = \nu = 0$ and stationary conditions prevail. In this case the relevant equations are

$$\sqrt{\frac{F(k)}{k^3}}\int_0^k dk' F(k')\,k'^2 + \sqrt{\frac{G(k)}{k^3}}\int_0^k dk' G(k')\,k'^2 - F(k)\,k^2\int_k^\infty \frac{dk''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}] - G(k)\,k^2\int_k^\infty \frac{dk''}{k''^{\frac{3}{2}}}\sqrt{F(k'')} = 0 \tag{24}$$

and $$\sqrt{\frac{F(k)}{k^3}}\int_0^k dk' G(k')\,k'^2 + \sqrt{\frac{G(k)}{k^3}}\int_0^k dk' F(k')\,k'^2 - G(k)\,k^2\int_k^\infty \frac{dk''}{k''^{\frac{3}{2}}}\sqrt{G(k'')} = 0. \tag{25}$$

These equations admit the integral (cf. equation (22))

$$\int_k^\infty \frac{dk''}{k''^{\frac{3}{2}}}[\sqrt{F(k'')}+\sqrt{G(k'')}]\int_0^k dk'[F(k')+G(k')]\,k'^2 = \text{constant.} \tag{26}$$

In seeking solutions of equations (24) and (25) we shall consider equation (26) together with the equation

$$\frac{\sqrt{F(k)}-\sqrt{G(k)}}{k^{\frac{3}{2}}}\int_0^k dk'[F(k')-G(k')]\,k'^2 - [F(k)+G(k)]\,k^2\int_k^\infty dk''\sqrt{\frac{F(k'')}{k''^3}} - [F(k)-G(k)]\,k^2\int_k^\infty dk''\sqrt{\frac{G(k'')}{k''^3}} = 0, \tag{27}$$

obtained by taking the difference of equations (24) and (25).

The structure of equations (26) and (27) suggests that we make the following transformations:

$$\left.\begin{aligned} X = \sqrt{F}+\sqrt{G}, \quad Y = \sqrt{F}-\sqrt{G} \\ F+G = \tfrac{1}{2}(X^2+Y^2) \quad \text{and} \quad F-G = XY. \end{aligned}\right\} \tag{28}$$

In terms of X and Y equations (26) and (27) take the forms

$$\int_0^k \mathrm{d}k(X^2+Y^2)\,k^2\int_k^\infty \frac{\mathrm{d}k}{k^{\frac{3}{2}}}X = \text{constant} \tag{29}$$

and $$\frac{Y}{k^{\frac{3}{2}}}\int_0^k \mathrm{d}kXYk^2 - \tfrac{1}{4}(X^2+Y^2)\,k^2\int_k^\infty \frac{\mathrm{d}k}{k^{\frac{3}{2}}}(X+Y) - \tfrac{1}{2}XYk^2\int_k^\infty \frac{\mathrm{d}k}{k^{\frac{3}{2}}}(X-Y) = 0. \tag{30}$$

These equations take more convenient forms if we now let

$$\tau = k^{-4}, \quad \mathrm{d}\tau = -4k^{-5}\mathrm{d}k \tag{31}$$

and
$$\left.\begin{aligned} \xi = Xk^{\frac{7}{2}} = (\sqrt{F}+\sqrt{G})\,k^{\frac{7}{2}} = (\sqrt{F}+\sqrt{G})\,\tau^{-\frac{7}{8}}, \\ \eta = Yk^{\frac{7}{2}} = (\sqrt{F}-\sqrt{G})\,k^{\frac{7}{2}} = (\sqrt{F}-\sqrt{G})\,\tau^{-\frac{7}{8}}. \end{aligned}\right\} \tag{32}$$

The equations then become

$$\int_\tau^\infty (\xi^2+\eta^2)\,\mathrm{d}\tau\,.\int_0^\tau \xi\,\mathrm{d}\tau = \text{constant} \tag{33}$$

and
$$\eta\int_\tau^\infty \xi\eta\,\mathrm{d}\tau = \tfrac{1}{4}(\xi^2+\eta^2)\int_0^\tau (\xi+\eta)\,\mathrm{d}\tau + \tfrac{1}{2}\xi\eta\int_0^\tau (\xi-\eta)\,\mathrm{d}\tau. \tag{34}$$

From equations (33) and (34) it is apparent that if ξ and η are solutions of these equations then so are $A\xi$ and $A\eta$, where A is an arbitrary constant. There is, therefore, no loss of generality if we set the 'constant' in equation (33) equal to 1; in the subsequent analysis we shall assume that this has been so chosen.

It is convenient to introduce the further change of variable

$$\psi = \frac{\eta}{\xi} = \frac{\sqrt{F}-\sqrt{G}}{\sqrt{F}+\sqrt{G}}, \tag{35}$$

or, alternatively,
$$\frac{\sqrt{G}}{\sqrt{F}} = \frac{1-\psi}{1+\psi}. \tag{36}$$

The permissible range of ψ is, therefore,

$$-1 < \psi \leqslant +1. \tag{37}$$

(Note particularly that $\psi = -1$ is excluded.)

Introducing the change of variable (35) we find that equations (33) and (34) take the forms

$$\int_\tau^\infty \xi^2(1+\psi^2)\,\mathrm{d}\tau\,.\int_0^\tau \xi\,\mathrm{d}\tau = 1 \tag{38}$$

and
$$4\int_\tau^\infty \xi^2\psi\,\mathrm{d}\tau = \xi\frac{1+\psi^2}{\psi}\int_0^\tau \xi(1+\psi)\,\mathrm{d}\tau + 2\xi\int_0^\tau \xi(1-\psi)\,\mathrm{d}\tau. \tag{39}$$

(a) The special solutions $\psi = 1$ and $\psi = 2-\sqrt{5}$

Equations (38) and (39) allow two special solutions with ψ = constant. To see this, we first differentiate equation (38) with respect to τ and obtain

$$\int_\tau^\infty \xi^2(1+\psi^2)\,\mathrm{d}\tau = \xi(1+\psi^2)\int_0^\tau \xi\,\mathrm{d}\tau. \tag{40}$$

Accordingly $$\psi = \psi_0 = \text{constant}, \tag{41}$$

will be a solution of equations (39) and (40) provided

$$4\psi_0 = \psi_0^{-1}(1+\psi_0^2)(1+\psi_0)+2(1-\psi_0); \tag{42}$$

for, in this case both equations reduce to

$$\int_\tau^\infty \xi^2\,\mathrm{d}\tau = \xi\int_0^\tau \xi\,\mathrm{d}\tau. \tag{43}$$

On further simplification, equation (42) becomes

$$(\psi_0-1)(\psi_0^2-4\psi_0-1) = 0. \tag{44}$$

The roots of this last equation are

$$1 \quad\text{and}\quad 2\pm\sqrt{5}; \tag{45}$$

of these roots $2+\sqrt{5}$ is outside the permissible range of ψ (cf. equation (37)). The allowed solutions are therefore

$$\psi_0 = 1 \quad\text{and}\quad \psi_0 = 2-\sqrt{5}. \tag{46}$$

For each of these values of ψ_0, equations (39) and (40) reduce to the same equation (43); and it can be readily verified that

$$\xi = \text{constant}\,\tau^{-\frac{2}{3}} \tag{47}$$

satisfies this equation and represents its unique solution. Normalizing the solution (47) so as to satisfy equation (38), we have

$$\xi = \frac{\tau^{-\frac{2}{3}}}{[9(1+\psi_0^2)]^{\frac{1}{3}}} = \frac{k^{\frac{8}{3}}}{[9(1+\psi_0^2)]^{\frac{1}{3}}}. \tag{48}$$

From equations (32) it now follows that

$$\sqrt{F}+\sqrt{G} = \xi k^{-\frac{7}{2}} = \frac{k^{-\frac{5}{6}}}{[9(1+\psi_0^2)]^{\frac{1}{3}}}, \tag{49}$$

and $$\sqrt{F}-\sqrt{G} = \frac{\psi_0 k^{-\frac{5}{6}}}{[9(1+\psi_0^2)]^{\frac{1}{3}}}. \tag{50}$$

Hence $$\left.\begin{aligned} F &= \frac{1}{4}\frac{(1+\psi_0)^2}{[9(1+\psi_0^2)]^{\frac{2}{3}}}k^{-\frac{5}{3}} \\ \text{and}\quad G &= \frac{1}{4}\frac{(1-\psi_0)^2}{[9(1+\psi_0^2)]^{\frac{2}{3}}}k^{-\frac{5}{3}}.\end{aligned}\right\} \tag{51}$$

For $\psi_0 = 1$, $G = 0$ and the solution for F represents the usual Kolmogoroff spectrum. Clearly, this solution must be included as a singular case of hydromagnetic turbulence. But the occurrence of the second solution $\psi_0 = 2-\sqrt{5}$ is unexpected. In this case

$$G = \left(\frac{\sqrt{5}-1}{3-\sqrt{5}}\right)^2 F = \tfrac{1}{2}(3+\sqrt{5})\,F = 2\cdot618F. \tag{52}$$

The energy in the magnetic field is therefore 2·6 times the energy in the turbulent motions. Nevertheless, the spectrum of both F and G follows the Kolmogoroff law.

It is to be particularly noted that the case of the equipartition of energy between the kinetic and the magnetic forms does not occur as a special solution of the equations. *Equipartition requires* $\psi = 0$; *and this is not a solution of equations* (38) *and* (39) *except in the trivial case* $\xi = 0$.

(*b*) *The reduction of equations* (38) *and* (39) *to a system of ordinary differential equations*

Returning to equations (38) and (39) we shall now show how these integral equations can be reduced to a system of ordinary differential equations. The differential equation equivalent to (38) can be directly written down by making use of the following elementary lemma:

Lemma. The integral equation

$$\int_\tau^\infty f(\tau)\,\mathrm{d}\tau \int_0^\tau g(\tau)\,\mathrm{d}\tau = 1 \tag{53}$$

is equivalent to either of the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{f}{g}} = -f \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{g}{f}} = +g. \tag{54}$$

Thus
$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\xi(1+\psi^2)]^{\frac{1}{2}} = -\xi^2(1+\psi^2), \tag{55}$$

or, alternatively,
$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\xi(1+\psi^2)] = -2\xi^{\frac{5}{2}}(1+\psi^2)^{\frac{3}{2}}. \tag{56}$$

Next differentiating equation (39) with respect to τ and rearranging we obtain

$$\frac{\xi^2(1+\psi)^3}{\psi} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\xi\frac{1+\psi^2}{\psi}\right)\int_0^\tau \xi(1+\psi)\,\mathrm{d}\tau + 2\frac{d\xi}{\mathrm{d}\tau}\int_0^\tau \xi(1-\psi)\,\mathrm{d}\tau = 0. \tag{57}$$

Now letting
$$x = \int_0^\tau \xi(1+\psi)\,\mathrm{d}\tau \quad \text{and} \quad y = \int_0^\tau \xi(1-\psi)\,\mathrm{d}\tau, \tag{58}$$

we can rewrite equation (57) in the form

$$x\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\xi\frac{1+\psi^2}{\psi}\right) + 2y\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{\xi^2(1+\psi)^3}{\psi}, \tag{59}$$

while equations (58) are themselves equivalent to the pair of differential equations

$$\frac{dx}{d\tau} = \xi(1+\psi) \tag{60}$$

and

$$\frac{dy}{d\tau} = \xi(1-\psi). \tag{61}$$

Multiplying equations (56), (59) and (61) by $d\tau/dx$ in accordance with equation (60) we obtain the following system of differential equations:

$$x\frac{d}{dx}\left(\xi\frac{1+\psi^2}{\psi}\right)+2y\frac{d\xi}{dx} = -\frac{\xi(1+\psi)^2}{\psi}, \tag{62}$$

$$\frac{d}{dx}[\xi(1+\psi^2)] = -2\frac{[\xi(1+\psi^2)]^{\frac{3}{2}}}{1+\psi} \tag{63}$$

and

$$\frac{dy}{dx} = \frac{1-\psi}{1+\psi}. \tag{64}$$

Equation (63) suggests the substitution

$$\xi(1+\psi^2) = \theta^{-2}. \tag{65}$$

Making this substitution and eliminating ξ, we find

$$\left[(1+\psi^2)^2+4\frac{y}{x}\psi^3\right]x\frac{d\psi}{dx} = \frac{\psi(1+\psi^2)}{1+\psi}\left[(1+\psi)^3-2\frac{x}{\theta}\left(1+\psi^2+2\frac{y}{x}\psi\right)\right], \tag{66}$$

$$\frac{d\theta}{dx} = \frac{1}{1+\psi} \quad \text{and} \quad \frac{dy}{dx} = \frac{1-\psi}{1+\psi}. \tag{67}$$

This system of equations is completed by (cf. equations (60) and (65))

$$\frac{dx}{d\tau} = \frac{1+\psi}{\theta^2(1+\psi^2)}, \tag{68}$$

or

$$\tau = \int_0^x \frac{\theta^2(1+\psi^2)}{1+\psi}\,dx. \tag{69}$$

In writing this last equation we have made use of the boundary condition, $\tau \to 0$ as $x \to 0$, which is justified below (equation (74)).

(c) *The behaviour of the solutions of equations* (66) *and* (67)

An important property of the solutions of equations (66) and (67) is that they allow a homology transformation. Thus, *if* $\psi(x)$, $\theta(x)$ *and* $y(x)$ *are solutions, then so are* $\psi(Ax)$, $A^{-1}\theta(Ax)$ *and* $A^{-1}y(Ax)$, *where* A *is an arbitrary constant.*

Next we may note that the special solutions obtained in § (*a*) above can be recovered from the present equations. For if

$$\psi = \psi_0 = \text{constant}, \tag{70}$$

equations (67) allow the solutions

$$\theta = \theta_0 = \frac{x}{1+\psi_0} \quad \text{and} \quad y = y_0 = \frac{1-\psi_0}{1+\psi_0}x, \tag{71}$$

and the vanishing of the right-hand side of equation (65) requires (if $\psi_0 \neq 0$) that

$$(1+\psi_0)^2 = 2\left(1+\psi_0^2+2\frac{1-\psi_0}{1+\psi_0}\psi_0\right). \tag{72}$$

On simplifying this last equation we find that it is the same as equation (44). We are thus led to the same special solutions as those considered in § (a). [Note, however, that $\psi = 0$, $\theta = y = x$ is also a solution of equations (66) and (67); but as we have already pointed out (cf. remarks at the end of § (a)) $\psi = 0$ is not a solution of the original equations; it has been 'manufactured on the way' by successive differentiations which were necessary to derive equations (66) and (67).]

We shall now show that for a solution of equations (66) and (67) to be physically acceptable, ψ must tend to either of the two special solutions, $\psi_0 = 1$ and $\psi_0 = 2-\surd 5$, as $x \to 0$.

First, we may observe that the physical conditions of the problem require that the integrals defining x and y (equations (58)) exist and are convergent for all finite τ. This is apparent for example from equations (37) and (38): the latter requiring the convergence of the integral

$$\int_\epsilon^\tau \xi \, \mathrm{d}\tau \tag{73}$$

for $\epsilon \to 0$ and the former ensuring the boundedness of ψ. Hence (cf. equation (31))

$$x \to 0 \quad \text{and} \quad y \to 0 \quad \text{as} \quad \tau \to 0 \quad \text{and} \quad k \to \infty. \tag{74}$$

At the same time $\qquad \psi \to \psi_* \quad \text{(say)} \quad (-1 < \psi_* \leqslant 1). \qquad (75)$

Hence (cf. equations (67) and (74))

$$y \to \frac{1-\psi_*}{1+\psi_*}x \quad (x \to 0). \tag{76}$$

Since (cf. equation (67))

$$\left.\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}x} &\to \frac{1}{1+\psi_*} \quad (x \to 0), \\ \theta &\to a + \frac{x}{1+\psi_*}, \end{aligned}\right\} \tag{77}$$

where a is a constant. But the boundedness of ψ as $x \to 0$ requires that $a = 0$; otherwise, it would follow from equations (66) and (75), that

$$\left[(1+\psi_*^2)^2 + 4\frac{1-\psi_*}{1+\psi_*}\psi_*^3\right] x\frac{\mathrm{d}\psi}{\mathrm{d}x} \to \psi_*(1+\psi_*^2)(1+\psi_*)^2 \tag{78}$$

and $\qquad \psi \to \text{constant} \ln x \quad (x \to 0), \qquad (79)$

contradicting the boundedness of ψ at $x = 0$. Hence

$$\theta \to \frac{x}{1+\psi_*} \quad \text{as} \quad x \to 0. \tag{80}$$

[Note that this last implies that $\xi \to \infty$ as $x \to 0$ (cf. equation (65)); but this singularity of ξ at $x = 0$ should be such as to leave the convergence of the integral (73) for $\epsilon \to 0$ unaffected.]

With $\theta \to 0$ as $x \to 0$, it follows from equation (66) that if ψ_* is not one of the values which make the right-hand side of this equation vanish (i.e. if ψ_* is not one of the special values 1 and $2-\sqrt{5}$; we are excluding the values 0 and $2+\sqrt{5}$), then we shall again be led to an equation of the form

$$x\frac{\mathrm{d}\psi}{\mathrm{d}x} \to \text{constant}, \tag{81}$$

and the same contradiction (79) would follow. Thus, we have shown that the physically acceptable solutions of equations (66) and (67) must be such that

$$\left.\begin{aligned} &\psi \to \psi_0 \quad (= 1 \text{ or } 2-\sqrt{5}),\\ &\theta \to \theta_0 = \frac{x}{1+\psi_0} \quad \text{and} \quad y = y_0 = \frac{1-\psi_0}{1+\psi_0}x \quad \text{as} \quad x \to 0. \end{aligned}\right\} \tag{82}$$

The behaviour of the solutions which tend to either of the singular points, $\psi_0 = 1$ and $\psi_0 = 2-\sqrt{5}$, of the differential equations (66) and (67) can be determined by making the substitutions

$$\psi = \psi_0 + \psi_1, \quad \theta = \theta_0 + \theta_1 \quad \text{and} \quad y = y_0 + y_1, \tag{83}$$

and regarding ψ_1, θ_1 and y_1 as small compared to ψ_0, θ_0 and y_0, respectively. In this manner we can linearize the equations (66) and (67) and we obtain after some straightforward reductions that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left[(1+\psi_0^2)^2 + 4\frac{1-\psi_0}{1+\psi_0}\psi_0^3\right]x^2\frac{\mathrm{d}\psi_1}{\mathrm{d}x} + \psi_0(1+\psi_0^2)\frac{\psi_0^2 - 6\psi_0 + 1}{1+\psi_0}x\psi_1\right\}\\ = -\psi_0\frac{1+\psi_0^2}{(1+\psi_0)^2}[(1+\psi_0)^3 - 8\psi_0]\psi_1. \end{aligned} \tag{84}$$

We shall now distinguish the two cases $\psi_0 = 1$ and $\psi_0 = 2-\sqrt{5}$.

Case (i): $\psi_0 = 1$. In this case equation (84) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\psi_1}{\mathrm{d}x} - x\psi_1\right) = 0. \tag{85}$$

The general solution of this equation is

$$\psi = C_1 x + C_2 x^{-1}, \tag{86}$$

where C_1 and C_2 are arbitrary constants. For the physically acceptable solutions $C_2 = 0$; also since ψ cannot exceed 1, C_1 must be chosen negative. Thus, in this case, the solution for ψ at the origin must have the behaviour

$$\psi = 1 - ax + O(x^2) \quad (a > 0 \text{ and } x \to 0). \tag{87}$$

In view of the homology property of the solutions of equations (66) and (67) it will suffice to consider the solution for any particular a; the solution for any other value of a can be derived by a simple homology transformation.

Expressing ψ as a power series in x in which the first two terms are 1 and $-x$ we find that

$$\psi = 1 - x + \tfrac{1}{4}x^2 + \tfrac{1}{24}x^3 + \dots \tag{88}$$

and
$$\left.\begin{aligned}\theta &= \tfrac{1}{2}x(1+\tfrac{1}{4}x+\tfrac{1}{24}x^2-\tfrac{1}{192}x^3+\ldots),\\ y &= \tfrac{1}{4}x^2(1+\tfrac{1}{6}x-\tfrac{1}{48}x^2+\ldots).\end{aligned}\right\} \tag{89}$$

Case (ii): $\psi_0 = 2-\sqrt{5}$. In this case equation (84) reduces to

$$x^2\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}+1{\cdot}216\,542\,4x\frac{\mathrm{d}\psi_1}{\mathrm{d}x}-1{\cdot}751\,864\,5\psi_1=0; \tag{90}$$

and the general solution of this equation is found to be

$$\psi_1 = ax^{\beta}+bx^{-\gamma}, \tag{91}$$

where a and b are arbitrary constants and

$$\beta = 1{\cdot}219\,730\,0 \quad \text{and} \quad \gamma = 1{\cdot}436\,272\,3. \tag{92}$$

For the physically acceptable solutions $b = 0$; also the boundedness of ψ for increasing x requires that a be positive. Again, in view of the homology property of the solutions of equations (66) and (67) it will suffice to consider the solution for any particular a.

Expressing ψ as a power series in x^{β} in which the first two terms are $2-\sqrt{5}$ and $0{\cdot}1x^{\beta}$, we find that

$$\psi = -0{\cdot}236\,067\,98+0{\cdot}1x^{\beta}-0{\cdot}062\,319\,204x^{2\beta}+0{\cdot}046\,457\,707x^{3\beta}+\ldots, \tag{93}$$

and

$$\left.\begin{aligned}\theta &= x(1{\cdot}309\,017\,0-0{\cdot}077\,195\,222x^{\beta}+0{\cdot}037\,568\,654x^{2\beta}-0{\cdot}023\,716\,451x^{3\beta}+\ldots)\\ y &= x(1{\cdot}618\,034\,0-0{\cdot}154\,390\,44x^{\beta}+0{\cdot}075\,137\,309x^{2\beta}-0{\cdot}047\,432\,903x^{3\beta}+\ldots).\end{aligned}\right\} \tag{94}$$

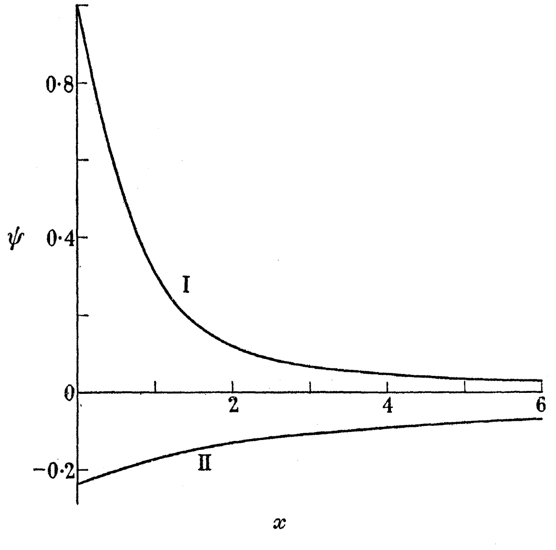

Figure 6. The two fundamental solutions of equations (66) and (67) which tend to $\psi = 1$ and $\psi = 2-\sqrt{5}$ $(= -0{\cdot}2361)$ as $x \to 0$. (The scale of x is arbitrary.)

It can be readily verified that solutions of equations (66) and (67) which have the behaviours (88) and (93) at the origin tend to $\psi = 0$ as $x\to\infty$ and $k\to 0$ (see figure 6). Since $\psi = 0$ corresponds to an equipartition among the two forms of energy, we may summarize the results of the present discussion as follows:

There exist two distinct modes of hydromagnetic turbulence which we may distinguish as the *velocity mode* and the *magnetic mode*, respectively. For both modes there is an equipartition of energy between the kinetic and the magnetic forms as $k \to 0$ (i.e. among the largest eddies). On the other hand, in the velocity mode the energy in the magnetic field tends to zero as $k \to \infty$ (i.e. among the smallest eddies), while in the magnetic mode the energy in the magnetic field tends to $\frac{1}{2}(3+\sqrt{5})$ $(=2\cdot61\ldots)$ times the energy in the turbulent motions in the smallest eddies.

(*d*) *The numerical forms of solutions*

Solutions of equations (66) and (67) whose behaviours at the origin are those described by the series expansions (88) and (89), and (93) and (94) have been obtained by numerical integration. In each case the integration was started with the aid of the series expansions and then continued forward by standard methods. The solution was then completed by the further quadrature (equation (69)) needed to relate τ (and hence k $(=\tau^{-\frac{1}{2}})$) with x. The results of these numerical calculations are summarized in tables 2 and 3.

TABLE 2. THE SPECTRAL CHARACTERISTIC OF THE VELOCITY MODE OF HYDROMAGNETIC TURBULENCE

x	ψ	k	$C(k)$	G/F
0	1·0000	∞	0·582387	0
0·04	0·9604	20·81	0·5820	$4\cdot080\times10^{-4}$
0·08	0·9216	12·37	0·5811	$1\cdot663\times10^{-3}$
0·12	0·8837	9·123	0·5795	$3\cdot810\times10^{-3}$
0·16	0·8467	7·349	0·5771	$6\cdot890\times10^{-3}$
0·20	0·8106	6·212	0·5740	$1\cdot094\times10^{-2}$
0·24	0·7755	5·414	0·5701	$1\cdot599\times10^{-2}$
0·28	0·7413	4·818	0·5655	$2\cdot206\times10^{-2}$
0·32	0·7082	4·354	0·5602	$2\cdot918\times10^{-2}$
0·36	0·6761	3·981	0·5542	$3\cdot735\times10^{-2}$
0·40	0·6450	3·673	0·5474	$4\cdot656\times10^{-2}$
0·48	0·5862	3·192	0·5322	$6\cdot806\times10^{-2}$
0·56	0·5318	2·832	0·5150	$9\cdot343\times10^{-2}$
0·64	0·4819	2·549	0·4964	0·1222
0·72	0·4365	2·321	0·4773	0·1539
0·80	0·3955	2·132	0·4581	0·1876
0·90	0·3502	1·937	0·4350	0·2316
1·0	0·3109	1·775	0·4135	0·2763
1·2	0·2480	1·521	0·3767	0·3631
1·4	0·2015	1·332	0·3483	0·4416
1·6	0·1670	1·185	0·3268	0·5094
1·8	0·1410	1·068	0·3107	0·5667
2·0	0·1210	0·9736	0·2984	0·6147
2·5	0·08767	0·7997	0·2782	0·7036
3·0	0·06769	0·6817	0·2666	0·7625
3·5	0·05466	0·5964	0·2592	0·8034
4·0	0·04562	0·5318	0·2542	0·8331
5·0	0·03402	0·4401	0·2480	0·8727
6·0	0·02698	0·3780	0·2443	0·8977
8·0	0·01896	0·2984	0·2402	0·9269
∞	0	0	0·231120	1·0000

In figure 6 the two solutions for $\psi(x)$ are illustrated. In view of the homology transformation which equations (66) and (67) admit, the solutions illustrated (and tabulated) represent *all* the solutions which tend to $\psi = 1$ or $\psi = 2-\sqrt{5}$ (as $x \to 0$) if we regard the scale of x as arbitrary.

TABLE 3. THE SPECTRAL CHARACTERISTICS OF THE MAGNETIC MODE OF HYDROMAGNETIC TURBULENCE

x	ψ	k	$C(k)$	G/F
0	−0·236068	∞	0·130091	2·61802
0·04	−0·2341	11·87	0·1309	2·597
0·08	−0·2316	7·069	0·1319	2·569
0·12	−0·2289	5·223	0·1330	2·540
0·16	−0·2260	4·216	0·1341	2·509
0·20	−0·2231	3·572	0·1353	2·479
0·24	−0·2202	3·120	0·1365	2·449
0·28	−0·2173	2·784	0·1377	2·419
0·32	−0·2144	2·523	0·1389	2·390
0·36	−0·2116	2·313	0·1400	2·361
0·40	−0·2087	2·141	0·1412	2·334
0·50	−0·2019	1·818	0·1440	2·268
0·60	−0·1954	1·592	0·1467	2·207
0·70	−0·1892	1·423	0·1493	2·151
0·80	−0·1833	1·292	0·1517	2·099
1·0	−0·1726	1·100	0·1563	2·008
1·2	−0·1630	0·9654	0·1603	1·930
1·6	−0·1466	0·7865	0·1672	1·805
2·0	−0·1333	0·6715	0·1729	1·710
2·5	−0·1199	0·5736	0·1787	1·619
3·0	−0·1089	0·5045	0·1833	1·549
4·0	−0·09231	0·4121	0·1905	1·448
5·0	−0·08024	0·3523	0·1957	1·379
6·0	−0·07105	0·3099	0·1997	1·329
8·0	−0·05795	0·2530	0·2054	1·261
10·0	−0·04903	0·2161	0·2093	1·217
12·0	−0·04254	0·1898	0·2121	1·186
∞	0	0	0·231120	1·000

The quantities which are of the greatest interest are, of course, the spectral functions $F(k)$ and $G(k)$. From the analysis of the preceding sections it follows that F is a Kolmogoroff spectrum both for $k \to 0$ and for $k \to \infty$. In view of the consequent great variation in $F(k)$ it is convenient to consider instead $F(k)\,k^{\frac{5}{3}}$. According to the definitions of the various quantities (cf. equations (31), (32) and (35))

$$4Fk^{\frac{5}{3}} = \xi^2(1+\psi)^2\,k^{-\frac{16}{3}} = \xi^2(1+\psi)^2\,\tau^{\frac{4}{3}}$$
$$= C(k) \quad \text{(say)}. \tag{95}$$

Since by setting the 'constant' on the right-hand side of equation (33) equal to unity and by considering the scale of x (and, therefore, also of k) as arbitrary, we have allowed ourselves complete freedom in the choice of units for measuring F and k, we can regard $C(k)$ as specifying the variation, in hydromagnetics, of what is a constant in hydrodynamics. The functions $C(k)$ for the two modes of turbulence are given in tables 2 and 3; they are further illustrated in figure 7. It may be noted here that corresponding to the behaviour (82) of ψ, θ and y we have

$$C(k) \to \frac{(1+\psi_0)^2}{[9(1+\psi_0^2)]^{\frac{2}{3}}} \quad (x \to 0,\ k \to \infty). \tag{96}$$

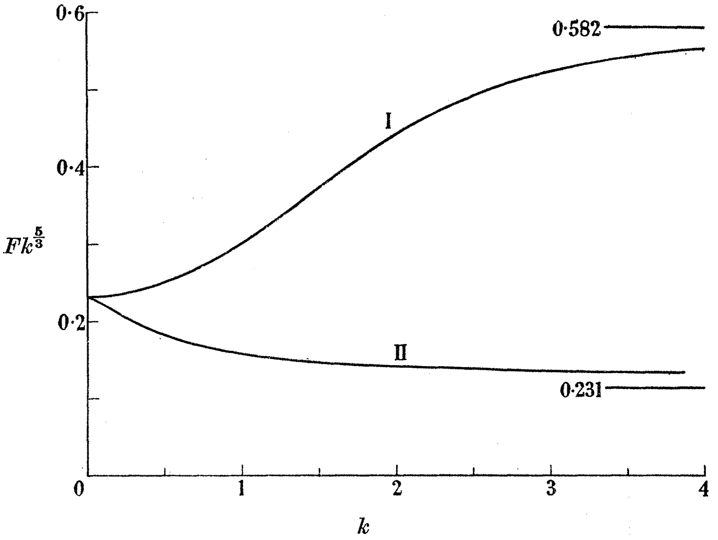

FIGURE 7. The variation of $Fk^{\frac{5}{3}}$ with k for the two modes of hydromagnetic turbulence (I, velocity mode; II, magnetic mode). The values to which these curves become asymptotic for $k \to \infty$ are shown. (The scales of F and k are arbitrary.)

On the other hand, from the Kolmogoroff behaviour of $F(k)$ for $k \to 0$ and $k \to \infty$, it follows that (cf. equations (51))

$$C(0) = C(\infty)\frac{(1+\psi_0^2)^{\frac{2}{3}}}{(1+\psi_0)^2} = \frac{1}{3^{\frac{4}{3}}} = 0{\cdot}2311\ldots, \tag{97}$$

which is the same for the two modes of turbulence.

Finally, considering the spectrum, $G(k)$, of the turbulent magnetic field, we have for both modes a Kolmogoroff spectrum for $k \to 0$ where equipartition prevails. The two modes differ in their behaviour for $k \to \infty$. In the velocity mode the density of energy in the magnetic field tends to zero as $k \to \infty$, while in the magnetic mode it tends to about 2·6 times the density of energy in the turbulent motions. In the latter case the spectrum has a Kolmogoroff behaviour also for $k \to \infty$. A quantity of special interest when considering the spectrum of the magnetic energy is therefore the ratio, G/F $(= (1-\psi)^2/(1+\psi)^2)$, of the energy in the two forms as a function of the wave number k. This ratio for the two modes of turbulence is listed in tables 2 and 3 and illustrated in figures 8 and 9.

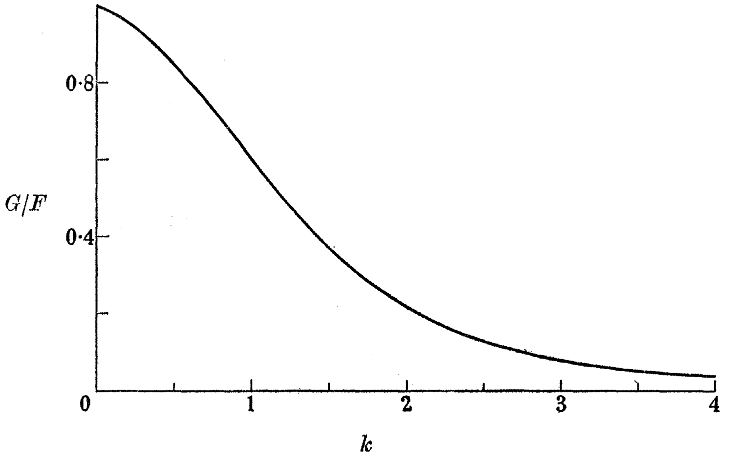

FIGURE 8. The ratio of the magnetic energy to the kinetic energy as a function of the wave number in the velocity mode of turbulence. This ratio tends to zero as $k \to \infty$. (The scale of k is arbitrary.)

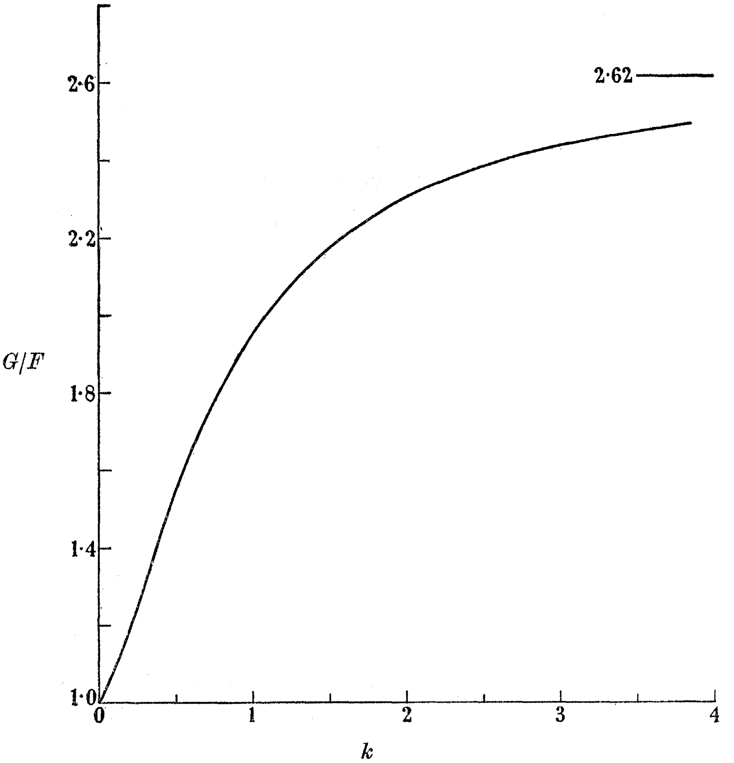

FIGURE 9. The ratio of the magnetic energy to the kinetic energy as a function of the wave number in the magnetic mode of turbulence. The value to which this ratio tends as $k \to \infty$, is shown. (The scale of k is arbitrary.)

6. The two modes of hydromagnetic turbulence

In hydrodynamics it is generally supposed that under stationary conditions the spectrum of turbulence in the limit of infinite Reynolds number is uniquely determined apart from permissible changes of scale. Thus, Kolmogoroff's law,

$$F(k) = \text{constant}\, k^{-\frac{5}{3}}, \tag{98}$$

is assumed to be universally applicable though the units in which F and k are measured will differ from case to case depending on initial conditions. In hydromagnetic turbulence one might, therefore, have expected that under stationary conditions the spectral distributions of $\overline{\mathbf{u}^2}$ and $\overline{\mathbf{h}^2}$ are similarly uniquely determined (again, apart from permissible changes of scale) with the reservation that the case of zero magnetic energy occur as a singular member of a homologous family of solutions. These expectations are realized in what we have called the velocity mode. For, the solutions describing this mode ($\psi \to 1$ as $k \to 0$) do form a homologous family and the case of zero magnetic energy does occur as a singular member of the family (by the choice $a = 0$ in equation (87) when $\psi \equiv 1$). But there is one element of surprise. While equipartition prevails among the largest eddies, the magnetic energy relative to the kinetic energy tends to zero very rapidly as we go to smaller eddies. This contradicts what has sometimes been argued from the close similarity of the equation governing the magnetic field and the equation governing the vorticity in hydrodynamics (cf. Batchelor 1950), namely, that $G(k)$ should be similar to $k^2F(k)$. However, it should be mentioned here that contrary to these views, Fermi (1949) and Schlüter and Biermann (cf. Biermann 1953) have always maintained that among the large eddies equipartition must prevail whether or not it prevails among the small eddies.

While the occurrence of the velocity mode is in general conformity with one's expectations, the occurrence of the magnetic mode is unexpected. In this second mode the magnetic energy in the small eddies predominates over the kinetic energy of motions though equipartition prevails among the large eddies. A further point to be noticed is that the case of zero magnetic energy is not included in this second family of solutions.

Lacking detailed analysis, we may surmise that the origin of these two modes of turbulence is the following: In hydromagnetic turbulence the dissipation of energy takes place by viscosity and by electrical conductivity. The spectral functions $F(k)$ and $G(k)$ must therefore depend on the manner of energy dissipation which dominates. In particular, we may expect that the solution obtained by first setting $\lambda = 0$ and then letting $\nu \to 0$ will be different from the solution obtained by first setting $\nu = 0$ and then letting $\lambda \to 0$. In the latter case since energy can be dissipated only by Joule heating, it is clear that the magnetic energy must predominate over the kinetic energy at all wave numbers. This is not necessary if $\lambda = 0$ and viscosity provides the only means of energy dissipation. The velocity and the magnetic modes of turbulence may originate in this way. By considering equations (17) and (18) under stationary conditions and retaining the terms in λ and ν one should be able to decide whether or not the two modes originate in the manner suggested.

7. On the possible character of the interstellar magnetic field

In discussing the role of interstellar magnetic fields for the problem of the origin of the cosmic rays, Fermi (1949, 1954) has taken one of two points of view dictated largely by what the astronomical evidence at the moment seemed to indicate as the most probable. Since the astronomical evidence continues to be inconclusive, it is important that we keep in mind both his points of view. In estimating the strength of the interstellar magnetic fields on either point of view, we shall use astronomical data which appear most likely at the present time.* The data we need are the root-mean-square velocity, v, of the interstellar clouds, the density, ρ, of the interstellar matter and the density ρ_t (of stars, gas and dust) in the local spiral arm of the galaxy. The best estimates of these quantities appear to be

$$v = 14\,\text{km/s}, \quad \rho = 2{\cdot}4\times 10^{-24}\,\text{g/cm}^3 \quad \text{and} \quad \rho_t = 6\times 10^{-24}\,\text{g/cm}^3. \tag{99}$$

Now Fermi's two points of view have been the following:

(i) *The magnetic field is a random turbulent field and equipartition exists between the two forms of energy.* On this view

$$\tfrac{1}{2}\rho v^2 = H^2/8\pi \quad \text{or} \quad H = (4\pi\rho)^{\frac{1}{2}}\,v. \tag{100}$$

With the estimated values of ρ and v, we find

$$H_e = 7{\cdot}0\times 10^{-6}\,\text{gauss}, \tag{101}$$

where we have used a subscript e to denote that this is the equipartition value of the field strength.

(ii) *The field is essentially uniform along the spiral arm.* In taking this point of view one is largely guided by the results on interstellar polarization (Hiltner 1951) which exhibit a surprising alinement of the directions of polarization over wide regions of the sky. But appreciable and apparently irregular fluctuations in the directions of polarization do exist; and this would indicate that the magnetic lines of force are not strictly straight and that appreciable deviations are caused by the motions of the interstellar clouds attached to the lines of force. On this picture the mean angular deviation, α, of the directions of polarization should be related to the strength of the field, the root-mean-square velocity of the clouds and the density ρ by (cf. Leverett Davis, Jr. 1951; Chandrasekhar & Fermi 1953, equation (7))

$$H = (\tfrac{4}{3}\pi\rho)^{\frac{1}{2}}\frac{v}{\alpha}. \tag{102}$$

The observations suggest that $\alpha \sim 0{\cdot}2$ radian. With this value of α and the earlier estimates of ρ and v, we find

$$H_u = 2{\cdot}0\times 10^{-5}\,\text{gauss}, \tag{103}$$

where we have used a subscript u to denote that this is the field strength on the uniform field hypothesis.

It has been pointed out by Fermi (cf. Chandrasekhar & Fermi 1953; see also Leverett Davis Jr. 1954) that there is a further condition which one might impose

* I am indebted to Dr A. Blaauw for steering me through the often conflicting astronomical evidence.

as a check on the derived values of the magnetic fields. This condition arises from the requirement of gravitational equilibrium of the spiral arm. If we idealize the spiral arm as a cylinder of radius, R, of uniform density, the required condition is

$$\begin{aligned} p_{\text{grav.}} &= \pi G \rho \rho_t R^2 = p_{\text{kin.}} + p_{\text{mag.}} \\ &= \tfrac{1}{3}\rho v^2 + \frac{H^2}{8\pi}. \end{aligned} \tag{104}$$

Inserting for v, ρ and ρ_t their estimated values, we obtain

$$2{\cdot}4 \times 10^{-13}(R/100pc)^2 = 1{\cdot}65 \times 10^{-12} + H^2/(8\pi). \tag{105}$$

Now the 'radius' of a spiral arm is a very indefinite notion, since the observations indicate not only considerable non-uniformity of the arms but also that the different spiral arms have widely different cross-sections; and the cross-sections are by no means circular. Nevertheless, it appears safe to estimate that

$$200pc < R < 500pc. \tag{106}$$

Since the gravitational pressure depends on the square of the assumed radius, it is perhaps best that we determine R from equation (105) and the two estimates, (100) and (103), for the strength of the magnetic field. We find that

$$\left.\begin{aligned} R &= 400pc \quad \text{when} \quad H = H_e = 7 \times 10^{-6}\,\text{gauss} \\ \text{and} \qquad R &= 800pc \quad \text{when} \quad H = H_u = 2 \times 10^{-5}\,\text{gauss.} \end{aligned}\right\} \tag{107}$$

The large value of the magnetic field (103) leading to $R = 800pc$ would seem to be in contradiction with the astronomical evidence (106). If on the strength of this evidence one reverts to Fermi's original picture of a turbulent magnetic field, how then are we to account for the polarization results? The theory of hydromagnetic turbulence presented in this paper would seem to provide a way for reconciling Fermi's two pictures.

First we may observe that since for both modes of turbulence equipartition prevails among the largest eddies, the root-mean-square field can in both cases be estimated by (100), since the principal contribution to the kinetic energy density of turbulence always comes from the largest eddies. If we now suppose that the interstellar magnetic fields are turbulent fields belonging to the velocity mode, then small-scale Fourier components in the magnetic field will be absent and the magnetic field would in fact be relatively uniform over large regions of space. At the same time there could be occasional regions in which the turbulence is of the second type; in this case we shall have strong local fields and these might be the regions, which Fermi (1954) had to postulate, where the direction of the magnetic field changes relatively abruptly. In other words, Fermi's two pictures appear as two extremes of idealization; and the actual picture may in reality have features common with both.

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