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Killing Spinors and Supersymmetric AdS Orbifolds

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ABSTRACT

We examine the behaviour of Killing spinors on AdS_5 under various discrete symmetries of the spacetime. In this way we discover a number of supersymmetric orbifolds, reproducing the known ones and adding a few novel ones to the list. These orbifolds break the $SO(4,2)$ invariance of AdS_5 down to subgroups. We also make some comments on the non-compact Stiefel manifold $W_{4,2}$.

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1. Introduction

Supersymmetric compactifications of type IIB string theory on spacetimes of the form $AdS_5 \times X_5$ have yielded a number of interesting results about superconformal gauge theory in four dimensions following the discovery[1] of the AdS/CFT correspondence. Here the compact manifold X_5 can be the sphere S^5, or one of many possible "non-spherical horizons" including spherical orbifolds and other Einstein spaces like the conifold base.

The presence of an AdS_5 factor guarantees conformal invariance of the dual field theory, while varying the X_5 affects the spectrum of the field theory and in particular the total number of supersymmetries. Recently a number of situations have been discussed where instead one keeps X_5 fixed (for example, to be S^5) and chooses different noncompact Einstein spaces in lieu of AdS_5. Some examples of this are the five-dimensional Stiefel manifold $W_{4,2}$[2], spaces with a nontrivial $H = dB$[3,4] and orbifolds of AdS_p[5,6,7,8]. The physical interpretation of all these spaces is not completely clear at present – for example, some of the spaces discussed in [2,3,4] have singular behaviour at infinity leading to boundaries of dimension less than 4.

Orbifolds of S^5[9] are interesting because they allow us to design spacetimes that are dual to a wide class of conformally invariant, supersymmetric field theories in 4 dimensions. Some orbifolds of AdS_5 have been interpreted as topological black holes[5] generalizing the famous BTZ black hole in 3 dimensions[10], while other orbifolds represent cosmological solutions[6][1]. A particular supersymmetric AdS_5 orbifold was discussed in Ref.[8] where it was proposed to be dual to a 3-brane field theory with a *pp*-wave propagating on it. Hence in this example one finds a type IIB supergravity background that is dual to a 3-brane worldvolume theory, not in its ground state but in a BPS excited state. This is an intriguing direction in which to generalize the AdS/CFT correspondence.

The purpose of this note is to examine conditions under which orbifolds of AdS_5 (with or without fixed points) preserve some supersymmetry. The analogous conditions for S^5 have been analyzed in some depth in Refs.[11,12]. One key result that helped in that classification was a theorem relating Killing spinors on Einstein 5-manifolds to parallel spinors on a 6d cone above them. However, one could also reproduce many of those results by directly studying the transformation properties of Killing spinors on S^5 under the orbifolding action.

[1] Much of the previous work on AdS orbifolds deals with non-supersymmetric cases, and hence our discussion below will not be closely related to it.

In what follows, we construct Killing spinors on AdS_5 in three different coordinate systems and examine their behaviour under various possible orbifolding actions. This enables us to construct a number of supersymmetric orbifolds, including some known ones and some that are apparently new. As we will see, different orbifolds can be conveniently studied in different coordinates systems on AdS_5. A complete classification of orbifolds of AdS_5, on the lines of Refs.[11,12], would be interesting to attempt. This would perhaps follow if one could prove a theorem relating Killing spinors on a (non-compact) 5-dimensional Einstein space to spinors on a "cone" over it with two timelike directions.

We also compute Killing spinors for $W_{4,2}$ and discuss some supersymmetric orbifolds of this space. The physical meaning of this space and its relevance to the AdS/CFT correspondence are not very clear, and the same holds for the AdS orbifolds we consider except in a few cases. We leave the detailed analysis of this question, along with the study of the global structure of these orbifold spacetimes, for the future.

Like the cases discussed in Refs.[8,2], the orbifolds discussed here break the $SO(4,2)$ invariance of AdS_5 down to subgroups, while preserving the S^5 factor and hence the $SO(6)$ symmetry associated to R-symmetry of the boundary CFT. One can of course combine the orbifolds discussed here with the ones proposed in Ref.[9] to get compactifications with still lower symmetry and supersymmetry.

2. Killing Spinors on AdS^5

AdS^5 spacetime can be described as a hyperboloid in a 6-dimensional spacetime with 2 timelike directions. Labelling the coordinates of this ambient spacetime as $X_{-1}, X_0, X_1, \ldots X_4$, the metric is

$$ds^2 = -(dX_{-1})^2 - (dX_0)^2 + (dX_1)^2 + \ldots + (dX_4)^2 \tag{2.1}$$

and the equation of the hyperboloid is:

$$-1 = -(X_{-1})^2 - (X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 \tag{2.2}$$

The metric on AdS^5 is the one induced from the ambient space.

We will find it convenient to work in three different sets of coordinates.

2.1. Light-Cone Type Coordinates

These consist of two pairs of lightlike coordinates and one complex coordinate. It is defined by

$$z_1^{\pm} = X_0 \pm X_1, \quad z_2^{\pm} = X_2 \pm X_{-1}, \quad w = (X_3 + iX_4) \tag{2.3}$$

and the hyperboloid is

$$-1 = -z_1^+ z_1^- + z_2^+ z_2^- + w\overline{w} \tag{2.4}$$

For this set of coordinates, it is convenient to choose an explicit basis for the Gamma-matrices as:

$$\Gamma_1 = \sigma_2 \otimes 1, \quad \Gamma_2 = i\sigma_3 \otimes \sigma_1, \quad \Gamma_3 = \sigma_3 \otimes \sigma_2, \quad \Gamma_4 = \sigma_3 \otimes \sigma_3, \quad \Gamma_5 = \sigma_1 \otimes 1 \tag{2.5}$$

Note that $(\Gamma_2)^2 = -1$, while the other Γ-matrices square to $+1$.

From the light-cone type coordinates we go to a set of five independent coordinates $\theta_1, \theta_2, \alpha, \beta, \delta$ where $0 \leq \theta_2 \leq \pi$, $0 \leq \beta \leq 2\pi$ and α,δ and θ_1 are non-compact. These coordinates are defined by:

$$\begin{aligned} z_1^{\pm} &= \cosh\frac{\theta_1}{2} e^{\pm\delta} \\ z_2^{\pm} &= \sinh\frac{\theta_1}{2}\,\cos\frac{\theta_2}{2} e^{\pm\alpha} \\ w &= \sinh\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2} e^{i\beta} \end{aligned} \tag{2.6}$$

By abuse of notation we will refer to these also as light-cone type coordinates, though they are actually an angular parametrization of those coordinates which solves the hyperboloid constraint. The metric on AdS^5 in these coordinates is:

$$ds^2 = -\sinh^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}d\alpha^2 + \cosh^2\frac{\theta_1}{2}d\delta^2 + \frac{1}{4}d\theta_1^2 + \sinh^2\frac{\theta_1}{2}(\frac{1}{4}d\theta_2^2 + \sin^2\frac{\theta_2}{2}d\beta^2) \tag{2.7}$$

For fixed θ_2 and β, the metric is proportional to that of AdS^3.

For this case, we have the vielbeins:

$$\begin{aligned} &e^{\underline{1}} = \frac{1}{2}\sinh\frac{\theta_1}{2}d\theta_2 \qquad e^{\underline{2}} = \sinh\frac{\theta_1}{2}\,\cos\frac{\theta_2}{2}\,d\alpha \\ &e^{\underline{3}} = \cosh\frac{\theta_1}{2}\,d\delta \qquad e^{\underline{4}} = \frac{1}{2}\,d\theta_1 \qquad e^{\underline{5}} = \sinh\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}\,d\beta \end{aligned} \tag{2.8}$$

and the spin connections:

$$\begin{aligned} &\omega^{\underline{12}} = \sin\frac{\theta_2}{2}\, d\alpha \qquad \omega^{\underline{14}} = \frac{1}{2}\cosh\frac{\theta_1}{2}\, d\theta_2 \qquad \omega^{\underline{15}} = -\cos\frac{\theta_2}{2} d\beta \\ &\omega^{\underline{24}} = \cos\frac{\theta_2}{2}\, \cosh\frac{\theta_1}{2}\, d\alpha \qquad \omega^{\underline{34}} = \sinh\frac{\theta_1}{2}\, d\delta \qquad \omega^{\underline{45}} = -\sin\frac{\theta_2}{2}\, \cosh\frac{\theta_1}{2}\, d\beta \end{aligned} \tag{2.9}$$

Note that the tangent-space metric has $\eta_{\underline{22}} = -1$ while the other components are $+1$.

We are interested in studying the Killing spinors on AdS^5 in this coordinate basis. The relevant equation is:

$$(\partial_\mu + \frac{1}{4}\omega_\mu^{\underline{ab}}\Gamma_{\underline{ab}} - \frac{1}{2}e_\mu^{\underline{a}}\Gamma_{\underline{a}})\epsilon = 0 \tag{2.10}$$

It is fairly straightforward to compute the solutions to this equation, which are given by:

$$\epsilon = e^{\frac{1}{4}\Gamma_4\theta_1} e^{-\frac{1}{4}\Gamma_{14}\theta_2} e^{-\frac{1}{2}\Gamma_{24}\alpha} e^{\frac{1}{2}\Gamma_3\delta} e^{\frac{1}{2}\Gamma_{15}\beta}\epsilon_0 \tag{2.11}$$

where ϵ_0 is an arbitrary constant spinor.

2.2. Complex Coordinates

Another coordinate system will turn out to be useful to investigate a different class of orbifolds. These will be called complex coordinates – they are actually complex coordinates of the ambient 6-dimensional spacetime, a complex time and two complex space dimensions. Thus we define:

$$u = X_{-1} + iX_0, \quad v = X_1 + iX_2, \quad w = (X_3 + iX_4) \tag{2.12}$$

in terms of which the hyperboloid is

$$-1 = -u\overline{u} + v\overline{v} + w\overline{w} \tag{2.13}$$

The coordinate w is the same as was used for the light-cone type coordinates. This time it is convenient to go to five independent coordinates $\theta_1, \theta_2, \alpha', \beta, \delta'$ where $0 \leq \theta_2 \leq \pi$, $0 \leq \beta, \alpha', \delta' \leq 2\pi$ and θ_1 is non-compact. These coordinates are defined by:

$$\begin{aligned} u &= \cosh\frac{\theta_1}{2} e^{i\delta'} \\ v &= \sinh\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2} e^{i\alpha'} \\ w &= \sinh\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2} e^{i\beta} \end{aligned} \tag{2.14}$$

Again, although these angles parametrize the complex coordinates in a way which solves the hyperboloid constraint, we will refer to the angles themselves as complex coordinates.

Note that these coordinates can be obtained from the previous ones by the formal replacement $\alpha = i\alpha'$ and $\delta = i\delta'$, which interchanges one space with one time direction. This replacement in Eq.(2.7) also gives us the metric in these coordinates. It is evident that the vielbeins are formally the *same* as before, though the tangent-space metric now has $\eta_{\underline{33}} = -1$. Careful inspection shows that the spin connections also turn out to be exactly the same as in Eq.(2.9), as the sign changes introduced by the interchange of a space and a time direction eventually cancel out. The change of tangent space metric necessitates a slightly different basis of Γ-matrices. We multiply Γ_2 of Eq.(2.5) by $-i$ and Γ_3 by i. Hence the new set of Γ-matrices (which we label Γ' to avoid confusion with the previous set) becomes:

$$\Gamma_1' = \sigma_2 \otimes 1, \quad \Gamma_2' = \sigma_3 \otimes \sigma_1, \quad \Gamma_3' = i\sigma_3 \otimes \sigma_2, \quad \Gamma_4' = \sigma_3 \otimes \sigma_3, \quad \Gamma_5' = \sigma_1 \otimes 1 \tag{2.15}$$

This time, $(\Gamma_3')^2 = -1$ while the others square to $+1$. The advantage of this choice is that we find (formally) the same Killing spinor as in Eq.(2.11), but now with the Γ'-matrices:

$$\epsilon = e^{\frac{1}{4}\Gamma_4'\theta_1} e^{-\frac{1}{4}\Gamma_{14}'\theta_2} e^{-\frac{1}{2}\Gamma_{24}'\alpha} e^{\frac{1}{2}\Gamma_3'\delta} e^{\frac{1}{2}\Gamma_{15}'\beta} \epsilon_0 \tag{2.16}$$

2.3. Horospherical Coordinates

Let us finally recall[13] the Killing spinors in horospherical coordinates, which consist of five independent real coordinates, r, x^1, x^2, x^3, x^4 in terms of which the metric is:

$$ds^2 = (dr)^2 + e^{2r}(-(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2) \tag{2.17}$$

Choosing the Γ matrices as

$$\Gamma_1 = i\sigma_3 \otimes \sigma_2, \quad \Gamma_2 = \sigma_3 \otimes \sigma_1, \quad \Gamma_r = \sigma_3 \otimes \sigma_3, \quad \Gamma_3 = \sigma_1 \otimes 1, \quad \Gamma_4 = \sigma_2 \otimes 1 \tag{2.18}$$

the Killing spinor is found to be[13]:

$$\epsilon = e^{\frac{1}{2}r\Gamma_r}(1+\frac{1}{2}x^\alpha\Gamma_\alpha(1-\Gamma_r))\epsilon_0$$
$$= \begin{pmatrix} e^{\frac{r}{2}}(\epsilon_0^{(1)} + x^+\epsilon_0^{(2)} + (x^3 - ix_4)\epsilon_0^{(3)}) \\ e^{-\frac{r}{2}}\epsilon_0^{(2)} \\ e^{-\frac{r}{2}}\epsilon_0^{(3)} \\ e^{\frac{r}{2}}(\epsilon_0^{(4)} + x^-\epsilon_0^{(3)} + (x^3 + ix^4)\epsilon_0^{(2)}) \end{pmatrix} \tag{2.19}$$

where $\alpha = 1,2,3,4$, $x^+ = x^1 + x^2$, $x^- = x^1 - x^2$ and

$$\epsilon_0 = \begin{pmatrix} \epsilon_0^{(1)} \\ \epsilon_0^{(2)} \\ \epsilon_0^{(3)} \\ \epsilon_0^{(4)} \end{pmatrix} \tag{2.20}$$

The transformation between these horospherical coordinates and the light-cone type coordinates $z_1^\pm, z_2^\pm, w$ is: is

$$e^r = z_2^+, \quad x^+ = \frac{z_1^+}{z_2^+}, \quad x^- = \frac{z_1^-}{z_2^+}, \quad x^3 = \frac{(w+\overline{w})}{2z_2^+}, \quad x^4 = \frac{(w-\overline{w})}{2iz_2^+}, \tag{2.21}$$

3. Orbifolds of AdS_5

3.1. Half-supersymmetric Orbifolds

Now we will examine the orbifolding actions which are natural in the various coordinates. In the light-cone type coordinates, one natural action follows from the following transformation:[8]:

$$z_1^\pm \to e^{\pm 2\pi/k} z_1^\pm, \qquad z_2^\pm \to e^{\pm 2\pi/k} z_2^\pm \tag{3.1}$$

which can be expressed as a simple translation:

$$\delta \to \delta + \frac{2\pi}{k}, \qquad \alpha \to \alpha + \frac{2\pi}{k} \tag{3.2}$$

From Eq.(2.4), this clearly has no fixed points, since the hyperboloid does not pass through $z_1^\pm = z_2^\pm = 0$. In order for the Killing spinor ϵ to be invariant under the above transformation we must require that the constant spinor ϵ_0 satisfy

$$e^{\frac{\pi}{k}(-\Gamma_{24}+\Gamma_3)}\epsilon_0 = \epsilon_0 \tag{3.3}$$

which means the matrix $(-\Gamma_{24} + \Gamma_3)$ must annihilate ϵ_0. In the basis chosen in Eq.(2.5), we have:

$$-\Gamma_{24} + \Gamma_3 = 2\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{3.4}$$

and hence the Killing spinors that are preserved, in this basis, are the ones for which

$$\epsilon_0 = \begin{pmatrix} \epsilon_0^{(1)} \\ \epsilon_0^{(2)} \\ 0 \\ 0 \end{pmatrix} \tag{3.5}$$

This in particular gives a direct proof that the orbifold discussed in Ref.[8] preserves half the supersymmetries.

The above orbifold action is generated by an $SU(1,1)$ matrix in the full isometry group $SO(4,2)$ of AdS_5, hence the surviving symmetry group is the commutant of $SU(1,1)$ in $SO(4,2)$ which is $SU(1,1) \times U(1)$. This is analogous to the fact that the simplest half-supersymmetric orbifold of S^5 (corresponding to D3-branes at an ALE singularity) has an R-symmetry group $SU(2) \times U(1)$.

Turning now to the complex coordinates, it is natural to consider orbifold actions of the type

$$u \to \gamma^d u, \qquad v \to \gamma^a v, \qquad w \to \gamma^b w \tag{3.6}$$

where $\gamma = \exp(2\pi i/k)$ and a, b, d are some integers. These are quite analogous to corresponding orbifolds of S^5. The result is also analogous: the orbifolding action above leaves the Killing spinor invariant if

$$(-a\Gamma'_{24} + d\Gamma'_3 + b\Gamma'_{15})\,\epsilon_0 = 0 \tag{3.7}$$

The above matrix has eigenvalues $(a + b - d)$, $-(a + b + d)$, $(a - b + d)$ and $-(a - b - d)$. If one of a, b, d is zero then we have two vanishing eigenvalues and $\frac{1}{2}$-supersymmetry.

Note, however, that because of the signature of the spacetime, all the $\frac{1}{2}$-supersymmetric cases are not equivalent. The case with $d = 0$ has a circle of fixed points $u\overline{u} = 1$, while the cases with $a = 0$ or $b = 0$ have no fixed points and are equivalent to each other.

In the case $d = 0$, the orbifold generator lies in an $SU(2)$ subgroup of $SO(4) \subset SO(4,2)$ hence the symmetry of the quotient space is $SU(2) \times U(1)$. On the other hand, for $a = 0$ or $b = 0$ the orbifold is generated by an element in an $SU(1,1)$ subgroup of $SO(2,2) \subset SO(4,2)$ and the surviving symmetry is $SU(1,1) \times U(1)$.

Next, it is useful to examine the orbifold described in Eq.(3.1), in horospherical coordinates. The action becomes:

$$r \to r + a, \quad x^+ \to x^+, \quad x^- \to e^{-2a} x^-, \quad x^3 \to e^{-a} x^3, \quad x^4 \to e^{-a} x^4, \tag{3.8}$$

Hence the Killing spinor transforms as

$$\epsilon \to e^{-\frac{1}{2}a(1\otimes\sigma_3)} \begin{pmatrix} e^{\frac{(r+a)}{2}}(\epsilon_0^{(1)} + x^+\epsilon_0^{(2)} + e^{-a}(x^3 - ix^4)\epsilon_0^{(3)}) \\ e^{-\frac{(r+a)}{2}}\epsilon_0^{(2)} \\ e^{-\frac{(r+a)}{2}}\epsilon_0^{(3)} \\ e^{\frac{(r+a)}{2}}(\epsilon_0^{(4)} + e^{-2a}x^-\epsilon_0^{(3)} + e^{-a}(x^3 + ix^4)\epsilon_0^{(2)}) \end{pmatrix} \tag{3.9}$$

Thus the Killing spinors that are preserved by this orbifold, in this basis, are the ones for which

$$\epsilon_0 = \begin{pmatrix} \epsilon_0^{(1)} \\ \epsilon_0^{(2)} \\ 0 \\ 0 \end{pmatrix} \tag{3.10}$$

Another apparently trivial kind of $\frac{1}{2}$-supersymmetric orbifold is apparent from Eqn.(2.19). Suppose we choose $\epsilon_0^{(2)} = \epsilon_0^{(3)} = 0$. Then the Killing spinor becomes independent of $x^\pm, x^3, x^4$. As a result, periodic identifications in these coordinates preserve the Killing spinor. This is essentially what was noted in Ref.[13], and corresponds to the fact that the identification of these coordinates breaks conformal invariance by introducing a scale, hence the conformal part of the superconformal invariance goes away. Thus such orbifolds preserve half the supersymmetries. (One can further deform the space in the x^3, x^4 directions and add a nontrivial B-field, preserving the remaining supersymmetry, as was done in Ref.[3,4]. In this case one does not expect the deformed manifold to have a Killing spinor, since the field strength dB also contributes to the supersymmetry variation.)

3.2. One-fourth Supersymmetry

We have already considered orbifold actions, in complex coordinates, of the general type

$$u \to \gamma^d u, \qquad v \to \gamma^a v, \qquad w \to \gamma^b w \tag{3.11}$$

where $\gamma = \exp(2\pi i/k)$ and a, b, d are some integers. We saw that the relevant matrix acting on Killing spinors has eigenvalues $(a + b - d)$, $-(a + b + d)$, $(a - b + d)$ and $-(a - b - d)$. Hence if all of a, b, d are nonzero then at most one of these eigenvalues can be zero and in that case we have a $\frac{1}{4}$-supersymmetric orbifold. For the $\frac{1}{4}$ supersymmetric cases we have no fixed points for the orbifold action.

The orbifold generator lies in an $SU(2,1)$ subgroup of $SO(4,2)$, hence it preserves only a $U(1)$ symmetry. This is the analogue of the $U(1)$ symmetry preserved by $\frac{1}{4}$-supersymmetric orbifolds of S^5, which is realized as a $U(1)$ R-symmetry in the boundary theory.

Another class of $\frac{1}{4}$-supersymmetric AdS_5 orbifolds comes from quotienting by a pair of transformations each of which preserves half the supersymmetry. If (k, k') are co-prime there are two inequivalent cases:

$$\begin{aligned} u &\to \gamma u, \qquad v \to \gamma^{-1} v \\ v &\to \gamma' v, \qquad w \to \gamma'^{-1} w \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} u &\to \gamma u, \qquad v \to \gamma^{-1} v \\ u &\to \gamma' u, \qquad w \to \gamma'^{-1} w \end{aligned} \tag{3.13}$$

where $\gamma^k = (\gamma')^{k'} = 1$.

Another interesting $\frac{1}{4}$-supersymmetric orbifold arises by combining the periodic identification in x^2, x^3, x^4 in the horospherical coordinates, with the orbifolding action of Eq.(3.8). Here the constant Killing spinor satisfies $\epsilon_0^{(2)} = \epsilon_0^{(3)} = \epsilon_0^{(4)} = 0$.

We have encountered a number of supersymmetric orbifolds, but it turns out that each of them is natural in a certain coordinate system and not so easy to describe in another. Thus, it becomes hard to combine the different actions discussed in the previous section and find more general orbifolds. For example, one of the simplest orbifolds described in complex coordinates in Eq.(3.6) arises by choosing $k = 2, d = a = 1, b = 0$. This just corresponds to the reflection $u \to -u$, $v \to -v$. In terms of the light cone type coordinates this means $z_i^\pm \to -z_i^\pm$, which cannot be carried out using the independent coordinates defined in Eq.(2.6), which only cover the region $z_1^\pm > 0$.

In contrast, the orbifold corresponding to $k = 2, a = b = 1, d = 0$ can be realized in the light cone type coordinates. In this case we have $v \to -v$, $w \to -w$. This corresponds to the action

$$z_1^+ \to z_1^- , \qquad z_2^+ \to -z_2^- , \qquad w \to -w \tag{3.14}$$

which in terms of the independent coordinates in Eq.(2.6) is just

$$\delta \to -\delta, \qquad \alpha \to -\alpha, \qquad \theta_1 \to -\theta_1 \tag{3.15}$$

This acts on the Killing spinor in Eq.(2.11) as follows. In our basis, this Killing spinor is explicitly given by

$$\epsilon = \begin{pmatrix} \cos\frac{\theta_2}{4} e^{\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha-\delta}{2}\epsilon_0^{(1)} + i\sinh\frac{\alpha-\delta}{2}\epsilon_0^{(2)}\right\} e^{-i\frac{\beta}{2}} \\ -i\sin\frac{\theta_2}{4} e^{\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha+\delta}{2}\epsilon_0^{(3)} + i\sinh\frac{\alpha+\delta}{2}\epsilon_0^{(4)}\right\} e^{i\frac{\beta}{2}} \\ \\ \cos\frac{\theta_2}{4} e^{-\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha-\delta}{2}\epsilon_0^{(2)} - i\sinh\frac{\alpha-\delta}{2}\epsilon_0^{(1)}\right\} e^{-i\frac{\beta}{2}} \\ +i\sin\frac{\theta_2}{4} e^{-\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha+\delta}{2}\epsilon_0^{(4)} - i\sinh\frac{\alpha+\delta}{2}\epsilon_0^{(3)}\right\} e^{i\frac{\beta}{2}} \\ \\ \cos\frac{\theta_2}{4} e^{-\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha+\delta}{2}\epsilon_0^{(3)} + i\sinh\frac{\alpha+\delta}{2}\epsilon_0^{(4)}\right\} e^{i\frac{\beta}{2}} \\ -i\sin\frac{\theta_2}{4} e^{-\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha-\delta}{2}\epsilon_0^{(1)} + i\sinh\frac{\alpha-\delta}{2}\epsilon_0^{(2)}\right\} e^{-i\frac{\beta}{2}} \\ \\ \cos\frac{\theta_2}{4} e^{\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha+\delta}{2}\epsilon_0^{(4)} - i\sinh\frac{\alpha+\delta}{2}\epsilon_0^{(3)}\right\} e^{i\frac{\beta}{2}} \\ +i\sin\frac{\theta_2}{4} e^{\frac{\theta_1}{4}} \left\{\cosh\frac{\alpha-\delta}{2}\epsilon_0^{(2)} - i\sinh\frac{\alpha-\delta}{2}\epsilon_0^{(1)}\right\} e^{-i\frac{\beta}{2}} \end{pmatrix} \tag{3.16}$$

Now one finds that, setting $\epsilon_0^{(1)} = \epsilon_0^{(2)}$ and $\epsilon_0^{(3)} = -\epsilon_0^{(4)}$, the expression for ϵ above goes over to ϵ' satisfying

$$\epsilon' = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \epsilon = (\sigma_3 \otimes \sigma_1)\epsilon \tag{3.17}$$

This coincides with the Lorentz transformation of ϵ under the action in Eq.(3.15)[2]

[2] The action in Eq.(3.15) inverts the sign of the vielbeins $e^{\underline{1}}, e^{\underline{3}}, e^{\underline{4}}, e^{\underline{5}}$, hence it must be represented on spinors by a matrix P which anticommutes with $\Gamma_{\underline{1}}, \Gamma_{\underline{3}}, \Gamma_{\underline{4}}, \Gamma_{\underline{5}}$. Thus P is proportional to $\Gamma_2 = i\sigma_3 \otimes \sigma_1$. Since $P^2 = 1$, it follows that $P = \sigma_3 \otimes \sigma_1$.

Thus we see (as expected from the analysis of the same orbifold in complex coordinates) that this orbifold preserves half the supersymmetries. But now that we know the preserved Killing spinors in the basis appropriate to light-cone coordinates, we can combine this orbifold with the orbifolding along lightlike directions defined in Eq.(3.1) and determine if there is any residual supersymmetry. Indeed we have seen that Eq.(3.1) preserves Killing spinors of the form Eq.(2.11) where the constant spinor ϵ_0 has the form:

$$\epsilon_0 = \begin{pmatrix} \epsilon_0^{(1)} \\ \epsilon_0^{(2)} \\ 0 \\ 0 \end{pmatrix} \tag{3.18}$$

while Eq.(3.15) preserves Killing spinors where ϵ_0 has the form

$$\epsilon_0 = \begin{pmatrix} \epsilon_0^{(1)} \\ \epsilon_0^{(1)} \\ \epsilon_0^{(3)} \\ -\epsilon_0^{(3)} \end{pmatrix} \tag{3.19}$$

Choosing now $\epsilon_0^{(1)} = \epsilon_0^{(2)}$ in the first of these equations and $\epsilon_0^{(3)} = 0$ in the second, we find that both the transformations together preserve $\frac{1}{4}$ of the supersymmetry, namely the Killing spinor for which

$$\epsilon_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.20}$$

This is a $Z \times Z_2$ orbifold of AdS_5. It is much more difficult, if not impossible, to express the Z_k orbifold given in complex coordinates by $v \to \gamma v, w \to \gamma^{-1} w, \gamma^k = 1$ in terms of light-cone coordinates. Luckily it is not necessary to do this. These orbifolds preserve the same Killing spinors for all k. Since we have shown that for $k = 2$ there is a $\frac{1}{4}$-supersymmetric $Z \times Z_2$ orbifold obtained by combining with the action Eq.(3.1), it follows that there is also a $Z \times Z_k$ orbifold with $\frac{1}{4}$-supersymmetry for all k.

4. The Stiefel Manifold $W_{4,2}$

Although not directly related to orbifolds of AdS_5, the noncompact Stiefel manifold $W_{4,2}$[2] is an interesting $\frac{1}{4}$-supersymmetric coset spacetime. We will speculate later on its possible relation with AdS_5 orbifolds. In the present section we will compute its single Killing spinor and comment on orbifolds of this spacetime.

This case corresponds to an analytic continuation of the compact manifold variously denoted $T_{1,1}$ or $V_{4,2}$, which is the base of the conifold geometry and hence appears in the study of D3-branes at conifolds[14]. It can be thought of as the coset space $(AdS_3 \times AdS_3)/U(1)$, and also as a (timelike) $U(1)$ fibration above *Euclidean* $AdS_2 \times AdS_2$. The natural coordinates for $W_{4,2}$ are $(\theta_1, \phi_1, \theta_2, \phi_2, \psi)$ where $0 \leq \psi < 4\pi$, $0 \leq \phi_i < 2\pi$ while θ_i are noncompact. In terms of these, the metric of $W_{4,2}$ is

$$\begin{aligned} ds^2 = &-\frac{1}{9}(d\psi + \cosh\theta_1\, d\phi_1 + \cosh\theta_2\, d\phi_2)^2 \\ &+ \frac{1}{6}(d\theta_1^2 + \sinh^2\theta_1\, d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sinh^2\theta_2\, d\phi_2^2) \end{aligned} \tag{4.1}$$

which explicitly exhibits the $U(1)$ fibration, with ψ being the fibre coordinate.

This spacetime breaks the maximal $SO(4,2)$ isometry to $SO(2,2) \times SO(2)$, much as its compact version breaks the maximal $SO(6)$ isometry of S^5 to $SO(4) \times SO(2)$. From the metric one can read off the vielbeins:

$$\begin{aligned} e^{\underline{1}} &= \frac{1}{\sqrt{6}} d\theta_1 \qquad e^{\underline{2}} = \frac{1}{\sqrt{6}} \sinh\theta_1\, d\phi_1 \\ e^{\underline{3}} &= \frac{1}{\sqrt{6}} d\theta_2 \qquad e^{\underline{4}} = \frac{1}{\sqrt{6}} \sinh\theta_2\, d\phi_2 \\ e^{\underline{5}} &= \frac{1}{3}(d\psi + \cosh\theta_1\, d\phi_1 + \cosh\theta_2\, d\phi_2) \end{aligned} \tag{4.2}$$

and compute the spin connections:

$$\begin{aligned} \omega^{\underline{12}} &= -\frac{2}{3}\cosh\theta_1\, d\phi_1 + \frac{1}{3}d\psi + \frac{1}{3}\cosh\theta_2\, d\phi_2 \\ \omega^{\underline{13}} &= \omega^{\underline{14}} = \omega^{\underline{23}} = \omega^{\underline{24}} = 0 \\ \omega^{\underline{34}} &= -\frac{2}{3}\cosh\theta_2\, d\phi_2 + \frac{1}{3}d\psi + \frac{1}{3}\cosh\theta_1\, d\phi_1 \\ \omega^{\underline{15}} &= -\frac{1}{\sqrt{6}}\sinh\theta_1\, d\phi_1 \qquad \omega^{\underline{25}} = \frac{1}{\sqrt{6}} d\theta_1 \\ \omega^{\underline{35}} &= -\frac{1}{\sqrt{6}}\sinh\theta_2\, d\phi_2 \qquad \omega^{\underline{45}} = \frac{1}{\sqrt{6}} d\theta_2 \end{aligned} \tag{4.3}$$

A convenient basis for the Γ-matrices this time is

$$\Gamma_1 = \sigma_1 \otimes 1, \quad \Gamma_2 = \sigma_2 \otimes 1, \quad \Gamma_3 = \sigma_3 \otimes \sigma_1, \quad \Gamma_4 = \sigma_3 \otimes \sigma_2, \quad \Gamma_5 = i\sigma_3 \otimes \sigma_3 \tag{4.4}$$

In this basis, the Killing spinor equations become

$$\frac{\partial}{\partial\theta_1}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} = \begin{pmatrix}0 & 0 & \frac{1}{\sqrt{6}} & 0\\ 0 & 0 & 0 & 0\\ \frac{1}{\sqrt{6}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} \tag{4.5}$$

$$\frac{\partial}{\partial\phi_1}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} = \begin{pmatrix}\frac{i}{3}\cosh\theta_1 & 0 & \frac{i}{\sqrt{6}}\sinh\theta_1 & 0\\ 0 & \frac{i}{3}\cosh\theta_1 & 0 & 0\\ -\frac{i}{\sqrt{6}}\sinh\theta_1 & 0 & -\frac{2i}{3}\cosh\theta_1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} \tag{4.6}$$

$$\frac{\partial}{\partial\theta_2}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{\sqrt{6}} & 0 & 0\\ \frac{1}{\sqrt{6}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} \tag{4.7}$$

$$\frac{\partial}{\partial\phi_2}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} = \begin{pmatrix}\frac{i}{3}\cosh\theta_2 & \frac{i}{\sqrt{6}}\sinh\theta_2 & 0 & 0\\ -\frac{i}{\sqrt{6}}\sinh\theta_2 & -\frac{2i}{3}\cosh\theta_2 & 0 & 0\\ 0 & 0 & \frac{i}{3}\cosh\theta_2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} \tag{4.8}$$

$$\frac{\partial}{\partial\psi}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} = -\frac{i}{6}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix}\begin{pmatrix}\epsilon^{(1)}\\ \epsilon^{(2)}\\ \epsilon^{(3)}\\ \epsilon^{(4)}\end{pmatrix} \tag{4.9}$$

A remarkable simplification takes place on observing that a spinor with $\epsilon^{(1)} = \epsilon^{(2)} = \epsilon^{(3)} = 0$ automatically satisfies the first four equations. Inserting this form in the last equation, we find

$$\frac{\partial}{\partial\psi}\epsilon^{(4)} = \frac{i}{2}\epsilon^{(4)} \tag{4.10}$$

from which the corresponding Killing spinor is

$$\epsilon = e^{\frac{i}{2}\psi}\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix} \tag{4.11}$$

It is easy to see that there are no other solutions to the coupled set of equations. For each of the components $\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}$, there is an irrational factor $\sqrt{6}$ in one or other equation, which implies that we can never get a solution that is single-valued in the angles ϕ_1, ϕ_2.

Thus, as expected, there is a single Killing spinor for this manifold, proving explicitly that it has $\frac{1}{4}$-supersymmetry. The Killing spinor has the very simple form given in Eq.(4.11), and is quite similar to a solution obtained in Ref.[15] in the context of 7-dimensional Einstein spaces.

Because this Killing spinor depends only on ψ, any orbifold of $W_{4,2}$ by an action under which spinors transform trivially (such as a translation in the angles) will preserve it. One example is provided by the $Z_k \times Z_{k'}$ transformation:

$$\theta_1 \to \theta_1 + \frac{2\pi}{k}, \quad \theta_2 \to \theta_2 + \frac{2\pi}{k'} \tag{4.12}$$

This introduces conical singularities into the Euclidean AdS_2 factors that make up the base of $W_{4,2}$, but as usual one expects that string propagation on this space is smooth.

5. Discussion

We have constructed Kiling spinors in various coordinate systems and thereby discovered a number of supersymmetric orbifolds of AdS_5. One should attempt to understand the global properties and causal structure of such spacetimes. Some of them are already known to be topological black holes.

One interesting proposal emerges from our discussion. There is an AdS_5/Z_2 orbifold with a circle of fixed points, very similar to the S^5/Z_2 orbifold obtained by placing D3-branes at a Z_2 ALE singularity. Both are cases of $\frac{1}{2}$-supersymmetry. Now for the latter, it is known[14] that blowing up the circle of fixed points is a relevant deformation which causes the theory to flow to the $\frac{1}{4}$-supersymmetric theory obtained by replacing S^5/Z_2 with the Stiefel manifold $V_{4,2}$. The corresponding conformal theories flow from $N=2$ to $N=1$ and can be obtained very simply by rotating branes in a brane construction[16,17]. One could perhaps expect an analogous blowup of our AdS_5/Z_2 orbifold to lead to the non-compact Stiefel manifold $W_{4,2}$ discussed in Ref.[2]. The physics of this would be quite different from the compact case and possibly very interesting[3].

In comparing our results with those of Ref.[8], we find that we have reproduced the Z-orbifold discussed there and explicitly shown that it is $\frac{1}{2}$-supersymmetric, which is important for the identification proposed with *pp*-waves on a brane. However, it is not clear if we have found the $Z \times Z$ orbifold that also finds brief mention in their work. This is supposed to be dual to a 3-brane with a pp-wave together with a D-instanton, and one would expect it to be $\frac{1}{4}$-supersymmetric. We have found two $\frac{1}{4}$-supersymmetric orbifolds

[3] This proposal arose in discussions with Debashis Ghoshal.

that include the action in Eq.(3.1). One is found by adjoining the Z action which compactifies one or more of the spatial coordinates in the brane, as discussed below Eq.(3.13). The other is obtained by adjoining a Z_k action, as discussed after Eq.(3.20). In neither of these cases does the second orbifold group appear symmetrically with the first, while the authors of Ref.[8] seem to suggest that such an example should exist. We hope to return to this point in the future, along with a study of the physical interpretation, in terms of the brane worldvolume theory, for the various orbifolds we have constructed here.

The compactification of type IIB string theory on $AdS_5 \times S^5$ possesses a remarkable "symmetry" between the two factors. Geometrically, the only difference is that while S^5 solves a quadratic equation in R^6, AdS_5 solves a quadratic equation in $R^{4,2}$. This "symmetry" is not reflected in the emergence of this background as the near-horizon geometry of D3-branes, where R^6 has a physical interpretation as the flat space transverse to the 3-branes, but $R^{4,2}$ does not appear. One may speculate that such a symmetry may be visible or partly visible in F-theory, which can sometimes be given a 12-dimensional interpretation. This philosophy has been partially explored in Ref.[8] using earlier observations in Ref.[18]. It would be interesting to find out whether such a symmetry can be exploited to systematically classify the supersymmetric orbifolds of AdS_5.

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