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Gauge-Invariant Couplings of Noncommutative Branes to Ramond-Ramond Backgrounds

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ABSTRACT

We derive the couplings of noncommutative D-branes to spatially varying Ramond-Ramond fields, extending our earlier results in hep-th/0009101. These couplings are expressed in terms of $*_n$ products of operators involving open Wilson lines. Equivalence of the noncommutative to the commutative couplings implies interesting identities as well as an expression for the Seiberg-Witten map that was previously conjectured. We generalise our couplings to include transverse scalars, thereby obtaining a Seiberg-Witten map relating commutative and noncommutative descriptions of these scalars. RR couplings for unstable non-BPS branes are also proposed.

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1. Introduction

In a previous paper[1], we examined the Chern-Simons terms on noncommutative branes in the "background independent" description $\Phi = -B$[2,3]. These terms describe the couplings of D-branes to Ramond-Ramond (RR) gauge potentials of various form degrees. In that work, the noncommutative description for D-branes with a B-field was given for both BPS and non-BPS branes, with the restriction that the RR potentials to which they couple are constant in space. It was shown subsequently[4] that our expressions can be rederived using the fact that D-branes on a torus with a B-field turn into slanted branes under T-duality.

The noncommutative couplings of Ref.[1] were shown to be consistent with the interpretation that a noncommutative brane is a configuration of infinitely many lower dimensional branes. In particular, the well-known terms which couple RR backgrounds to non-abelian scalars on a collection of BPS D-branes[5,6,7] were reproduced. Generalisations of these terms to unstable non-BPS D-branes were also obtained and compared with results in Ref.[8].

In the present paper, our goal is to extend these couplings to nonconstant RR backgrounds. We will propose formulae for these couplings, which involve the open Wilson lines ("Wilson tails") that have recently attracted much interest in the study of noncommutative field theory[9,10,11,12]. The generalized $*'$ and $*_n$ products that have been discussed in Refs.[13,14,15,16,17] will play an important role [1].

[1] A recently developed approach to write smeared operators and compute correlation functions directly in coordinate space[18] might provide a useful alternative framework for this kind of investigation.

This investigation is of interest first of all because RR couplings characterise much of the physical behaviour of D-branes in superstring theory, both BPS and non-BPS. Hence it is important to know how these couplings generalise to the noncommutative setting. Additionally, we will find that the equivalence of noncommutative couplings to commutative ones gives rise to a variety of interesting identities. Some of these, which are novel, amount to nontrivial properties of the $*$ product. We will prove these identities explicitly in special cases, although one can also turn the logic around and claim that the new identities follow from the equivalence of commutative and noncommutative RR couplings of D-branes. Other identities that we will encounter are actually maps from noncommutative fields to their commutative counterparts. These embody the change of variables between these two types of fields, known as the Seiberg-Witten map. Again, one can try to confirm the expressions for the Seiberg-Witten map that we find (one of them was conjectured earlier by Liu[16]) from independent computations. Alternatively one can claim that they follow from equivalence of commutative and noncommutative RR couplings. It therefore emerges that this equivalence is a powerful tool to derive properties of noncommutative field theory. Finally, a third type of identity that we will come across holds only in the DBI approximation of slowly-varying fields. In this case it will be interesting to examine how the derivative corrections are incorporated, a point to which we hope to return in the future.

When this manuscript was nearly complete, we learned of forthcoming papers by H. Liu and J. Michelson, and by Y. Okawa and H. Ooguri[19], having substantial overlap with our work. We are grateful to Jeremy Michelson and Hirosi Ooguri for informing us about their results.

2. Non-constant RR fields

In this section we write down the couplings of a noncommutative brane to spatially varying Ramond-Ramond potentials. We will neglect transverse scalars, to which we return in the following section. Thus the results of this section hold only when the transverse scalars are set to zero, alternatively they hold in complete generality for a Euclidean D9-brane.

One may be concerned that the RR couplings of a commutative D-brane can have derivative corrections when the RR field is spatially varying. However, the coupling of a Dp-brane to the top form C^{p+1} and to the next lower form C^{p-1} are exactly known in the commutative case, because for these two it has been argued[20] that derivative corrections are absent. For the other RR forms, the derivative corrections are only partially determined, and in these cases our results for noncommutative couplings should agree with the standard commutative couplings only for slowly-varying fields.

2.1. Coupling to the RR Top-Form C^{p+1}

Let us consider a noncommutative Euclidean Dp-brane with an even number $p+1$ of world-volume directions. As is well-known, in a noncommutative theory one does not have gauge-invariant local operators, because of the non-locality induced by the noncommutativity. One does, however, have gauge-invariant operators of definite momentum. Hence, in order to study the coupling of spatially varying RR gauge potentials to noncommutative branes, we will choose the RR potential to be evaluated at a definite momentum k^i. The RR couplings to noncommutative branes for $k^i = 0$ were found in Ref.[1].

Now we will use the following important results from Refs.[16,17]. Given a collection of local operators $\mathcal{O}_I(x)$ on the brane world-volume which transform in the adjoint under gauge transformations, one can obtain a natural gauge-invariant operator of fixed momentum k^i by smearing the locations of these operators along a straight contour given by $\xi^i(\tau) = \theta^{ij} k_j\, \tau$ with $0 \le \tau \le 1$, and multiplying the product by a Wilson line $W(x, C)$ along the same contour,

$$W(x,C) \equiv \exp\left(i \int_0^1 d\tau \frac{\partial \xi^i(\tau)}{\partial \tau} \hat{A}_i\Big(x + \xi(\tau)\Big)\right) \tag{2.1}$$

Here and in what follows, $\hat{A}$ denotes the noncommutative gauge field, and $\hat{F}$ the corresponding field strength.

The resulting formula for the gauge-invariant operator is:

$$\begin{aligned} \mathcal{Q}(k) &= \int \frac{d^{p+1}x}{(2\pi)^{p+1}} \left(\prod_{I=1}^{n} \int_0^1 d\tau_I \right) P_* \left[W(x,C) \prod_{I=1}^{n} \mathcal{O}_I\Big(x + \xi(\tau_I)\Big)\right] * e^{ik.x} \\ &= \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\ L_* \left[W(x,C) \prod_{I=1}^{n} \mathcal{O}_I(x) \right] * e^{ik.x} \end{aligned} \tag{2.2}$$

where P_* denotes path-ordering with respect to the $*$-product, while L_* is an abbreviation for the combined path-ordering and integrations over τ_I. In this formula the operators $\mathcal{O}_I$ are smeared over the straight contour of the Wilson line. This prescription arises by starting with the symmetrised-trace action for infinitely many D-instantons and expanding it around the configuration describing a noncommutative Dp-brane.

Expanding the Wilson line and using manipulations described in Refs.[14,16,17], we get

$$\mathcal{Q}(k) = \sum_{m=0}^{\infty} \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\ \mathcal{Q}_m(x)\, e^{ik.x} \tag{2.3}$$

where

$$\mathcal{Q}_m(x) = \frac{1}{m!}(\theta\partial)^{i_1}....(\theta\partial)^{i_m}\langle \mathcal{O}_1(x), ..., \mathcal{O}_n(x), \hat{A}_{i_1}(x), ..., \hat{A}_{i_m}(x)\rangle_{*_{m+n}} \tag{2.4}$$

Here we have introduced the notation $\langle f_1(x), f_2(x), ..., f_p(x)\rangle_{*_p}$ for the $*_p$ product of p functions, as defined for example in the appendix of Ref.[16]. We note here the simple formula for $*_2$:

$$\langle f(x), g(x)\rangle_{*_2} \equiv f(x)\frac{\sin(\frac{1}{2}\overleftarrow{\partial_p}\theta^{pq}\overrightarrow{\partial_q})}{\frac{1}{2}\overleftarrow{\partial_p}\theta^{pq}\overrightarrow{\partial_q}}g(x) \tag{2.5}$$

The above procedure for defining gauge-invariant operators is useful when applied to couplings between closed-string and open-string modes, namely couplings of a noncommutative D-brane to bulk fields. Once we know the coupling of a generic closed-string supergravity Fourier mode[2] $\widetilde{\Sigma}(k)$ to a D-brane for $k = 0$, then we can derive the coupling at nonzero momentum by suitably inserting an open Wilson line as above. More precisely, if the zero-momentum coupling is

$$\widetilde{\Sigma}(0) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, \frac{1}{\mathrm{Pf}\,\theta}\mathcal{O}_\Sigma(A, X) \tag{2.6}$$

where A is the gauge field and X are the transverse scalars, then the coupling at nonzero momentum is given by

$$\widetilde{\Sigma}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, \frac{1}{\mathrm{Pf}\,\theta}\, L_*\Big[\mathcal{O}_\Sigma(A, X)\, W(x, C)\Big] * e^{ik.x} \tag{2.7}$$

The constant factor $\mathrm{Pf}\,\theta \equiv \sqrt{\det \theta^{ij}}$ has been written explicitly, instead of absorbing it into the definition of $\mathcal{O}_\Sigma$, for later convenience.

[2] Fourier modes will always be denoted with a tilde.

In our case the relevant closed string mode is the RR gauge potential $\widetilde{C}^{p+1}(k)$, and, as shown in Ref.[1], the role of $\mathcal{O}_\Sigma$ is played by the operator $\mu_p \,\mathrm{Pf}\, Q$ where μ_p is the brane tension and

$$Q^{ij} \equiv \theta^{ij} - \theta^{ik} \hat{F}_{kl}\, \theta^{lj} \tag{2.8}$$

Hence we deduce the coupling of this brane to the form $C^{(p+1)}$, in momentum space, to be:

$$\mu_p\, \epsilon^{i_1 \ldots i_{p+1}}\, \widetilde{C}^{(p+1)}_{i_1 \ldots i_{p+1}}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\; L_* \Big[\sqrt{\det(1-\theta\hat{F})}\; W(x,C) \Big] * e^{ik.x} \tag{2.9}$$

On the other hand, we know that the coupling of a Dp-brane to a $C^{(p+1)}$ form is given in the commutative description by

$$\mu_p\, \epsilon^{i_1 \ldots i_{p+1}}\, \widetilde{C}^{(p+1)}_{i_1 \ldots i_{p+1}}(-k)\; \delta^{(p+1)}(k) \tag{2.10}$$

As equations (2.9) and (2.10) describe the same system in two different descriptions, they must be equal. Thus we predict the identity[3]:

$$\delta^{(p+1)}(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\; L_* \big[\sqrt{\det(1-\theta\hat{F})}\; W(x,C) \big] * e^{ik.x} \tag{2.11}$$

This amounts to saying that the right hand side is actually independent of $\hat{A}$, a rather nontrivial fact.

Let us show explicitly that this identity holds to order $\mathcal{O}(\hat{A}^3)$ and to all orders in θ. First, the operator multiplying the open Wilson line is expanded as:

$$\begin{aligned} \sqrt{\det(1-\theta\hat{F})} =& 1 - \frac{1}{2}\mathrm{tr}\,(\theta\hat{F}) - \frac{1}{4}\mathrm{tr}\,(\theta\hat{F}\theta\hat{F}) + \frac{1}{8}(\mathrm{tr}\,(\theta\hat{F}))^2 \\ & - \frac{1}{48}(\mathrm{tr}\,(\theta\hat{F}))^3 + \frac{1}{8}(\mathrm{tr}\,\theta\hat{F})\mathrm{tr}\,(\theta\hat{F}\theta\hat{F}) - \frac{1}{6}\mathrm{tr}\,(\theta\hat{F}\theta\hat{F}\theta\hat{F}) + \mathcal{O}(\hat{F}^4) \end{aligned} \tag{2.12}$$

[3] The idea that this identity should hold arose in discussions with Sumit Das.

Then the first four terms in $\mathcal{Q}(x)$ (Eq.(2.3)) can be explicitly computed, giving:

$$
\begin{aligned}
\mathcal{Q}_0(x) &= 1 - \frac{1}{2}\theta^{ij}\hat{F}_{ji} - \frac{1}{4}\theta^{ij}\theta^{kl}\langle\hat{F}_{jk},\hat{F}_{li}\rangle_{*_2} + \frac{1}{8}\theta^{ij}\theta^{kl}\langle\hat{F}_{ji},\hat{F}_{lk}\rangle_{*_2} \\
&\quad - \frac{1}{6}\theta^{ij}\theta^{kl}\theta^{mn}\langle\hat{F}_{jk},\hat{F}_{lm},\hat{F}_{ni}\rangle_{*_3} + \frac{1}{8}\theta^{ij}\theta^{kl}\theta^{mn}\langle\hat{F}_{ji},\hat{F}_{lm},\hat{F}_{nk}\rangle_{*_3} \\
&\quad - \frac{1}{48}\theta^{ij}\theta^{kl}\theta^{mn}\langle\hat{F}_{ji},\hat{F}_{lk},\hat{F}_{nm}\rangle_{*_3} + \mathcal{O}(\hat{A}^4) \\
\mathcal{Q}_1(x) &= \theta^{ij}\partial_j\hat{A}_i - \frac{1}{2}\theta^{ij}\theta^{kl}\partial_l\langle\hat{F}_{ji},\hat{A}_k\rangle_{*_2} \\
&\quad - \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{mn}\partial_n\langle\hat{F}_{jk},\hat{F}_{li},\hat{A}_m\rangle_{*_3} + \frac{1}{8}\theta^{ij}\theta^{kl}\theta^{mn}\partial_n\langle\hat{F}_{ji},\hat{F}_{lk},\hat{A}_m\rangle_{*_3} + \mathcal{O}(\hat{A}^4) \\
\mathcal{Q}_2(x) &= \frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l\langle\hat{A}_i,\hat{A}_k\rangle_{*_2} - \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{mn}\partial_l\partial_n\langle\hat{F}_{ji},\hat{A}_k,\hat{A}_m\rangle_{*_3} + \mathcal{O}(\hat{A}^4) \\
\mathcal{Q}_3(x) &= \frac{1}{6}\theta^{ij}\theta^{kl}\theta^{mn}\partial_j\partial_l\partial_n\langle\hat{A}_i,\hat{A}_k,\hat{A}_m\rangle_{*_3} + \mathcal{O}(\hat{A}^4)
\end{aligned}
\tag{2.13}
$$

It is a straightforward, though lengthy, exercise to check that

$$
\mathcal{Q}_0(x) + \mathcal{Q}_1(x) + \mathcal{Q}_2(x) + \mathcal{Q}_3(x) = 1 + \mathcal{O}(\hat{A}^4) \tag{2.14}
$$

This proves our identity Eq.(2.11), expressing the equivalence of the coupling in Eq.(2.9) to that in Eq.(2.10), up to $\mathcal{O}(\hat{A}^4)$ terms.

In fact, we can prove that this identity holds to all orders in $\hat{A}$ for the special case where the noncommutativity parameter θ is of rank two. In this case one can write

$$
\sqrt{\det(1-\theta\hat{F})} = 1 + \theta^{12}\hat{F}_{12} \tag{2.15}
$$

which is a considerable simplification of one factor in the formula. The proof of Eq.(2.11) for the rank two case is given in Appendix A. At present we do not have an explicit proof in the most general case.

It is illuminating to express Eq.(2.11) in the operator formalism. We use the fact that

$$
\int \frac{d^{p+1}x}{(2\pi)^{\frac{p+1}{2}}}\,\frac{1}{\mathrm{Pf}\,\theta} \to \mathrm{tr} \tag{2.16}
$$

to rewrite the LHS of Eq.(2.11) as follows:

$$\begin{aligned}\delta^{p+1}(k) &= \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, e^{ik.x} \\ &= \frac{1}{(2\pi)^{\frac{p+1}{2}}}\, \mathrm{tr}\,(\mathrm{Pf}\,\theta\; e^{ik.\mathbf{x}})\end{aligned} \tag{2.17}$$

The RHS of Eq.(2.11) can be converted to a symmetrised trace involving $\mathbf{X}^i \equiv \mathbf{x}^i + \theta^{ij}\hat{A}_j(\mathbf{x})$, following arguments in Ref.[17], and it becomes:

$$\frac{1}{(2\pi)^{\frac{p+1}{2}}}\, \mathrm{Str}\,(\mathrm{Pf}\, Q\; e^{ik.\mathbf{X}}) \tag{2.18}$$

where Str denotes the symmetrised trace. Finally, we use $[\mathbf{x}^i, \mathbf{x}^j] = i\theta^{ij}$ and $[\mathbf{X}^i, \mathbf{X}^j] = iQ^{ij}(\mathbf{x})$. Then, Eq.(2.11) takes the elegant form:

$$\mathrm{tr}\,\left(\mathrm{Pf}\,[\mathbf{x}^i, \mathbf{x}^j]\; e^{ik.\mathbf{x}}\right) = \mathrm{Str}\,\left(\mathrm{Pf}\,[\mathbf{X}^i, \mathbf{X}^j]\; e^{ik.\mathbf{X}}\right) \tag{2.19}$$

In this form, it is easy to see that Eq.(2.11) holds for constant $\hat{F}$, or equivalently for constant Q. In this special case it can be proved by pulling $\mathrm{Pf}\, Q$ out of the symmetrised trace on the RHS, and then using Eq.(2.16) with θ replaced by Q.

For the more general case where $\hat{F}$ and therefore Q is spatially varying, a suggestive line of argument runs as follows. To prove this identity, we basically need to make the replacement:

$$\mathrm{Str}\,\mathrm{Pf}\, Q(X) \rightarrow \int \frac{d^{p+1}X}{(2\pi)^{\frac{p+1}{2}}} \tag{2.20}$$

with the integrand on the RHS involving some suitable generalization of the Moyal $*$ product. For constant Q^{ij} this is valid with the usual Moyal $*$ product, while for varying Q^{ij} it requires the deformation quantization of a Poisson structure with variable coefficients. The existence of such a quantization is in fact guaranteed by the work of Kontsevich[21] (see also Ref.[22,23]), and was noted more recently in a context similar to the present one in Ref.[16]. Now given such a quantisation and its associated $*$ product, the RHS of Eq.(2.19) reduces to the integral over X of a simple exponential $e^{ik.X}$, and the result will presumably be a delta-function with any reasonable $*$ product. It is important to find a rigorous proof of Eq.(2.19) along these lines, and also to investigate whether the generalised associative product associated to $Q^{ij}(x)$ is helpful in writing down noncommutative brane couplings.

2.2. Coupling to the RR Form C^{p-1} and the Seiberg-Witten Map

Next let us turn to the coupling of a noncommutative p-brane to the RR form $C^{(p-1)}$. In the commutative case this form appears in a wedge product with the 2-form $B+F$. For the noncommutative brane in a constant RR background, it was observed in Ref.[1] that $B+F$ must be replaced by the 2-form Q^{-1} with Q^{ij} given by Eq.(2.8). It follows that the coupling in the general noncommutative case (with varying $C^{(p-1)}$) is:

$$\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,(Q^{-1})_{i_p i_{p+1}} W(x,C)\Big] * e^{ik.x} \tag{2.21}$$

For comparison, the coupling of a Dp-brane to the form $C^{(p-1)}$ in terms of commutative variables is given by:

$$\begin{aligned}&\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\; (B+F)_{i_p i_{p+1}}(x)\, e^{ik.x}\\ &= \;\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k) \left[\delta^{p+1}(k) B_{i_p i_{p+1}} + \widetilde{F}_{i_p i_{p+1}}(k)\right]\end{aligned} \tag{2.22}$$

Next, rewrite Q^{-1} as

$$\begin{aligned}Q^{-1} &= \theta^{-1}\left[1+\theta\hat{F}(1-\theta\hat{F})^{-1}\right]\\ &= B+\hat{F}(1-\theta\hat{F})^{-1}\end{aligned} \tag{2.23}$$

where we have used the relation $B=\theta^{-1}$. Using this relation and also Eq.(2.11), we can rewrite Eq.(2.21) as:

$$\begin{aligned}&\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k) \Bigg\{\delta^{p+1}(k)\, B_{i_p i_{p+1}} +\\ &\int \frac{d^{p+1}x}{(2\pi)^{p+1}}\; L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,\big(\hat{F}(1-\theta\hat{F})^{-1}\big)_{i_p i_{p+1}}\, W(x,C)\Big] * e^{ik.x}\Bigg\}\end{aligned} \tag{2.24}$$

Equating this to Eq.(2.22), we find that

$$\widetilde{F}_{ij}(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\; L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,\big(\hat{F}(1-\theta\hat{F})^{-1}\big)_{ij}\, W(x,C)\Big] * e^{ik.x} \tag{2.25}$$

This relates the commutative field strength F to the non-commutative field strength $\hat{F}$, therefore it amounts to a closed-form expression for the Seiberg-Witten map. This identity was previously conjectured starting from a Poisson approximation by Liu[16], who checked that to order $(\hat{A})^3$, the RHS agrees with the $\mathcal{O}(\hat{A})^3$ result of Ref.[14]. Here we see that it follows from the equivalence of commutative and noncommutative Chern-Simons couplings of a p-brane to a Ramond-Ramond $(p-1)$-form.

2.3. Coupling to the RR Form C^{p-3} and Lower Forms

The coupling of a noncommutative Dp-brane to the RR form $C^{(p-3)}$ was written down in Ref.[1] for the case of constant RR field:

$$\frac{1}{2}\,\mu_p\,\epsilon^{i_1\ldots i_{p+1}}\,\widetilde{C}^{(p-3)}_{i_1\ldots i_{p-3}}(0)\int\frac{d^{p+1}x}{(2\pi)^{p+1}}\,\sqrt{\det(1-\theta\hat{F})}\,(Q^{-1})_{i_{p-2}i_{p-1}}(Q^{-1})_{i_pi_{p+1}} \tag{2.26}$$

where the 2-form Q^{-1} is given in Eq.(2.23). For spatially varying $C^{(p-3)}$ we can therefore write the coupling as:

$$\begin{aligned}&\frac{1}{2}\,\mu_p\,\epsilon^{i_1\ldots i_{p+1}}\,\widetilde{C}^{(p-3)}_{i_1\ldots i_{p-3}}(-k)\;\times\\ &\int\frac{d^{p+1}x}{(2\pi)^{p+1}}\,L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,(Q^{-1})_{i_{p-2}i_{p-1}}(Q^{-1})_{i_pi_{p+1}}W(x,C)\Big]*e^{ik.x}\end{aligned} \tag{2.27}$$

This is to be compared with the commutative coupling. We do not expect this to give us new information about the relation between commutative and noncommutative gauge fields, since the Seiberg-Witten map has already been obtained in the previous subsection by comparing the couplings of $C^{(p-1)}$. Therefore, comparing the couplings of $C^{(p-3)}$ can at best provide a consistency check of what we have already deduced, unless we have further information about derivative corrections.

In the DBI approximation of slowly varying fields, the commutative coupling is:

$$\frac{1}{2}\,\mu_p\,\epsilon^{i_1\ldots i_{p+1}}\,\widetilde{C}^{(p-3)}_{i_1\ldots i_{p-3}}(-k)\int\frac{d^{p+1}x}{(2\pi)^{p+1}}\,(B+F)_{i_{p-2}i_{p-1}}(B+F)_{i_pi_{p+1}}e^{ik.x} \tag{2.28}$$

Inserting Eq.(2.23) for Q^{-1} in Eq.(2.27), and comparing with Eq.(2.28), we find that in the DBI approximation we must have:

$$\begin{aligned}&\int d^{p+1}k'\,\widetilde{F}_{ij}(k')\,F_{kl}(k-k')\\ &=\int\frac{d^{p+1}x}{(2\pi)^{p+1}}\,L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,\Big(\hat{F}(1-\theta\hat{F})^{-1}\Big)_{ij}\Big(\hat{F}(1-\theta\hat{F})^{-1}\Big)_{kl}W(x,C)\Big]*e^{ik.x}\end{aligned} \tag{2.29}$$

To arrive at this expression we have made use of the identities Eqs.(2.11) and (2.25).

First of all, for strictly constant F, the two sides match since in this case we have

$$F = \hat{F}(1 - \theta\hat{F})^{-1}$$

and $\widetilde{F}(k) \sim \delta(k)$. Then we can pull all the F and $\hat{F}$ out of the integrals, leaving $\delta(k)$ on both sides.

For slowly-varying F, we can use a procedure described in Ref.[1] and used in Ref.[16] where it leads to Eq.(5.8) of that paper. This consists of the replacement

$$\int \frac{d^{p+1}x}{(2\pi)^{p+1}} \sqrt{\det(1 - \theta\hat{F})} \to \int \frac{d^{p+1}X}{(2\pi)^{p+1}} \tag{2.30}$$

which, in the present case, results in the equation:

$$F_{ij}(X(x))F_{kl}(X(x)) = \left(\hat{F}(1 - \theta\hat{F})^{-1}\right)_{ij} \left(\hat{F}(1 - \theta\hat{F})^{-1}\right)_{kl} \tag{2.31}$$

This is just the square of equation (5.8) in Ref.[16].

This can be extended in a similar way to couplings involving the lower RR forms. In all these cases, it is interesting to examine how the derivative corrections match up, a point which we intend to address in a subsequent work.

3. Inclusion of Transverse Scalars

In this section we include RR couplings to the scalars $\hat{\Phi}^a$, $a = p+1, \ldots, 9$ that represent the transverse degrees of freedom of the noncommutative brane. One important effect of these scalars is to modify the Wilson lines by a term depending on an arbitrary momentum q_a. Another source of coupling between these scalars and the RR field comes about through the noncommutative analogue of Myers terms[1].

3.1. Modification of the Wilson line

We start by considering the scalar-dependence of the open Wilson lines. We will extract some q-dependent couplings to transverse scalars arising from this dependence. One interesting consequence will be a derivation of the Seiberg-Witten map for transverse scalars.

The open Wilson line including transverse scalars is given by:

$$W'(x,C) = P_* \exp\left[i\int_0^1 d\tau \Big(\frac{\partial\xi^i(\tau)}{\partial\tau}\hat{A}_i(x+\xi(\tau)) + q_a\hat{\Phi}^a(x+\xi(\tau))\Big)\right] \tag{3.1}$$

Inserting this definition in place of $W(x,C)$ in Eq.(2.3), and denoting the LHS by $\mathcal{Q}'(k)$, one finds that the couplings to a general spatially varying supergravity mode are:

$$\mathcal{Q}'(k) = \sum_{m=0}^{\infty}\int \frac{d^{p+1}x}{(2\pi)^{p+1}}\ \mathcal{Q}'_m(x)\ e^{ik.x} \tag{3.2}$$

where the $\mathcal{Q}'_m$ are given by

$$\begin{aligned}\mathcal{Q}'_m(x) = \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(\theta\partial)^{i_1}\ldots(\theta\partial)^{i_k}(iq)_{a_{k+1}}\ldots(iq)_{a_m}\ \times \\ \langle\mathcal{O}_1(x),\ldots,\mathcal{O}_n(x),\hat{A}_{i_1}(x),\ldots,\hat{A}_{i_k}(x),\hat{\Phi}^{a_{k+1}}(x),\ldots,\hat{\Phi}^{a_m}(x)\rangle_{*_{n+m}}\end{aligned} \tag{3.3}$$

For the coupling to the RR top form C^{p+1}, the operator $\sqrt{\det(1-\theta\hat{F})}$ that must be smeared over the Wilson line has to be generalised by the addition of terms coming from the pullback of the RR field components transverse to the brane. We will return to this in the following subsection. For now we will ignore such terms and just focus on the q-dependence of the coupling.

Let us plug in this operator into the expansion of the Wilson line of Eq.(3.1). To the order we are working, we need not consider the terms beyond $\mathcal{Q}'_3(x)$. The terms which do not contain any power of $\hat{\Phi}$ are exactly the same as the ones considered in the earlier section. Now let us collect the terms which contain one power of $\hat{\Phi}$ from $\mathcal{Q}'_1$, $\mathcal{Q}'_2$ and $\mathcal{Q}'_3$.

$$\begin{aligned}\mathcal{Q}'_1(x) &: iq_{a_1}\big\{\hat{\Phi}^{a_1} - \frac{1}{2}\theta^{kl}\langle\hat{F}_{lk},\hat{\Phi}^{a_1}\rangle_{*_2} - \frac{1}{4}\theta^{ij}\theta^{kl}\langle\hat{F}_{jk},\hat{F}_{li},\hat{\Phi}^{a_1}\rangle_{*_3} \\ &\qquad + \frac{1}{8}\theta^{ij}\theta^{kl}\langle\hat{F}_{ji},\hat{F}_{lk},\hat{\Phi}^{a_1}\rangle_{*_3}\big\} \\ \mathcal{Q}'_2(x) &: iq_{a_1}\big\{\theta^{ij}\partial_j\langle\hat{A}_i,\hat{\Phi}^{a_1}\rangle_{*_2} - \frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\langle\hat{F}_{lk},\hat{A}_i,\hat{\Phi}^{a_1}\rangle_{*_3}\big\} \\ \mathcal{Q}'_3(x) &: iq_{a_1}\frac{1}{2!}\theta^{ij}\theta^{kl}\partial_j\partial_l\langle\hat{A}_i,\hat{A}_k,\hat{\Phi}^{a_1}\rangle_{*_3}\end{aligned} \tag{3.4}$$

The contribution of these terms can be easily shown to be:

$$
\begin{aligned}
iq_{a_1}\big\{\hat\Phi^{a_1} + \frac{i}{2}\theta^{kl}\langle \hat A_l, [\hat A_k, \hat\Phi^{a_1}]\rangle_{*_2} + \frac{1}{2}\theta^{ij}\theta^{kl}\langle \hat A_i, \partial_l \hat A_j, \partial_k\hat\Phi^{a_1}\rangle_{*_3} + \theta^{ij}\langle \hat A_i, \partial_j\hat\Phi^{a_1}\rangle_{*_2} \\
- \theta^{ij}\theta^{kl}\langle \hat A_i, \partial_j \hat A_l, \partial_k\hat\Phi^{a_1}\rangle_{*_3} + \frac{1}{2}\theta^{ij}\theta^{kl}\langle \hat A_j, \hat A_l, \partial_i\partial_k\hat\Phi^{a_1}\rangle_{*_3}\big\}
\end{aligned}
\tag{3.5}
$$

After carrying out the Seiberg-Witten map, we expect that the coupling written in terms of noncommutative variables should give rise to the coupling in commutative variables in the DBI approximation, which is given by:

$$
\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\Big\{\widetilde C^{p+1}_{i_1\ldots i_{p+1}}(-k,q)\delta^{(p+1)}(k) + iq_a\widetilde\Phi^a(k)\, C^{p+1}_{i_1\ldots i_{p+1}}(-k,q) + \ldots\Big\} \tag{3.6}
$$

This is just the Taylor series expansion of the RR field considered as a functional of the transverse scalars. Here Φ is the SW transform of the noncommutative $\hat\Phi$. Hence the linear coupling in $\hat\Phi$, after the SW map, should become $iq_{a_1}\Phi^{a_1}$.

It follows that the SW map of a transverse scalar, to this order, is given by the quantity in braces in Eq.(3.5) above. One can read off the SW map for $\hat\Phi$ from that of $\hat A_b$ (Ref.[14]) by dimensional reduction[24]. It is easy to show that both of them match exactly to this order as expected.

3.2. Noncommutative Myers Terms

In this section we study the couplings of noncommutative p-branes to RR fields of rank greater than $p+1$. Our strategy will be as follows. For a collection of N D-instantons, we know the coupling [5,6] of the non-Abelian scalars ϕ^i to all the RR p-form fields. In this coupling we may substitute $\phi^i = X^i \equiv x^i + \theta^{ij}\hat A_j(x)$, $i = 0, \ldots, p$, representing the classical solution for a noncommutative p-brane along with fluctuations. In this way we find the corresponding couplings of the noncommutative brane, valid for constant RR fields. As in the preceding sections of this paper, we then smear these operators over an open Wilson line to derive the couplings to spatially varying RR fields. Since we know that a single commutative p-brane has no couplings to RR forms of rank greater than $p+1$, we will find another interesting identity similar to that in Eq.(2.11), but this time involving the transverse scalars.

For simplicity let us start with the case of a Euclidean D1-brane coupling to the RR 4-form in type IIB. The coupling of N D-instantons to the RR 4-form is

$$
\mathrm{tr}\left(\frac{1}{2!2^2}\left(-i[\phi^{i_1},\phi^{i_2}]\right)\left(-i[\phi^{i_3},\phi^{i_4}]\right)C^{(4)}_{i_1i_2i_3i_4}\right) \tag{3.7}
$$

Here ϕ^i represent all 10 transverse scalars on a D-instanton. Now insert $\phi^1 = X^1,\ \phi^2 = X^2$. The remaining ϕ^i are renamed $\hat{\Phi}^a$, they represent the scalars transverse to the noncommutative D1-brane. Thus we find the coupling:

$$\frac{1}{2!2}\,\epsilon_{ij}\,\mathrm{tr}\,\Big(\big(-i[X^i,X^j]\big)\big(-i[\hat{\Phi}^a,\hat{\Phi}^b]\big)-\big(-i[X^i,\hat{\Phi}^a]\big)\big(-i[X^j,\hat{\Phi}^b]\big)\Big)C^{(4)}_{12ab} \tag{3.8}$$

Making the replacements

$$\begin{aligned}-i[X^1,X^2] &= Q^{12}=\theta^{12}(1+\theta^{12}\hat{F}_{12})\\ [X^i,\hat{\Phi}^a] &= i\theta^{ij}D_j\hat{\Phi}^a\end{aligned} \tag{3.9}$$

the operator turns into:

$$\theta^{12}\Big((1+\theta^{12}\hat{F}_{12})\big(-i[\hat{\Phi}^a,\hat{\Phi}^b]\big)+\theta^{ij}D_j\hat{\Phi}^a D_i\hat{\Phi}^b\Big) \tag{3.10}$$

Now this can be smeared over the Wilson line as in Eq.(2.7) to find the operator coupling to $C_{12ab}(-k)$. The result should be compared with the corresponding coupling on a commutative brane. However, it is well-known that a single commutative p-brane does not couple to forms of rank greater than $p+1$, so the expression that we obtain must be equal to zero. As a result we find that:

$$0=\int\frac{d^{p+1}x}{(2\pi)^{p+1}}L_*\Big[\Big((1+\theta^{12}\hat{F}_{12})\big(-i[\hat{\Phi}^a,\hat{\Phi}^b]\big)+\theta^{ij}D_j\hat{\Phi}^a D_i\hat{\Phi}^b\Big)W'(x,C)\Big]*e^{ik.x} \tag{3.11}$$

This is a new identity. An explicit proof of this, for the case where $q_a=0$ in Eq.(3.1) (and hence $W'(x,C)=W(x,C)$) is given in Appendix B.

A more general version of the above identity can be obtained by inserting into the coupling Eq.(3.7) the classical solution and fluctuations for a system of n Dp-branes constructed out of infinitely many D-instantons. In this case, the commutative branes are non-Abelian and therefore they do couple to the higher RR forms. These known couplings have to be equal to the noncommutative couplings obtained by following through the above procedure. Our identity stating that Eq.(3.11) vanishes will then arise as the special case for $n=1$.

4. Non-BPS branes and RR couplings

It is well-known that non-BPS branes in superstring theory also couple to RR forms. In commutative variables, these couplings for a single non-BPS brane are given by:

$$\hat{S}_{CS} = \frac{\mu_{p-1}}{2T_0} \int dT \wedge \sum_n C^{(n)} \wedge e^{B+F} \tag{4.1}$$

where T is the tachyon field and T_0 is its value at the minimum of the tachyon potential. (More general tachyon couplings have been found in Refs.[25,26,27,28], but here we will only deal here with the term linear in T). In this section, we would like to express the couplings of RR fields to a non-BPS brane in a constant B-field, in terms of noncommutative variables in the background independent $\Phi = -B$ description.

In a previous paper[1], we found these couplings for the case of constant RR fields. Here we generalize them to non-constant RR fields. As we have seen above for BPS D-branes, RR forms couple to gauge invariant operators in the noncommutative world-volume gauge theory of the D-brane. These operators are obtained by smearing the operators which couple to constant RR fields, over a straight Wilson line. Here we follow the same prescription for non-BPS branes. It is important to note that there is no direct matrix-theory derivation of this prescription in this case. We will be able to show that it nevertheless gives rise to the correct commutative couplings, which is strong a posteriori justification for it.

Consider the coupling of a Euclidean non-BPS Dp-brane with an even number of world-volume directions, to the RR form $C^{(p)}$, in the commutative description:

$$\begin{aligned}\frac{\mu_{p-1}}{2T_0} \int dT \wedge C^{(p)} &= \frac{\mu_{p-1}}{2T_0} \int d^{p+1}x \, \epsilon^{i_1 i_2 \ldots i_{p+1}} \, \partial_{i_1} T(x) \; C^{(p)}_{i_2 \ldots i_{p+1}}(x) \\ &= \frac{\mu_{p-1}}{2T_0} \int d^{p+1}k \; \epsilon^{i_1 \ldots i_{p+1}} (-ik_{i_1}) \, \widetilde{T}(k) \, \widetilde{C}^{(p)}_{i_2 \ldots i_{p+1}}(-k)\end{aligned} \tag{4.2}$$

In Ref.[1] it was argued that the noncommutative generalisation of this coupling, for constant RR fields, is

$$\frac{\mu_{p-1}}{2T_0} \, \epsilon^{i_1 i_2 \ldots i_{p+1}} \, \widetilde{C}^{(p)}_{i_2 \ldots i_{p+1}}(0) \int \frac{d^{p+1}x}{(2\pi)^{p+1}} \sqrt{\det(1 - \theta \hat{F})} \, \mathcal{D}_{i_1} \hat{T}(x) \tag{4.3}$$

where

$$\mathcal{D}_i \hat{T}(x) = -i \; Q^{-1}_{ij} [X^j, \hat{T}(x)] \tag{4.4}$$

Then, the same RR form $C^{(p)}$ couples to a noncommutative non-BPS Dp-brane through the following coupling for each momentum mode:

$$\frac{\mu_{p-1}}{2T_0}\,\epsilon^{i_1 i_2\ldots i_{p+1}}\,\widetilde{C}^{(p)}_{i_2\ldots i_{p+1}}(-k)\int\frac{d^{p+1}x}{(2\pi)^{p+1}}\,L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,\mathcal{D}_{i_1}\hat{T}(x)\,W(x,C)\Big]*e^{ik.x} \quad (4.5)$$

Now we will show that this coupling is identical to Eq.(4.2) on carrying out the SW map, to terms containing at most three open-string fields. For this, we need the SW map of the quantity $\partial_a T$ (which has previously been examined in Ref.[24], to second order in open-string fields). This can be read off straightforwardly from the SW map of F, and is given by:

$$\begin{aligned}\partial_a T = {} & [C_a,\hat{T}] + \theta^{ij}\langle \hat{A}_i,[C_a,\hat{T}]\rangle_{*_2} + \frac{1}{2}\theta^{ij}\langle \hat{F}_{ij},[C_a,\hat{T}]\rangle_{*_2} + \theta^{ij}\langle \hat{F}_{ai},[C_j,\hat{T}]\rangle_{*_2}\\ & + \frac{1}{2}\theta^{ij}\theta^{kl}\partial_i\partial_k\langle [C_a,\hat{T}],\hat{A}_l,\hat{A}_j\rangle_{*_3} - \frac{1}{2}\theta^{ij}\theta^{kl}\partial_k\langle \hat{F}_{ij},[C_a,\hat{T}],\hat{A}_l\rangle_{*_3}\\ & - \theta^{ij}\theta^{kl}\partial_k\langle \hat{F}_{ai},[C_j,\hat{T}],\hat{A}_l\rangle_{*_3} + \frac{1}{2}\theta^{ij}\theta^{kl}\langle \hat{F}_{ai},[C_j,\hat{T}],\hat{F}_{kl}\rangle_{*_3}\\ & + \frac{1}{8}\theta^{ij}\theta^{kl}\langle [C_a,\hat{T}],\hat{F}_{ij},\hat{F}_{kl}\rangle_{*_3} + \frac{1}{4}\theta^{ij}\theta^{kl}\langle [C_a,\hat{T}],\hat{F}_{jk},\hat{F}_{il}\rangle_{*_3}\\ & + \theta^{ij}\theta^{kl}\langle \hat{F}_{ik},\hat{F}_{al},[C_j,\hat{T}]\rangle_{*_3}\end{aligned} \quad (4.6)$$

where $C_a = -i\theta^{-1}_{ab}X^b$.

Now let us expand the coupling on the noncommutative side. For this we expand the operator that is smeared over the Wilson line in Eq.(4.5) to terms with one $\hat{T}(x)$ and at most two $\hat{F}$'s:

$$\begin{aligned}\mathcal{O}_i(x) \equiv {} & \sqrt{\det(1-\theta\hat{F})}\,(-i)\,Q^{-1}_{ij}[X^j,\hat{T}(x)]\\ = {} & \sqrt{\det(1-\theta\hat{F})}\left\{[C_i,\hat{T}] + \left(\frac{\hat{F}}{1-\theta\hat{F}}\right)_{ik}\theta^{kl}\,[C_l,\hat{T}]\right\}\\ = {} & \left\{1-\frac{1}{2}\mathrm{tr}\,(\theta\hat{F}) - \frac{1}{4}\mathrm{tr}\,(\theta\hat{F}\theta\hat{F}) + \frac{1}{8}\big(\mathrm{tr}\,(\theta\hat{F})\big)^2\right\}[C_i,\hat{T}]\\ & - \frac{1}{2}\mathrm{tr}\,(\theta\hat{F})\,\hat{F}_{ik}\theta^{kl}\,[C_l,\hat{T}] + \hat{F}_{ik}\theta^{kl}\,[C_l,\hat{T}] + \hat{F}_{ij}\theta^{jk}\hat{F}_{kl}\theta^{lm}\,[C_m,\hat{T}]\end{aligned} \quad (4.7)$$

With this expansion at hand, it is straightforward to show that the noncommutative coupling is equivalent to the commutative one using the SW map in Eq.(4.6).

Clearly one can extend this logic to obtain the coupling of a noncommutative non-BPS p-brane to lower and higher RR forms, though we will not work this out here. However, it is remarkable that the prescription formulated for BPS branes works for non-BPS branes in the case we have investigated. This fact might provide a clue to the open problem of constructing unstable non-BPS D-branes from matrix theory. Various results on the construction of brane-antibrane pairs from matrix theory can be found in Refs.[29,30,31].

5. Conclusions

We have seen that the computation of RR couplings for noncommutative branes, initiated in Ref.[1], can be elegantly extended to spatially varying RR fields using the ideas in Refs.[16,17]. Comparison of the noncommutative couplings to commutative ones gives rise to a number of interesting identities including the Seiberg-Witten map.

These ideas were also extended to incorporate transverse scalars, with analogous results. The generalisation to RR couplings of unstable, non-BPS branes also gives sensible results despite the fact that in this case the construction of the branes starting from matrix theory is on less solid ground.

Some interesting problems that we have not addressed include the question of whether one can gain some insight into the nonabelian SW map by these methods. One should also ask what interesting physical effects follow from noncommutative RR couplings, analogous for example to the Myers effect for nonabelian branes.

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Appendix A. Proof of Eq.(2.11) for θ of Rank Two

In this appendix we present a proof of the identity (2.11) for the case where θ is of rank two, namely a Euclidean D1-brane. For this, we write down all terms of $\mathcal{O}(\hat{A}^n)$ and show that for any $n > 0$, the sum of the contributions vanishes identically.

In the rank two case, the operator $\sqrt{\det(1-\theta\hat{F})}$ just becomes $1+\theta^{12}\hat{F}_{12}$. Recall that the quantities $\mathcal{Q}_n$ defined in Eq.(2.4) contain contributions of order $\hat{A}^n$ from the expansion of the Wilson line. In addition we can select terms of order $0,1,2$ in $\hat{A}$ from the operator $(1+\theta^{12}\hat{F}_{12})$. Thus, terms of order $\hat{A}^n$ can arise from $\mathcal{Q}_n(x)$, $\mathcal{Q}_{n-1}(x)$ and $\mathcal{Q}_{n-2}(x)$. These terms are:

$$\begin{aligned}
\text{From } \mathcal{Q}_n(x) :& \ \frac{1}{n!}\theta^{i_1j_1}...\theta^{i_nj_n}\partial_{j_1}...\partial_{j_n}\langle \hat{A}_{i_1},...,\hat{A}_{i_n}\rangle_{*_n} \\
\text{From } \mathcal{Q}_{n-1}(x) :& \ \frac{1}{(n-1)!}\theta^{i_1j_1}...\theta^{i_{n-1}j_{n-1}}\partial_{j_1}...\partial_{j_{n-1}}\langle \theta^{12}(\partial_1\hat{A}_2-\partial_2\hat{A}_1),\hat{A}_{i_1},...,\hat{A}_{i_{n-1}}\rangle_{*_n} \\
\text{From } \mathcal{Q}_{n-2}(x) :& \ \frac{1}{(n-2)!}\theta^{i_1j_1}...\theta^{i_{n-2}j_{n-2}}\partial_{j_1}...\partial_{j_{n-2}}\langle -i\theta^{12}[\hat{A}_1,\hat{A}_2],\hat{A}_{i_1},...,\hat{A}_{i_{n-2}}\rangle_{*_{n-1}}
\end{aligned} \tag{A.1}$$

The summed indices in the above expression can only take the values 1 and 2. Writing them explicitly, we find that the above contributions are:

$$\begin{aligned}
\text{From } \mathcal{Q}_n(x) :& \ (\theta^{12})^n\sum_{r=0}^{n}\frac{(-1)^r}{r!(n-r)!}\partial_1^r\partial_2^{n-r}\langle \hat{A}_1^{n-r},\hat{A}_2^r\rangle_{*_n} \\
\text{From } \mathcal{Q}_{n-1}(x) :& \ (\theta^{12})^n\Big\{\sum_{q=0}^{n-1}\frac{(-1)^q}{(q+1)!(n-q-1)!}\partial_1^q\partial_2^{n-q-1}\langle \hat{A}_1^{n-q-1},\partial_1(\hat{A}_2^{q+1})\rangle_{*_n} \\
&\qquad -\sum_{t=0}^{n-1}\frac{(-1)^t}{t!(n-t)!}\partial_1^t\partial_2^{n-t-1}\langle \partial_2(\hat{A}_1^{n-t}),\hat{A}_2^t\rangle_{*_n}\Big\} \\
\text{From } \mathcal{Q}_{n-2}(x) :& \ (\theta^{12})^n\Big\{\sum_{p=0}^{n-2}\frac{(-1)^p}{(p+1)!(n-p-1)!}\partial_1^p\partial_2^{n-p-2}\langle \partial_1(\hat{A}_1^{n-p-1}),\partial_2(\hat{A}_2^{p+1})\rangle_{*_n} \\
&\qquad -\sum_{s=0}^{n-2}\frac{(-1)^s}{(s+1)!(n-s-1)!}\partial_1^s\partial_2^{n-s-1}\langle \partial_2(\hat{A}_1^{n-s-1}),\partial_1(\hat{A}_2^{s+1})\rangle_{*_n}\Big\}
\end{aligned} \tag{A.2}$$

Here we have also used the recursive relation

$$\theta^{ij}\partial_j\langle f_1,\ldots,f_n,\partial_i g\rangle_{\star_n} = i\sum_{j=1}^{n-1}\langle f_1,\ldots,[f_j,g],\ldots,f_{n-1}\rangle_{\star_{n-1}} \tag{A.3}$$

to convert the $*_{n-1}$ products in $\mathcal{Q}_{n-2}(x)$ to $*_n$ products.

Now it is easy to check that the three contributions above add up to zero. Therefore we have proved the proposed identity Eq.(2.11) in the case where θ^{ij} is of rank two.

Appendix B. Proof of Eq.(3.11) for θ of Rank Two

This proof is similar in spirit to the one in the preceding Appendix. The main difference is that the operator to be smeared over the open Wilson line is the one in Eq.(3.10).

To simplify the proof we take $q_a = 0$ in Eq.(3.1), so that the $\mathcal{Q}'_m(x)$ in Eq.(3.3) reduce to the $\mathcal{Q}_m(x)$ in Eq.(2.4).

As before, we collect all the terms containing n powers of $\hat{A}$, and they can arise from $\mathcal{Q}_n, \mathcal{Q}_{n-1}$ and $\mathcal{Q}_{n-2}$. Thus we have the following three terms:

$$\begin{aligned}
\text{From } \mathcal{Q}_n : \ & \frac{1}{n!}\theta^{i_1 j_1} \ldots \theta^{i_n j_n} \partial_{j_1} \ldots \partial_{j_n} \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_{i_1}, \ldots, \hat{A}_{i_n} \rangle_{\star_{n+1}} \\
& + \theta^{ij} \langle \partial_j \hat{\Phi}^a, \partial_i \hat{\Phi}^b, \hat{A}_{i_1}, \ldots, \hat{A}_{i_n} \rangle_{\star_{n+2}} \Big\} \\
\text{From } \mathcal{Q}_{n-1} : \ & \frac{1}{(n-1)!}\theta^{i_1 j_1} \ldots \theta^{i_{n-1} j_{n-1}} \partial_{j_1} \ldots \partial_{j_{n-1}} \\
& \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], \theta^{12}(\partial_1 \hat{A}_2 - \partial_2 \hat{A}_1), \hat{A}_{i_1}, \ldots, \hat{A}_{i_{n-1}} \rangle_{\star_{n+1}} \\
& + \theta^{ij} \langle \partial_j \hat{\Phi}^a, -i[\hat{A}_i, \hat{\Phi}^b], \hat{A}_{i_1}, \ldots, \hat{A}_{i_{n-1}} \rangle_{\star_{n+1}} \\
& + \theta^{ij} \langle -i[\hat{A}_j, \hat{\Phi}^a], \partial_i \hat{\Phi}^b, \hat{A}_{i_1}, \ldots, \hat{A}_{i_{n-1}} \rangle_{\star_{n+1}} \Big\} \\
\text{From } \mathcal{Q}_{n-2} : \ & \frac{1}{(n-2)!}\theta^{i_1 j_1} \ldots \theta^{i_{n-2} j_{n-2}} \partial_{j_1} \ldots \partial_{j_{n-2}} \\
& \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], -i\theta^{12}[\hat{A}_1, \hat{A}_2], \hat{A}_{i_1}, \ldots, \hat{A}_{i_{n-2}} \rangle_{\star_n} \\
& + \theta^{ij} \langle -i[\hat{A}_j, \hat{\Phi}^a], -i[\hat{A}_i, \hat{\Phi}^b], \hat{A}_{i_1}, \ldots, \hat{A}_{i_{n-2}} \rangle_{\star_n}
\end{aligned} \tag{B.1}$$

Again, these expressions can be simplified using the fact that the summed indices take only the values 1 and 2:

$$\begin{aligned}
\text{From } \mathcal{Q}_n : \ & \frac{(\theta^{12})^n}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \partial_1^r \, \partial_2^{n-r} \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-r}, \hat{A}_2^r \rangle_{\star_{n+1}} \\
& + \theta^{12} \langle \partial_2 \hat{\Phi}^a, \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-r}, \hat{A}_2^r \rangle_{\star_{n+2}} - \theta^{12} \langle \partial_1 \hat{\Phi}^a, \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-r}, \hat{A}_2^r \rangle_{\star_{n+2}} \Big\}
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\text{From } \mathcal{Q}_{n-1} : \ & \frac{(\theta^{12})^n}{(n-1)!} \sum_{q=0}^{n-1} (-1)^q \binom{n-1}{q} \partial_1^q \, \partial_2^{n-q-1} \\
& \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], (\partial_1 \hat{A}_2 - \partial_2 \hat{A}_1), \hat{A}_1^{n-q-1}, \hat{A}_2^q \rangle_{\star_{n+1}} \\
& + \langle \partial_2 \hat{\Phi}^a, -i[\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-q-1}, \hat{A}_2^q \rangle_{\star_{n+1}} - \langle \partial_1 \hat{\Phi}^a, -i[\hat{A}_2, \hat{\Phi}^b], \hat{A}_1^{n-q-1}, \hat{A}_2^q \rangle_{\star_{n+1}} \\
& + \langle -i[\hat{A}_2, \hat{\Phi}^a], \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-q-1}, \hat{A}_2^q \rangle_{\star_{n+1}} - \langle -i[\hat{A}_1, \hat{\Phi}^a], \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-q-1}, \hat{A}_2^q \rangle_{\star_{n+1}} \Big\}
\end{aligned} \tag{B.3}$$

$$\text{From } \mathcal{Q}_{n-2} : \frac{(\theta^{12})^{n-1}}{(n-2)!} \sum_{p=0}^{n-2} (-1)^p \binom{n-2}{p} \partial_1^p \partial_2^{n-p-2} \Big\{ \langle -i[\hat{\Phi}^a, \hat{\Phi}^b], -i[\hat{A}_1, \hat{A}_2], \hat{A}_1^{n-p-2}, \hat{A}_2^p \rangle_{\star_n} + \langle -i[\hat{A}_2, \hat{\Phi}^a], -i[\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-p-2}, \hat{A}_2^p \rangle_{\star_n} - \langle -i[\hat{A}_1, \hat{\Phi}^a], -i[\hat{A}_2, \hat{\Phi}^b], \hat{A}_1^{n-p-2}, \hat{A}_2^p \rangle_{\star_n} \Big\} \tag{B.4}$$

Finally, we rewrite these expressions using the recursion relation Eq.(A.3), to get:

$$\text{From } \mathcal{Q}_n : -i(\theta^{12})^n \sum_{r=0}^{n} \frac{(-1)^r}{r!\,(n-r)!} \partial_1^r \partial_2^{n-r} \langle [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-r}, \hat{A}_2^r \rangle_{\star_{n+1}} + (\theta^{12})^{n+1} \sum_{s=0}^{n} \frac{(-1)^s}{s!\,(n-s)!} \partial_1^s \partial_2^{n-s} \{ \langle \partial_2 \hat{\Phi}^a, \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-s}, \hat{A}_2^s \rangle_{\star_{n+2}} - \langle \partial_1 \hat{\Phi}^a, \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-s}, \hat{A}_2^s \rangle_{\star_{n+2}} \} \tag{B.5}$$

$$\begin{aligned}
\text{From } \mathcal{Q}_{n-1} : & -i(\theta^{12})^n \Big\{ \sum_{a=0}^{n-1} \frac{(-1)^a}{a!\,(n-a-1)!} \partial_1^a \partial_2^{n-a-1} \Big(\langle \partial_2 \hat{\Phi}^a, [\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-a-1}, \hat{A}_2^a \rangle_{\star_{n+1}} \\
& - \langle [\hat{A}_1, \hat{\Phi}^a], \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-a-1}, \hat{A}_2^a \rangle_{\star_{n+1}} \Big) \\
& - (i\theta^{12}) \sum_{b=0}^{n-1} \frac{(-1)^b}{(b+1)!\,(n-b-1)!} \partial_1^{b+1} \partial_2^{n-b-1} \Big(\langle \partial_1 \hat{\Phi}^a, \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-b-1}, \hat{A}_2^{b+1} \rangle_{\star_{n+2}} \\
& - \langle \partial_2 \hat{\Phi}^a, \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-b-1}, \hat{A}_2^{b+1} \rangle_{\star_{n+2}} \Big) \\
& + \sum_{c=0}^{n-1} \frac{(-1)^c}{(c+1)!\,(n-c-1)!} \partial_1^c \partial_2^{n-c-1} \Big(\langle \partial_1 [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-c-1}, \hat{A}_2^{c+1} \rangle_{\star_{n+1}} \\
& + \langle [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-c-1}, \partial_1 (\hat{A}_2^{c+1}) \rangle_{\star_{n+1}} \Big) \\
& + \sum_{d=0}^{n-2} \frac{(-1)^d}{(d+1)!\,(n-d-2)!} \partial_1^d \partial_2^{n-d-1} \Big(\langle \partial_1 \hat{\Phi}^a, [\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-d-2}, \hat{A}_2^{d+1} \rangle_{\star_{n+1}} \\
& - \langle [\hat{A}_1, \hat{\Phi}^a], \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-d-2}, \hat{A}_2^{d+1} \rangle_{\star_{n+1}} \Big) \\
& - \sum_{e=0}^{n-1} \frac{(-1)^e}{e!\,(n-e)!} \partial_1^e \partial_2^{n-e-1} \langle [\hat{\Phi}^a, \hat{\Phi}^b], \partial_2 (\hat{A}_1^{n-e}), \hat{A}_2^e \rangle_{\star_{n+1}} \Big\}
\end{aligned} \tag{B.6}$$

$$
\begin{aligned}
\text{From } \mathcal{Q}_{n-2} : i(\theta^{12})^n \Big\{ & \sum_{f=0}^{n-2} \frac{(-1)^f}{(f+1)!\,(n-f-1)!} \partial_1^f \partial_2^{n-f-1} \Big(\langle \partial_1 [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-f-1}, \hat{A}_2^{f+1} \rangle_{\star_{n+1}} \\
& + \langle [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-f-1}, \partial_1 (\hat{A}_2^{f+1}) \rangle_{\star_{n+1}} \Big) \\
& - \sum_{h=0}^{n-2} \frac{(-1)^h}{(h+1)!\,(n-h-2)!} \partial_1^h \partial_2^{n-h-1} \Big(\langle [\hat{A}_1, \hat{\Phi}^a], \partial_1 \hat{\Phi}^b, \hat{A}_1^{n-h-2}, \hat{A}_2^{h+1} \rangle_{\star_{n+1}} \\
& - \langle \partial_1 \hat{\Phi}^a, [\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-h-2}, \hat{A}_2^{h+1} \rangle_{\star_{n+1}} \Big) \\
& - \sum_{i=0}^{n-2} \frac{(-1)^i}{(i+1)!\,(n-i-1)!} \partial_1^{i+1} \partial_2^{n-i-2} \Big(\langle \partial_2 [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-i-1}, \hat{A}_2^{i+1} \rangle_{\star_{n+1}} \\
& + \langle [\hat{\Phi}^a, \hat{\Phi}^b], \hat{A}_1^{n-i-1}, \partial_2 (\hat{A}_2^{i+1}) \rangle_{\star_{n+1}} \Big) \\
& + \sum_{k=0}^{n-2} \frac{(-1)^k}{(k+1)!\,(n-k-2)!} \partial_1^{k+1} \partial_2^{n-k-2} \Big(\langle [\hat{A}_1, \hat{\Phi}^a], \partial_2 \hat{\Phi}^b, \hat{A}_1^{n-k-2}, \hat{A}_2^{k+1} \rangle_{\star_{n+1}} \\
& - \langle \partial_2 \hat{\Phi}^a, [\hat{A}_1, \hat{\Phi}^b], \hat{A}_1^{n-k-2}, \hat{A}_2^{k+1} \rangle_{\star_{n+1}} \Big) \Big\}
\end{aligned}
\tag{B.7}
$$

The reader will readily verify that the three expressions in Eqs.(B.5), (B.6) and (B.7) above add up to 0. This completes the proof.

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