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Liouville D-branes in Two-Dimensional Strings and Open String Field Theory

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We study open strings in the noncritical $c = 1$ bosonic string theory compactified on a circle at self-dual radius. These strings live on D-branes that are extended along the Liouville direction (*FZZT* branes). We present explicit expressions for the disc two- and three-point functions of boundary operators in this theory, as well as the bulk-boundary two-point function. The expressions obtained are divergent because of resonant behaviour at self-dual radius. However, these can be regularised and renormalized in a precise way to get finite results. The boundary correlators are found to depend only on the differences of boundary cosmological constants, suggesting a fermionic behaviour. We initiate a study of the open-string field theory localised to the physical states, which leads to an interesting matrix model.

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1. Introduction

Non-critical string theories, describing strings propagating in two dimensions or less, were instrumental in shaping our understanding of the behaviour of string theory beyond the perturbative regime. The $\mathcal{O}(1/g_s)$ nonperturbative effect, so characteristic of D-branes, first emerged from the study of these systems[1]. Only recently, thanks to the advancement in our understanding of boundary Liouville dynamics[2–14] (following earlier work [15,16]), has a physical understanding of the nonperturbative effects begun to emerge[17–22]. (See the review [23] for an exhaustive list of references.)

In another development, the dynamics of tachyon condensation led Sen to propose a new duality between open and closed strings[24–26]. Noncritical string theories are likely to be ideally suited for understanding this duality and indeed they have already played an important role in the shaping of these ideas. Recently in an interesting work[22], Gaiotto and Rastelli applied this philosophy to Liouville theory coupled to $c = -2$ matter. This system has certain topological symmetries[27] constraining its dynamics. Using these symmetries the authors obtain the Kontsevich topological matrix model[28] describing the *closed-string* theory starting from the *open-string field theory*.

Among the non-critical string theories, the theory of a single scalar field coupled to worldsheet gravity has perhaps the richest structure. The matter theory has central charge $c = 1$, while the Liouville field with its central charge $c_L = 25$ provides an interpretation as a *critical* string theory with two-dimensional target space. Closed strings in this background have been studied in the past from quite a few different angles: matrix quantum mechanics (see Ref.[29] and references therein), worldsheet conformal field theory[30–32], topological field theory[33–35] and topological matrix models[36–39] related to the moduli space of Riemann surfaces (see [40] for a recent review).

The $c = 1$ closed-string theory has a marginal deformation, corresponding to changing the radius R of compactification of the scalar field. At a particular value of this radius, $R = 1$ in our conventions, the theory is self-dual under T-duality and an $SU(2) \times SU(2)$ symmetry gets restored as a result of which momentum and winding modes become degenerate with each other.

In this paper, we will consider the open-string version of this two-dimensional string theory – more precisely, a scalar field compactified at the self-dual radius on a worldsheet with the topology of a disc or an upper half plane, coupled to the Liouville mode. Various types of branes are possible depending on the choice of boundary condition on the fields.

We will choose to work with (generalized) Neumann boundary conditions on the Liouville field φ. On the matter field X we impose Dirichlet boundary conditions, as a result of which the brane is localised in X and there are no momentum modes in that direction, only winding modes. Because the radius is self-dual, one can equally well impose Neumann boundary conditions in X and then there are momentum but no winding modes. The physics is identical in the two cases.

The resulting branes are stable and are known as FZZT branes[2–4]. We will compute the two- and three-point disc correlation functions of the fields living on the FZZT branes, as well as the bulk-boundary two-point function of such fields with 'bulk' fields. The Liouville contributions to these correlators are non-trivial and general expressions are available in the literature[2,8,7,14]. Some of them are only known in the form of contour integrals. From the point of view of these theories, the $c_L = 25$ Liouville field coupled to $c = 1$ matter is at the 'boundary' of the theories studied in Refs.[2--16]. In the specific case of interest, we take a careful limit to obtain the desired correlators. In particular we are able to evaluate the relevant contour integrals in our case, leading to expressions that are much simpler and more explicit than those previously given in the literature for the more general $c \leq 1$ case.

The resulting expressions satisfy the expected consistency conditions and other recursion relations. When the Liouville theory is combined with matter, one gets a massless 'tachyon' field[1] labelled by integer winding numbers. The matter contribution to the tachyon correlators are just winding number conserving delta functions. In addition to the tachyons, there are discrete states at ghost numbers one and zero[41,42]. The former are the remnants of massless and massive states of critical strings and their correlators are determined by the SU(2) symmetry at the self-dual radius. (The latter class of operators are characteristic of non-critical theories and in particular, they form a ring on which the symmetry of the theory can be realized in a geometric way[43].) We have not attempted to study this ring in the FZZT brane background (for general results on the $c \leq 1$ boundary ground ring, see Ref.[44]). As mentioned above, the expressions for these correlators are divergent. As we will see, once we perform renormalizations of the bulk and boundary cosmological constants, the divergence is a common multiplicative factor for both the two- and three-point boundary correlators.

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[1] No operator in this paper is truly unstable on the worldsheet. The FZZT boundary conditions do not allow such modes. With this in mind, and since there is also no unstable operator in the bulk, we use the word tachyon as is conventional.

The simple and elegant form of the answers obtained is suggestive of a simple physical interpretation, perhaps in terms of fermions, as we will see. The answers share some of the properties of the (simpler) case of $c = -2$[22], notably that they are independent of the bulk cosmological constant. All this encourages us to try and understand the corresponding open-string field theory, following the ideas in Ref.[22]. Accordingly, in the last section of this paper, we begin to study the open-string field theory of the FZZT branes. Motivated by the fact that the disc path integral describing classical processes of non-critical string theory localizes to the BRS cohomology, we evaluate the action for the 'on-shell' states (tachyons and the discrete states). This results in a non-local theory of infinitely many matrices. We hope to analyse this theory in more detail in the future.

The organisation of this paper is as follows. Sec. 2 describes the background and sets up notation. The Liouville contributions to the two- and three-point functions are evaluated in Sections 3 and 4. In Sec. 5 we calculate the bulk-boundary two-point function. Sec. 6 is devoted to string field theory. We end with some comments in the final Sec. 7. Appendix A contains some properties of the special functions that appear in the Liouville correlators. Some details of the contour integral relevant for Sec. 4 are given in Appendix B.

2. Two-dimensional Open String Theory and the FZZT branes

The theory we are interested in is described by the worldsheet action[2]

$$\begin{aligned}&\frac{1}{4\pi}\int_{\mathcal{D}}\Big((\partial X)^2+(\partial\varphi)^2+Q\hat{R}\varphi+4\pi\mu_0\,e^{2b\varphi}\Big)\\&+\frac{1}{2\pi}\int_{\partial\mathcal{D}}\Big(Q\hat{K}\varphi+2\pi\mu_{B,0}\,e^{b\varphi}\Big)\,,\end{aligned}\tag{2.1}$$

where X, ϕ are the matter and Liouville fields and Q, b are numerical coefficients. In this action, $\mathcal{D}$ has the topology of a disc/UHP, $\hat{R}$ is the curvature of the (reference) metric, $\hat{K}$ the induced curvature of the boundary and μ_0 and $\mu_{B,0}$ are the (bare) bulk and boundary cosmological constants respectively.

With the action above, the matter sector has central charge $c = 1$ while the Liouville sector has $c_L = 1 + 6Q^2$. The coefficient b appearing in the exponents satisfies $Q = b + \frac{1}{b}$. Criticality requires the choice $c_L = 25$, from which we determine $Q = 2$ and $b = 1$. Because

[2] We work in $\alpha' = 1$ units.

of divergences that appear at $b = 1$, we will need to carefully take the limit $b \to 1$ and regularise the divergences appropriately.

On the field φ, we will impose

$$i(\partial - \overline{\partial})\varphi = 4\pi\mu_{B,0}\, e^{b\varphi}, \tag{2.2}$$

the generalized Neumann boundary condition.

The field X, which we take to be Euclidean is, in general, compactified on a circle of radius R. We can impose a suitable boundary condition on the field X; for instance, at a generic radius, we could impose Dirichlet or Neumann boundary conditions $(\partial \pm \overline{\partial})X = 0$ which, in conjunction with the boundary condition on the Liouville field above, describe the non-compact D-instanton and $D0$-brane respectively. As noted in the introduction, the two choices are physically equivalent at self-dual radius.

In the bulk, the observables of the theory are a massless scalar field in the two dimensional target space known as the 'tachyon' field, and an infinite set of quantum mechanical states which arise at special values of the momentum, known as the discrete states. The vertex operators[3] corresponding to the tachyon field take the following form at weak coupling:

$$\mathcal{T}_k = c\bar{c}\, \exp\left(ik(X \pm \overline{X}) + (2 - |k|)\varphi\right). \tag{2.3}$$

The tachyon vertex operator on the boundary on the other hand carries additional indices (σ_1, σ_2) corresponding to the boundary conditions on the two ends of the open string:

$$T_k^{\sigma_1\sigma_2} \equiv c\, \left[e^{ikX}\, V_\beta\right]^{\sigma_1\sigma_2} = c\, \left[\exp\left(ikX + \beta\varphi\right)\right]^{\sigma_1\sigma_2}, \tag{2.4}$$

where the second expression is the asymptotic form. From this we see that β labels the Liouville momentum, and the conformal dimension of the Liouville vertex operator is $\Delta = \beta(Q - \beta)$ where $Q = b + \frac{1}{b} = 2$. Requiring that the full vertex operator has dimension one, one finds the on-shell condition $\beta = 1 - |k|$. The boundary label σ is related to the (bare) cosmological constants μ_0 and $\mu_{B,0}$ by:

$$\cos 2\pi b\left(\sigma - \frac{Q}{2}\right) = \frac{\mu_{B,0}}{\sqrt{\mu_0}}\sqrt{\sin \pi b^2}. \tag{2.5}$$

As we shall discuss later, the cosmological constants require renormalisation in the $c = 1$ string theory.

[3] We are only considering local operators, which correspond to non-normalizable modes.

We are specifically interested in the theory at self-dual radius $R = 1$, where the worldsheet theory is an $SU(2)_L \times SU(2)_R$ current algebra at level 1. The symmetry of the closed-string theory is generated by $(J^{\pm}, J^3) = (e^{\pm i2X}, i\partial X)$ and their right moving counterparts. The physical vertex operators at ghost number one are[43]:

$$Z_{k;m,\overline{m}} = c\overline{c}\, V^{mat}(k,m)\overline{V}^{mat}(k,\overline{m}) \exp\left((2-k)\varphi\right). \tag{2.6}$$

where k is a non-negative integer or half integer; $V^{mat}(k,k) \equiv e^{ikX}$, and the operators $V^{mat}(k, m < k)$ are defined by acting with the $SU(2)_L$ lowering operator. Hence $m = k, k-1, \cdots, -k$. The corresponding right movers are defined in a similar manner. The physical content of the theory can also be summarized as a massless field $\mathcal{T}(\theta, \phi, \psi; \varphi)$ living on an S^3 times the non-compact Liouville direction.

The open string imposes a boundary condition relating the left and right moving currents J^a and $\overline{J}^a$. The branes in the $SU(2)_n$ theory are labelled by a half-integer $J = 0, \cdots, \frac{n}{2}$ which labels the conjugacy class in the group, and continuous moduli which take values in $SO(3)$ which label the origin of the 3-sphere viewed as a group manifold[45]. The conjugacy classes are topologically 2-spheres in the group manifold.

For our case, level $n = 1$, there are only two possible discrete labels $J = 0, 1/2$ and the full moduli space is $SU(2)$ which is topologically S^3[46]. A brane is simply a point on this sphere, which can be thought of as a degenerate S^2. It breaks the $SO(4) = SU(2) \times SU(2)$ symmetry of the 3-sphere to a diagonal $SU(2)$ symmetry group of the degenerate 2-sphere. The open-string modes are classified as representations of this $SU(2)$.

For example, the boundary states which correspond to Neumann and Dirichlet for generic radii are labelled by the two poles on the S^3, and are given respectively by $J^a = \pm\overline{J}^a$. The generators of the diagonal $SU(2)$ subgroup which is preserved are $J^a \pm \overline{J}^a$. The allowed representations of the diagonal SU(2) are $k \pm \overline{k}$ where both k and $\overline{k}$ are integer or half integer, so that the allowed representations of the diagonal subgroup are integer. Note that half the representations of $SU(2)$ (the half-integer spins) do not correspond to physical operators.

Thus the physical vertex operators of the open string at ghost number one are:

$$Y^{\sigma_1\sigma_2}(k,m) \equiv c\,\left[V^{mat}(k,m)V_\beta\right]^{\sigma_1,\sigma_2} = c\,\left[V^{mat}(k,m)\exp((1-k)\varphi)\right]^{\sigma_1\sigma_2}, \tag{2.7}$$

where, (k, m) are the usual $SU(2)$ labels with spin k an integer and $m = k, k-1, \ldots, -k$.

3. Boundary Two-Point Function

In this section, we shall compute the two-point function of the Liouville vertex operators $V_\beta^{\sigma_1\sigma_2}$ which enter the physical open-string vertex operators (2.7). The two-point function of boundary operators in Liouville theory, of arbitrary central charge $c_L = 1+6Q^2$, is given by[2]:

$$\left\langle V_{\beta_1}^{\sigma_1\sigma_2}(x) V_{\beta_2}^{\sigma_2\sigma_1}(0) \right\rangle \equiv \frac{\delta(\beta_1+\beta_2-Q) + d(\beta|\sigma_1,\sigma_2)\delta(\beta_1-\beta_2)}{|x|^{2\Delta_{\beta_1}}}, \tag{3.1}$$

where $d(\beta|\sigma_1,\sigma_2)$ is the *reflection amplitude*, the expression for which is given below. The delta functions can be understood as arising due to the reflection from the Liouville potential, and is not present in the higher-point functions. Every non-normalizable operator in the theory is related to a normalizable operator by this reflection, $V_\beta^{\sigma_1\sigma_2} = d(\beta|\sigma_1,\sigma_2) V_{Q-\beta}^{\sigma_1\sigma_2}$.

The reflection amplitude $d(\beta|\sigma_1,\sigma_2)$ is given by[2]:

$$\begin{aligned} d(\beta|\sigma_1,\sigma_2) &= \mathcal{A}_1\, \mathcal{A}_2\, \mathcal{A}_3, \\ \mathcal{A}_1 &= \left(\pi\mu_0\gamma(b^2) b^{2-2b^2}\right)^{\frac{Q-2\beta}{2b}}, \\ \mathcal{A}_2 &= \frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}, \\ \mathcal{A}_3 &= \frac{S_b(2Q-\sigma_1-\sigma_2-\beta)\, S_b(\sigma_1+\sigma_2-\beta)}{S_b(\beta+\sigma_1-\sigma_2)\, S_b(\beta-\sigma_1+\sigma_2)}. \end{aligned} \tag{3.2}$$

In the above, $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$ and the special functions $\Gamma_b(x)$ and $S_b(x)$ are defined in [2,14]. We record the relevant details in Appendix A.

As mentioned above, to specialise to $c = 1$ we must carefully take the limit $b \to 1, Q \to 2$. This limit is singular and requires us first of all to renormalize both the bulk and boundary cosmological constants. In the first line in Eq.(3.2), we set $b = 1 - \frac{\varepsilon}{2}$ and find that

$$\mathcal{A}_1 \to \left(\pi\mu_0\gamma(1-\varepsilon)\right)^{1-\beta}. \tag{3.3}$$

Using $\gamma(1-\varepsilon) \to \varepsilon$, we see that the above expression becomes finite if we define the renormalised[4] bulk cosmological constant by:

$$\mu = 4\pi\mu_0\, \epsilon. \tag{3.4}$$

[4] This differs by a factor of 4 from the normalisation used in Refs.[9,18]. However, it is more natural as the area of a unit 2-sphere is 4π.

Using this and recalling from (2.7) that $\beta = 1 - k$ with k a non-negative integer, it follows that the first factor in the two-point amplitude is:

$$\mathcal{A}_1 = \left(\frac{\mu}{4}\right)^{1-\beta} = \left(\frac{\sqrt{\mu}}{2}\right)^{2k}. \tag{3.5}$$

The renormalisation of the bare bulk cosmological constant μ_0 performed above is well-known, and leads to the result that the cosmological operator for $c = 1$ closed strings is not the naive one, $e^{2\varphi}$, but rather $\varphi\, e^{2\varphi}$.

Now coming back to Eq.(2.5) and taking $b = 1 - \frac{\varepsilon}{2}$ we have, for small ε:

$$\cos 2\pi\sigma = \sqrt{\pi\varepsilon}\,\frac{\mu_{B,0}}{\sqrt{\mu_0}} = 2\pi\varepsilon\,\frac{\mu_{B,0}}{\sqrt{\mu}}, \tag{3.6}$$

which means that we also need to define a renormalised[5] boundary cosmological constant $\mu_B = 2\pi\varepsilon\,\mu_{B,0}$. Hence finally the relation between the σ parameter and the renormalised (bulk and boundary) cosmological constants is:

$$\cos 2\pi\sigma = \frac{\mu_B}{\sqrt{\mu}}. \tag{3.7}$$

The parameter σ can be real or imaginary depending on whether $\mu_B < \sqrt{\mu}$ or $\mu_B > \sqrt{\mu}$. In what follows, we keep all the σ_i generic.

The factor $\mathcal{A}_2$ depends only on β and not on σ_i. Using Eq.(3.2), we find:

$$\mathcal{A}_2 = \frac{\Gamma_1(-2k)}{\Gamma_1(2k)}. \tag{3.8}$$

This expression is actually divergent. However, we can regulate it by going slightly off-shell. We can do this by shifting β from the integer value by an amount ϵ: $k \to k + \epsilon$ and extract the leading divergence. We could use a different regulator and deform b away from 1 to $1 - \epsilon$ and we get the same answer. As detailed in Appendix A, $\mathcal{A}_2$ is determined to be:

$$\mathcal{A}_2 = \frac{(-1)^k}{(2\pi)^{2k}\,\Gamma(2k+1)\Gamma(2k)}\,\frac{1}{\epsilon^{2k+1}}. \tag{3.9}$$

Finally we turn to the third factor in Eq.(3.1):

$$\mathcal{A}_3 = \frac{S_1(2Q - \sigma_1 - \sigma_2 - \beta)\,S_1(\sigma_1 + \sigma_2 - \beta)}{S_1(\beta + \sigma_1 - \sigma_2)\,S_1(\beta - \sigma_1 + \sigma_2)}. \tag{3.10}$$

[5] Once again this differs (now by a factor of 2) from the normalisations of Refs.[18,9], and is consistent with the length of a unit circle being 2π.

Now using the inversion relation $S_b(x)S_b(Q-x)=1$ and substituting $\beta=1-k$, $\mathcal{A}_3$ can be rewritten as:

$$\begin{aligned}\mathcal{A}_3 &= \frac{S_1(\sigma_1+\sigma_2-\beta)}{S_1(-Q+\beta+\sigma_1+\sigma_2)}\frac{S_1(Q-\beta+\sigma_1-\sigma_2)}{S_1(\beta+\sigma_1-\sigma_2)} \\ &= \frac{S_1(-1+k+\sigma_1+\sigma_2)}{S_1(-1-k+\sigma_1+\sigma_2)}\frac{S_1(1+k+\sigma_1-\sigma_2)}{S_1(1-k+\sigma_1-\sigma_2)}.\end{aligned} \tag{3.11}$$

Next we define the combinations $\sigma^\pm=\sigma_1\pm\sigma_2$ and invoke the recursion relation (see Appendix A) $S_1(x+1)=2\sin\pi x\, S_1(x)$ to write:

$$\begin{aligned}\mathcal{A}_3 &= \prod_{m=1}^{2k}\left(2\sin\pi(\sigma^+ + k-1-m)\right)\prod_{n=1}^{2k}\left(2\sin\pi(\sigma^- + k-1-n)\right) \\ &= \left(4\,\sin\,\pi\sigma^+\;\sin\,\pi\sigma^-\right)^{2k}.\end{aligned} \tag{3.12}$$

This can be rewritten in terms of the original boundary parameters σ_1 and σ_2:

$$\mathcal{A}_3 = \left(2\left(\cos 2\pi\sigma_1-\cos 2\pi\sigma_2\right)\right)^{2k} = \left(2\,\frac{\mu_{1B}-\mu_{2B}}{\sqrt{\mu}}\right)^{2k}. \tag{3.13}$$

Putting everything together, we finally get:

$$d(1-k|\mu_{1B},\mu_{2B}) = \frac{(-1)^k}{\epsilon^{2k+1}}\frac{(\mu_{1B}-\mu_{2B})^{2k}}{(2\pi)^{2k}\,\Gamma(2k+1)\Gamma(2k)}. \tag{3.14}$$

We will find it convenient to renormalize the open-string operators (2.7) as

$$\tilde{Y}^{\sigma_1,\sigma_2}_{k,m} = (2\pi\epsilon)^k\Gamma(2k)\,Y^{\sigma_1\sigma_2}_{k,m}. \tag{3.15}$$

This redefinition is different from the standard one found in the literature for closed strings, as it has an additional factor of $(2\pi\epsilon)^k$. For the cosmological operator ($k=0$) this extra factor is absent and the renormalisation is the standard one. The matter contribution to the two-point function being trivial, let us put the renormalization factor in the Liouville vertex operator alone and define

$$\tilde{V}^{\sigma_1,\sigma_2}_{1-k} = (2\pi\epsilon)^k\Gamma(2k)\,V^{\sigma_1,\sigma_2}_{1-k}. \tag{3.16}$$

Expressed in these variables, the reflection amplitude is:

$$\tilde{d}(1-k|\mu_{1B},\mu_{2B}) = \frac{(-1)^k}{\epsilon}\frac{(\mu_{2B}-\mu_{1B})^{2k}}{2k}. \tag{3.17}$$

Several features of this result are noteworthy. First, it is independent of the bulk cosmological constant μ. A similar feature was noticed[22] for the correlators of $c = 28$ Liouville theory (corresponding to strings propagating in a $c = -2$ matter background). Second, the result depends only on the difference of the two boundary cosmological constants μ_{1B}, μ_{2B}. We will see later that these features persist for the boundary three-point function. They are reminiscent of the identification of the extended B-type branes of topological field theories to fermions[47]. Finally, we see that after renormalization, the reflection amplitude has a simple pole singularity (as a function of ϵ). Again this turns out to be the case for the boundary three-point function as well. Later, when we use this in the string field theory action, we will need to absorb this singularity by a redefinition of the string coupling constant.

4. Boundary Three-Point Function

The three-point function in boundary Liouville theory is defined by:

$$\langle V_{\beta_1}^{\sigma_2\sigma_3}(x^1) V_{\beta_2}^{\sigma_3\sigma_1}(x^2) V_{\beta_3}^{\sigma_1\sigma_2}(x^3)\rangle = \frac{C_{\beta_1\beta_2\beta_3}^{\sigma_2\sigma_3\sigma_1}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2}}. \qquad (4.1)$$

An expression was found in Ref.[8] (see also Refs.[9,14] for subsequent discussions) for the coefficient C as a product of four factors:

$$\begin{aligned}
C_{\beta_1\beta_2\beta_3}^{\sigma_2\sigma_3\sigma_1} &= \mathcal{B}_1\,\mathcal{B}_2\,\mathcal{B}_3\,\mathcal{B}_4,\,, \\
\mathcal{B}_1 &= \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{1}{2}(Q-\beta_1-\beta_2-\beta_3)}, \\
\mathcal{B}_2 &= \frac{\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(Q-\beta_1-\beta_2+\beta_3)\Gamma_b(Q-\beta_1+\beta_2-\beta_3)}{\Gamma_b(Q)\,\Gamma_b(Q-2\beta_1)\,\Gamma_b(Q-2\beta_2)\,\Gamma_b(Q-2\beta_3)}, \\
\mathcal{B}_3 &= \frac{S_b(Q-\beta_3+\sigma_1-\sigma_3)\,S_b(2Q-\beta_3-\sigma_1-\sigma_3)}{S_b(\beta_2+\sigma_2-\sigma_3)\,S_b(Q+\beta_2-\sigma_2-\sigma_3)}, \\
\mathcal{B}_4 &= \frac{1}{i}\int_{-i\infty-0}^{+i\infty-0} ds\ \prod_{i=1}^{4}\frac{S_b(U_i+s)}{S_b(V_i+s)}.
\end{aligned} \qquad (4.2)$$

In the factor $\mathcal{B}_4$, the quantities U_i, V_i, $i = 1,\cdots,4$ are defined as follows:

$$\begin{aligned}
U_1 &= \sigma_1+\sigma_2-\beta_1, & V_1 &= 2Q+\sigma_2-\sigma_3-\beta_1-\beta_3, \\
U_2 &= Q-\sigma_1+\sigma_2-\beta_1, & V_2 &= Q+\sigma_2-\sigma_3-\beta_1+\beta_3, \\
U_3 &= \beta_2+\sigma_2-\sigma_3, & V_3 &= 2\sigma_2, \\
U_4 &= Q-\beta_2+\sigma_2-\sigma_3, & V_4 &= Q.
\end{aligned} \qquad (4.3)$$

We want to compute the above for the values $b = 1$, $\beta_i = 1 - k_i$ for our case of $c = 1$. In this section, we choose the kinematic regime $k_3 > k_1, k_2 > 0$. We shall later need to take a careful limit as k_i approach integers. The first two factors are evaluated as before, and we get

$$
\begin{aligned}
\mathcal{B}_1 &= \left(\frac{\mu}{4}\right)^{\frac{1}{2}(k_1+k_2+k_3-1)} = \left(\frac{\sqrt{\mu}}{2}\right)^{\sum_i k_i - 1}, \\
\mathcal{B}_2 &= \frac{\Gamma_1(1+\sum_i k_i)\Gamma_1(1+k_1+k_2-k_3)\Gamma_1(1+k_1-k_2+k_3)\Gamma_1(1+k_1-k_2-k_3)}{\Gamma_1(2)\,\Gamma_1(2k_1)\,\Gamma_1(2k_3)\,\Gamma_1(2k_2)} \\
&= \frac{(-1)^{\lfloor (k_2+k_3-k_1)/2 \rfloor}}{(2\pi\epsilon)^{k_2+k_3-k_1}} \times \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \\
&\quad \times \frac{\Gamma_1(1+k_1+k_2-k_3)\,\Gamma_1(1+k_1-k_2+k_3)\,\Gamma_1(1-k_1+k_2+k_3)}{\Gamma_1(2k_1)\,\Gamma_1(2k_2)\,\Gamma_1(2k_3)} \\
&= \frac{(-1)^{\lfloor (k_2+k_3-k_1)/2 \rfloor}}{(2\pi\epsilon)^{k_2+k_3-k_1}} \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \prod_{j=1}^{3} \frac{\Gamma_1\left(1+\sum_i k_i - 2k_j\right)}{\Gamma_1(2k_j)},
\end{aligned}
\tag{4.4}
$$

where $\lfloor x \rfloor$ is the integer part of x. We have used the properties of the special function $\Gamma_1(x)$ at integer arguments given in Appendix A to rewrite the last factor in the numerator of $\mathcal{B}_2$.

For the factor $\mathcal{B}_3$, we insert the values of the parameters to write it as:

$$
\mathcal{B}_3 = \frac{S_1(1+k_1+k_2+\sigma_1-\sigma_3)\,S_1(3+k_1+k_2-\sigma_1-\sigma_3)}{S_1(1-k_2+\sigma_2-\sigma_3)\,S_1(3-k_2-\sigma_2-\sigma_3)}. \tag{4.5}
$$

It turns out that this simplifies when combined with a similar factor in $\mathcal{B}_4$.

Finally we must evaluate the contribution $\mathcal{B}_4$. This is carried out in Appendix B, where the contour integral in the last line of Eq.(4.2) is evaluated explicitly. That is then combined with $\mathcal{B}_3$ of Eq.(4.5) above to give the following amazingly simple result for the product:

$$
\mathcal{B}_3\,\mathcal{B}_4 = \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(\frac{2\mu_{21}}{\sqrt{\mu}}\right)^{-1} \left\{ \left(\frac{2\mu_{23}}{\sqrt{\mu}}\right)^{\sum_i k_i} - \left(\frac{2\mu_{13}}{\sqrt{\mu}}\right)^{\sum_i k_i} \right\}. \tag{4.6}
$$

Putting everything together, we arrive at the three-point function (with $\beta_i = 1 - k_i$):

$$
\begin{aligned}
C^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_3} &= \mathcal{B}_1\,\mathcal{B}_2\,\mathcal{B}_3\,\mathcal{B}_4 \\
&= \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{(2\pi\epsilon)^{1+\Sigma_i k_i}} \frac{\mu_{23}^{\Sigma_i k_i} - \mu_{13}^{\Sigma_i k_i}}{\mu_{21}} \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \prod_{j=1}^{3} \frac{\Gamma_1\left(1+\sum_i k_i - 2k_j\right)}{\Gamma_1(2k_j)}.
\end{aligned}
\tag{4.7}
$$

In terms of the renormalised operators defined in Eq.(3.15) and (3.16), the three-point function becomes:

$$\tilde{C}^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_3} = \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{2\pi\epsilon} \frac{\mu_{23}^{\Sigma_i k_i} - \mu_{13}^{\Sigma_i k_i}}{\mu_{21}} \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \prod_{j=1}^{3} \frac{\Gamma_1\left(1+\sum_i k_i - 2k_j\right)\Gamma(2k_j)}{\Gamma_1(2k_j)}. \tag{4.8}$$

In the special case of tachyons, the momenta of the three operators obey $k_1 + k_2 = k_3$, and the three point function takes the simpler form

$$\tilde{C}^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_1+\beta_2-1} = \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{\epsilon} \frac{\mu_{23}^{2k_1+2k_2} - \mu_{13}^{2k_1+2k_2}}{\mu_{21}}. \tag{4.9}$$

Like the boundary reflection amplitude Eq.(3.17), the boundary three-point function obtained here also turns out to be independent of the bulk cosmological constant, depends only on pairwise differences of boundary cosmological constants, and has a simple pole singularity in ϵ. We conjecture that these three properties also hold for all n-point functions of boundary operators in this theory.

As a check, we consider the three-point function with momenta $k_1 = k$, $k_2 = 1$ and $k_3 = k$ for the three operators. In this case the middle operator has $\beta = 0$ and hence, if we choose $\sigma_1 = \sigma_3$ (which implies $\mu_{1B} = \mu_{3B}$), it reduces to the identity. Now the above correlator should reduce to the two-point function. From Eq.(4.7) we find:

$$C^{\mu_2\mu_1\mu_1}_{1-k,1,1-k} = \frac{(-1)^k}{(2\pi\epsilon)^{2k+2}} \frac{2\pi\,\mu_{21}^{2k}}{\Gamma(2k+1)\Gamma(2k)}. \tag{4.10}$$

Comparing with Eq.(3.14), we see that this is related to the (bare) reflection amplitude by:

$$C^{\mu_2\mu_1\mu_1}_{1-k,1,1-k} = \frac{1}{2\pi\epsilon}\, d(1-k|\mu_1,\mu_2). \tag{4.11}$$

If we interpret $\frac{1}{2\pi\epsilon}$ as the $\delta(0)$ factor arising from $\delta(\beta_1-\beta_2)$ in Eq.(3.1), we may conclude that in the special case being considered, the three-point function indeed reduces to the two-point function as expected.

5. Bulk-Boundary Two-Point Function

The bulk-boundary two-point function on the disc involves a boundary operator $V_\beta^{\sigma\sigma}$ and a bulk operator $\mathcal{V}_\alpha$. This was computed in Ref.[7] (see also Refs.[9,14]) and the result is:

$$\left\langle \mathcal{V}_\alpha(z,\bar{z}) V_\beta^{\sigma\sigma}(x) \right\rangle = \frac{A^\sigma_{\alpha\beta}}{|z-\bar{z}|^{2\Delta_\alpha - \Delta_\beta}\,|z-x|^{2\Delta_\beta}}, \tag{5.1}$$

where,

$$
\begin{aligned}
A^{\sigma}_{\alpha\beta} &= \mathcal{C}_1\,\mathcal{C}_2\,\mathcal{C}_3, \\
\mathcal{C}_1 &= 2\pi\left(\pi\mu_0\gamma(b^2)b^{2-2b^2}\right)^{(Q-2\alpha-\beta)/2}, \\
\mathcal{C}_2 &= \frac{\Gamma_1^3(Q-\beta)\Gamma_1(2\alpha-\beta)\Gamma_1(2Q-2\alpha-\beta)}{\Gamma_1(Q)\Gamma_1(Q-2\beta)\Gamma_1(\beta)\Gamma_1(2\alpha)\Gamma_1(Q-2\alpha)}, \\
\mathcal{C}_3 &= \frac{1}{i}\int_{-i\infty}^{i\infty} dt\; e^{2\pi i(2\sigma-Q)t}\,\frac{S_1\left(t+\frac{1}{2}\beta+\alpha-\frac{1}{2}Q\right)S_1\left(t+\frac{1}{2}\beta-\alpha+\frac{1}{2}Q\right)}{S_1\left(t-\frac{1}{2}\beta-\alpha+\frac{3}{2}Q\right)S_1\left(t-\frac{1}{2}\beta+\alpha+\frac{1}{2}Q\right)}.
\end{aligned}
\tag{5.2}
$$

We want to evaluate this in the $c=1$ string theory where, as usual, we need to take the singular limit $b=1$. Let us also recall that $\beta = 1-k$ and $\alpha = 1-\frac{1}{2}k$ — the bulk and boundary windings are related due to the winding number conservation condition from the matter sector. We will assume that $k>0$. The first factor $\mathcal{C}_1$ can be rewritten using the the by-now familiar renormalized bulk cosmological constant:

$$
\mathcal{C}_1 = 2\pi\left(\frac{\mu}{4}\right)^{k-\frac{1}{2}}, \tag{5.3}
$$

while the second factor is easily evaluated to be:

$$
\mathcal{C}_2 = \frac{(-1)^{1+k}}{2\pi\Gamma(2k)\left(\Gamma(k)\right)^2}\,(2\pi\epsilon)^{2k-1}. \tag{5.4}
$$

Finally, we come to the third factor $\mathcal{C}_3$ which involves an integral similar to the one encountered in the evaluation of the boundary three-point function. Specifically, we have to evaluate

$$
\mathcal{C}_3 = \frac{1}{i}\int_{-i\infty}^{i\infty} dt\;\exp\left(4\pi i(\sigma-1)t\right)\,\frac{S_1\left(t-k+\frac{1}{2}\right)S_1\left(t+\frac{1}{2}\right)}{S_1\left(t+k+\frac{3}{2}\right)S_1\left(t+\frac{3}{2}\right)}. \tag{5.5}
$$

For large imaginary values of t, the integrand falls off exponentially. This makes the integral (5.5) convergent. In the kinematic region where k is negative, all the poles of the integrand arising from the numerator are in the left half-plane while those from the denominator are in the right half-plane. For other values of k the integral is defined by analytic continuation described in detail in Appendix B. Once again, the integral is dominated by its singular part, which comes from the collision of the poles from the two half-planes. Denoting $t+\frac{1}{2}=n$, the conditions for collision are met for integer values of n between $1-k$ and k. Evaluating the (singular) residues at these poles we find

$$
\mathcal{C}_3 = \frac{1}{(2\pi\epsilon)^{2k+1}}\,e^{-2\pi i\sigma}\sum_{n=1-k}^{k} e^{4\pi i\sigma n} = \frac{1}{(2\pi\epsilon)^{2k+1}}\,\frac{\sin(4\pi\sigma k)}{\sin 2\pi\sigma}. \tag{5.6}
$$

Combining Eqs.(5.3), (5.4) and (5.6), the bulk-boundary two point function of tachyons is found to be:

$$
\begin{aligned}
A^{\sigma}_{\alpha\beta} &= \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1} \frac{(-1)^{k-1}}{\epsilon^2\,\Gamma(2k)\,\left(\Gamma(k)\right)^2}\,\frac{\sin(4\pi\sigma k)}{\sin(2\pi\sigma)} \\
&= \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1} \frac{(-1)^{k-1}}{\epsilon^2\,\Gamma(2k)\,\left(\Gamma(k)\right)^2}\,\sum_{\ell=1}^{k}(-1)^{\ell+1}\binom{2k-\ell}{\ell-1}\left(\frac{2\mu_B}{\sqrt{\mu}}\right)^{2k-2\ell+1}.
\end{aligned}
\tag{5.7}
$$

Like the previous correlators, this too is conveniently expressed in terms of renormalised bulk and boundary operators, the latter being given by Eqn.(3.15) and for the former we choose:

$$
\tilde{Z}(k;m,\overline{m}) = (2\pi\epsilon)^{-k}\frac{\Gamma(k)}{\Gamma(1-k)}\,Z(k;m,\overline{m}). \tag{5.8}
$$

Once again, this redefinition differs from the standard one and is chosen so as simplify the form of the renormalized expression. Specialising to the tachyons (2.3),(2.4) for simplicity,

$$
\left\langle \tilde{T}^{\sigma\sigma}(k)\tilde{\mathcal{T}}(k)\right\rangle = \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1}\,\frac{(-1)^{k-1}}{\epsilon^2}\frac{\sin(\pi k)\sin(4\pi\sigma k)}{\pi\sin(2\pi\sigma)}. \tag{5.9}
$$

Unlike the boundary two- and three-point functions, we see that the bulk-boundary correlator does depend explicitly on the bulk cosmological constant μ, through σ. It also lacks the translational symmetry in μ_B that we found in the boundary correlators. (This was to be expected, since there is only one boundary operator $V^{\sigma\sigma}_{\beta}$ and this is necessarily diagonal in the boundary cosmological constant. However, we suspect that with more boundary operators too, the bulk-boundary correlators will lack translational symmetry in the μ_B.) Finally, we see that this correlator has a double pole singularity in ϵ, unlike the simple pole found in the boundary correlators.

Specialising further to $k=0$ (the cosmological operators) we find:

$$
\left\langle \tilde{T}^{\sigma\sigma}(0)\tilde{\mathcal{T}}(0)\right\rangle = \left\langle \tilde{T}^{\sigma\sigma}(k)\tilde{\mathcal{T}}(k)\right\rangle\Big|_{k\to 0} = -\frac{2}{\sqrt{\mu}}\,\frac{4\pi\sigma}{\sin 2\pi\sigma}. \tag{5.10}
$$

Interestingly, in this case the correlator is non-singular.

As a consistency check, the bulk-boundary two-point function, if correctly normalised, should reduce to the bulk one point function when the boundary Liouville momentum vanishes, $\beta \to 0$. This corresponds to $k=1$ in our case. The bulk one-point function of Liouville theory is given by[2]:

$$
\langle \mathcal{V}_{\alpha}(z,\overline{z})\rangle_{\sigma} = \frac{U_{\sigma}(\alpha)}{|z-\overline{z}|^{2\Delta_{\alpha}}}, \tag{5.11}
$$

where,

$$U_\sigma(\alpha) = \frac{2}{b}\left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2\alpha}{2b}} \Gamma(2b\alpha - b^2)\,\Gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2} - 1\right) \\ \times\ \cos\left(\pi(2\alpha - Q)(2\sigma - Q)\right). \tag{5.12}$$

Due to the momentum conservation condition from the matter sector, this should strictly be evaluated only at $k = 0$. Nevertheless, let us keep k arbitrary at this stage. Putting $b = 1$ in (5.12) and performing the familiar renormalization of cosmological constants, as well as renormalization of the bulk tachyon as in Eq.(5.8), the one-point function is:

$$\tilde{U}_\sigma(k) = -\left(\frac{\sqrt{\mu}}{2}\right)^k (2\pi\epsilon)^{-k} \frac{2\pi\cos(2\pi\sigma k)}{k\sin(\pi k)}. \tag{5.13}$$

The bulk-one point function above is seen to satisfy the expected functional equation $\tilde{U}_{\sigma+\frac{1}{2}}(k) + \tilde{U}_{\sigma-\frac{1}{2}}(k) = 2\cos(\pi k)\,\tilde{U}_\sigma(k)$ rather trivially[9]. Substituting $k = 1$ (hence $\alpha = 1 - \frac{k}{2} = \frac{1}{2}$) formally,

$$\tilde{U}_\sigma(k = 1) = -\frac{\sqrt{\mu}}{2}\frac{\cos 2\pi\sigma}{\pi\epsilon^2}, \tag{5.14}$$

On the other hand, Eq.(5.9) evaluated at $k = 1$ gives:

$$\left\langle \tilde{T}^{\sigma\sigma}(1,1)\tilde{\mathcal{T}}(1)\right\rangle = \frac{\sqrt{\mu}}{2}\frac{2\cos 2\pi\sigma}{\epsilon}. \tag{5.15}$$

Recalling from Eq.(3.15) that $\tilde{T}(1) = 2\pi\epsilon\, T(1)$ we see that

$$\left\langle T^{\sigma\sigma}(1)\tilde{\mathcal{T}}(1)\right\rangle = \frac{\sqrt{\mu}}{2}\frac{\cos 2\pi\sigma}{\pi\epsilon^2}. \tag{5.16}$$

This agrees with Eq.(5.14) (upto a sign).

6. Physical Correlators and Open String Field Theory

According to a recent proposal of Sen[24–26], open-string field theory on D-branes in a certain background is dual to a theory of closed strings to which the branes in that background couple. In most known examples, the complete string field theory is extremely complicated, and lacking the necessary analytic tools, is only accessible through approximation schemes such as level truncation. Having examples of D-branes on which the full open-string field theory can be analysed is clearly important. Non-critical string theories, with their relatively simple yet rich physical content and high degree of symmetry, are string backgrounds where we may further our understanding of this duality[25,26]. Indeed in Ref.[22], the topological Kontsevich matrix model of topological gravity (equivalent to $c = -2$ closed-string theory) is shown to arise from the open-string field theory of the branes of $c = -2$ matter coupled to Liouville theory.

In this section, we take the first step in this direction for the case of the N FZZT branes in the $c = 1$ theory compactified at $R = 1$. Open-string field theory has an infinite number of fields, but it also has infinite gauge redundancy. The closed-string sector of the $c = 1$ theory at the self-dual radius, and indeed of all non-critical string theories, possesses a topological symmetry due to which only degenerate worldsheets at the boundary of the moduli space of Riemann surfaces contribute to correlators. In other words, only physical ('on-shell') states in the cohomology of the BRS operator Q_B contribute to quantum string amplitudes, at all genus. This is the well-known topological localisation.

When D-branes, *i.e.* open strings, are included, we lack a direct proof that this property continues to hold. However, an important source of intuition comes from the relation of the bulk theory to the topological SL(2)/U(1) coset[34] and the deformed conifold[48]. There has been progress in understanding localisation in the open-string sector of the topological SU(2)/U(1) cosets[49], closely related to the first description. On the other hand, a classic result due to Witten[50] tells us that the open-string field theory on D3-branes wrapping a 3-cycle of the deformed conifold localises to pure Chern-Simons theory (this considerably preceded the discovery of D-branes!). With this motivation, for the moment we simply assume that the string field theory localises onto the physical states (as defined by the BRS cohomology) and arrive at an action for these modes, postponing a detailed analysis of localisation and the resulting model for future work[6].

Let the CFT Hilbert space of the states of the first-quantized string between the ith and the jth brane be $\mathcal{H}_{ij}$ $(i, j = 1, \cdots, N)$. The open-string field $|\Psi_{(ij)}\rangle$ is a ghost-number one state in this Hilbert space. The action defining the classical string field theory is

$$S[\Psi] = -\frac{1}{2g_{s0}} \sum_{ij} \left\langle \Psi_{(ij)}, Q_B \Psi_{(ji)} \right\rangle - \frac{1}{3g_{s0}} \sum_{ijk} \left\langle \Psi_{(ij)} \Psi_{(jk)} \Psi_{(ki)} \right\rangle , \qquad (6.1)$$

where the quadratic and the cubic terms are given in terms of CFT correlators and Q_B is the BRST operator (see [51] for a review). The linearised equation of motion of the theory

[6] In [22], a powerful nilpotent symmetry of the gauge fixed quantum action of the $c = -2$ noncritical string theory is exploited for localisation. The existence of such a symmetry is stronger in that it takes into account the effect of worldsheet instantons. We note, however, that the absence of compact two-cycles in the deformed conifold geometry will forbid potential instanton correction.

$Q_B|\Psi\rangle = 0$ is the statement that the worldsheet configuration is physical in the free theory — the cubic term then describes interactions.

Our task now is to compute the OSFT action (6.1) on the FZZT branes localized onto the physical states. This amounts to the evaluation of correlators in the CFT of $c = 1$ matter plus Liouville plus the (b, c) ghosts. In Liouville theory, there is no sense in which the string coupling is weak, therefore we cannot really regard the cubic term as a perturbation. This is reflected in the fact that there is an infrared divergence in the two-point correlators of the physical states. We shall use the same regulator that we did earlier and see that the kinetic and the cubic term, evaluated on the on-shell states, contribute to the same order.

We have seen earlier that the physical states of the background CFT are summarized as an $N \times N$ matrix field living on a 2-sphere. The expansion of the open-string field in terms of these states is:

$$\left|\Psi_{(ij)}\right\rangle = \sum_{k,m} T_{ij}(k,m) \left|\tilde{Y}^{ij}(k,m)\right\rangle, \tag{6.2}$$

where $|\tilde{Y}^{ij}(k,m)\rangle$ is the ghost number one primary state in the boundary CFT corresponding to the open string with ends on branes (i, j) transforming as spherical harmonics $\mathcal{Y}_k^m(\theta, \phi)$ under rotations of S^2. Under the assumptions described above, the open-string field theory action reduces to an action for these matrices.

The coefficients of the string field in the OSFT action are determined by the CFT correlation functions for primary operators. The matter and ghost contribution to the two- and three-point correlators are very simple. As we saw in section 2, the physical operators behave exactly like the spherical harmonics. For the two-point function, the matter contribution is the condition $k_1 = k_2 \equiv k$ and the conservation condition $m_1 = -m_2 \equiv m$ of the J_3 component of angular momentum. The full two-point function is:

$$D_{ij}(k) \equiv \left\langle \tilde{Y}^{ij}(k,m), c_0 \tilde{Y}^{ji}(k,-m)\right\rangle = \frac{(-1)^k}{\epsilon^2} \frac{(z_j - z_i)^{2k}}{2k}, \tag{6.3}$$

where, we have introduced the notation $\mu_B \equiv z$. Let us note that for the special case of the cosmological operators, both the terms in Eq. (3.1) contribute.

The full three-point function is determined from the SU(2) addition condition from the matter sector and, using Eq.(4.8), its expression is:

$$
\begin{aligned}
C_{jki}(k_1,k_2,k_3) &\equiv \left\langle \tilde{Y}^{jk}(k_1,m_1)\tilde{Y}^{ki}(k_2,m_2)\tilde{Y}^{ij}(k_3,m_3)\right\rangle \\
&= \frac{(-1)^{\Sigma_i k_i/2}}{(2\pi\epsilon)}\left(\frac{z_{jk}^{\Sigma_i k_i}-z_{ik}^{\Sigma_i k_i}}{z_{ji}}\right)\frac{\Gamma_1(1+\Sigma_i k_i)}{\Gamma_1(2)}\prod_{j=1}^{3}\frac{\Gamma_1(1+\Sigma_i k_i-2k_j)\Gamma(2k_j)}{\Gamma_1(2k_j)} \\
&\qquad\times\int d(\cos\theta)d\phi\,\mathcal{Y}_{k_1}^{m_1}(\theta,\phi)\mathcal{Y}_{k_2}^{m_2}(\theta,\phi)\mathcal{Y}_{k_3}^{m_3}(\theta,\phi),
\end{aligned}
\tag{6.4}
$$

where $\mathcal{Y}_m^k(\theta,\phi)$ are the spherical harmonics of S^2. Let us recall that the three-point function is evaluated with the condition $k_3 > k_1, k_2$, a choice made in evaluating the Liouville correlators in Sec. 4. We would also like to point out that the extra divergence present in the two-point function (6.3) compared to the three-point function (6.4) is due to the delta function in (3.1), which can be understood as an infra-red divergence arising from the volume of the target space.

It is now straightforward to evaluate the action (6.1) localised on the physical states. The kinetic operator Q_B simplifies to c_0L_0 in the Siegel gauge. This has a zero acting on the physical states $|\tilde{Y}^{ij}(k,m)\rangle$, (which are dimension zero primaries of the underlying CFT):

$$
\begin{aligned}
\left\langle \tilde{Y}^{ij}(k',m'),Q_B\tilde{Y}^{ji}(k,m)\right\rangle &= \left\langle \tilde{Y}^{ij}(k',m')\right| c_0L_0\left|\tilde{Y}^{ji}(k,m)\right\rangle \\
&= \left\langle \tilde{Y}^{ij}(k',m')\right| c_0\left(k^2+\beta_2(2-\beta_2)-1\right)\left|\tilde{Y}^{ji}(k,m)\right\rangle \\
&= 2k\epsilon\left\langle \tilde{Y}_{ij}(k',m')\right| c_0\left|\tilde{Y}^{ji}(k,m)\right\rangle \\
&= 2k\epsilon\, D_{ij}(k)\,\delta_{k,k'}\delta_{m+m',0}.
\end{aligned}
\tag{6.5}
$$

This zero absorbs the volume divergence in the two point function. Using (6.3) in (6.5), we get the coefficient of the non-local kinetic term

$$
\left\langle \tilde{Y}^{ij}(k,m),Q_B\tilde{Y}^{ji}(k,-m)\right\rangle = \frac{(-1)^k}{\epsilon}\left(z_j-z_i\right)^{2k}, \tag{6.6}
$$

which has a simple pole in ϵ. The coefficient of the cubic term is simply the three-point function (6.4) with the same singularity. Thus, the two terms in the action (6.1) have an identical singular coefficient. Then we can renormalise the string coupling as

$$
g_s \equiv \epsilon g_{s0}, \tag{6.7}
$$

to get a sensible matrix theory with a finite action. The novelty of this matrix model, compared to the existing ones in the literature, is that it has the $SU(2)$ symmetry of the theory manifest from the beginning.

Let us note that the (singular) renormalisation (6.7) of the string coupling was also necessary in Ref.[22] in order to get the Kontsevich model. It is actually implicit in [22], where the $1/\epsilon$ singularity of the three-point function as well as the delta-functions in the two-point functions have been suppressed[52].

The complete action involving all the modes in (6.2) is a little cumbersome. However, if we restrict to the tachyons (thereby giving up $SU(2)$ symmetry), we find the action $\mathcal{S} = \mathcal{S}_2 + \mathcal{S}_3$, where

$$\begin{aligned}
\mathcal{S}_2 &= -\frac{1}{2g_s} \sum_{k=0}^{\infty} (-1)^k \sum_{ij} \tilde{T}^{ij}(k) \, (z_j - z_i)^{2k} \, \tilde{T}^{ji}(-k) \\
\mathcal{S}_3 &= -\frac{1}{3g_s} \sum_{k_1,k_2} (-1)^{k_1+k_2} \sum_{ijl} \frac{z_{jl}^{2k_2+2k_2} - z_{il}^{2k_2+2k_2}}{z_{ji}} \tilde{T}^{jl}(k_1) \tilde{T}^{li}(k_2) \tilde{T}^{ij}(-k_1-k_2).
\end{aligned} \tag{6.8}$$

In terms of a matrix $Z = \mathrm{diag}\,(iz_1, iz_2, \cdots)$, the kinetic term may be written as

$$\mathcal{S}_2 \sim \sum_k \mathrm{Tr}\, \tilde{T}(k) \left[Z, \cdots \left[Z, \tilde{T}(k) \right] \cdots \right]. \tag{6.9}$$

The fact that our Liouville correlators depend only on the difference of boundary cosmological constants shows up as a symmetry of the above term under a shift of the matrix $\tilde{T}$ by an arbitrary diagonal matrix. This symmetry is shared by the cubic term which can also be written down similarly.

7. Discussion

We have studied correlators of the boundary Liouville theory in the limit that the Liouville central charge c_L tends to 25, or equivalently $c \to 1$. The results are embodied in Eqs.(3.14)–(3.17), (4.8)–(4.9) and (5.9). The principal motivation to present these results is that they are far more explicit than the boundary correlators known for the $c < 1$ theory (as embodied in Eqs.(3.2),(4.2) and (5.2)). The latter are given in terms of special functions $S_b(x), \Gamma_b(x)$ and some of the correlators are known only as contour integrals over products of such functions. These contour integrals can be explicitly evaluated for $c = 1$

only, as far as we know, at the self-dual radius[7]. The boundary correlators we obtain in this way are all divergent, but as we have noted, the divergence factors out from the two- and three-point functions and can be absorbed in a rescaling of the string coupling leading to a well-defined open-string field theory action.

The fact that the boundary correlators are independent of the bulk cosmological constant is reminiscent of a similar fact in Ref.[22]. There, the dependence of the two-point function on μ_B is crucial in recovering the Kontsevich model[28], where the different $\mu_{B,i}$ turn into the eigenvalues of the Kontsevich matrix. In similar vein, our matrix model depends only on $\mu_{B,i}$ which are the eigenvalues of a constant matrix Z.

We did not find a proof that the boundary correlators at $c = 1$ and selfdual radius are all independent of the bulk cosmological constant. However, if we assume this to be true, then we can see that the n-point tree-level boundary correlators must scale with the boundary cosmological constant μ_B as:

$$\langle V(k_1)V(k_2)\cdots V(k_n)\rangle \sim \mu_B^{\sum_{i=1}^{n} k_i - n + 2} \tag{7.1}$$

where a factor of $(k_i - 1)$ comes from each Liouville vertex operator and an additional 2 comes from the linear dilaton factor in the path integral. This scaling is satisfied by the two- and three-point correlators that we computed. It is tempting to also conjecture that the n-point correlators will depend only on the pairwise differences μ_{ij} of boundary cosmological constants.

The natural matrix model that we might have expected to find from our computations, which is the analogue of the Kontsevich model for $c = 1$ at self-dual radius, is the model of Ref.[39]. But this is a one-matrix model, and here we find a model with infinitely many matrices. Moreover the model of [39] incorporates amplitudes for (closed-string) tachyon external states only, based as it is on the amplitudes computed in Ref.[38] from matrix quantum mechanics, in which the other discrete states have not yet been constructed. So there is in fact no candidate matrix model presently available that incorporates the full $SU(2)$ symmetry of the $c = 1$ string at self-dual radius. In contrast, the approach in the present paper does lead to such a model, presented in embryonic form in Eqs.(6.4)–(6.6) More work is needed to understand this model and confirm whether open/closed duality works as expected.

[7] Of course, rational multiples of this radius which correspond to orbifolds of the theory also have a similar behaviour.

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Appendix A. Special Functions at $c = 1$

The correlators of Liouville theory are expressed in terms of some special functions[2,14]. In the case of $c_L = 25$, *i.e.*, $b = 1$, they are:

$$\begin{aligned} \ln \Gamma_1(x) &= \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-t}}{(1-e^{-t})^2} - \frac{(1-x)^2}{2e^t} - \frac{2(1-x)}{t} \right), \\ \ln S_1(x) &= \int_0^\infty \frac{dt}{t} \left(\frac{\sinh 2t(1-x)}{2\sinh^2 t} - \frac{1-x}{t} \right). \end{aligned} \tag{A.1}$$

Both are meromorphic functions and are related to each other via:

$$S_1(x) = \frac{1}{S_1(2-x)} = \frac{\Gamma_1(x)}{\Gamma_1(2-x)}, \tag{A.2}$$

where, we have also made use of the unitarity relation $S_1(x)S_1(2-x) = 1$. The function Γ_1 has poles at zero and negative integer arguments. Therefore, from Eq.(A.2), $S_1(x)$ has poles at these arguments and zeroes at integers larger than 1.

The functions $\Gamma_1(x)$ and $S_1(x)$ satisfy the recursion relations:

$$\begin{aligned} \Gamma_1(x+1) &= \frac{\sqrt{2\pi}}{\Gamma(x)}\, \Gamma_1(x), \\ S_1(x+1) &= 2\sin(\pi x)\, S_1(x); \end{aligned} \tag{A.3}$$

where, $\Gamma(x)$ is the usual Euler gamma function. The values of these special functions at (half)-integer arguments turn out to be of interest. In particular, we would need the ratio $\Gamma_1(-n)/\Gamma_1(n)$, which, as a matter of fact, is divergent. However, using the recursion relations above, one can show that the leading divergence, near an integer n is

$$\begin{aligned}\frac{\Gamma_1(-n)}{\Gamma_1(n)} &\equiv \lim_{\epsilon\to 0+} \frac{\Gamma_1(-n-\epsilon)}{\Gamma_1(n+\epsilon)} \\ &= \frac{(-1)^{n(n+1)/2}}{(2\pi)^n\Gamma(n)\Gamma(n+1)}\,\frac{1}{\epsilon^{n+1}}.\end{aligned} \tag{A.4}$$

However, for half-integer arguments:

$$\frac{\Gamma_1\left(-\frac{2m+1}{2}\right)}{\Gamma_1\left(\frac{2m+1}{2}\right)} = \frac{(-1)^{(m+1)(m+2)/2}\sqrt{2}}{\pi^{m+\frac{3}{2}}\,(2m-1)!!\,(2m+1)!!}, \quad m \in \mathbf{Z}, \tag{A.5}$$

the corresponding ratio is finite.

Likewise, using the relations above, $S_1(1-x) = \frac{\Gamma(x)\Gamma_1(-x)}{\Gamma(-x)\Gamma_1(x)}$. Therefore, one finds that

$$\begin{aligned} S_1(1-n) &= \frac{(-1)^{n(n-1)/2}}{(2\pi\epsilon)^n}, \\ S_1\left(1-\frac{2n+1}{2}\right) &= \frac{(-1)^{-n(n+1)/2}}{2^{2n+\frac{1}{2}}\pi^{n+\frac{3}{2}}}, \end{aligned} \tag{A.6}$$

for an integer n. Once again, the first of the above is to be defined as a limit.

Appendix B. Evaluation of a Contour Integral for the Three-Point Function

Here we will evaluate the contour integral

$$\mathcal{B}_4 = \frac{1}{i}\int_{-i\infty-0}^{+i\infty-0} ds \;\prod_{i=1}^{4} \frac{S_b(U_i+s)}{S_b(V_i+s)} \tag{B.1}$$

where U_i, V_i are given in Eq.(4.3). The definition of the contour integral as an analytic function of the momenta is explained in Ref. [14]. We shall summarize and use that prescription for our case in which k_i approach positive integers, and $b \to 1$. We shall use an off-shell parameter ϵ here which is the deformation b away from $b = 1$. As mentioned in section 3, an equivalent deformation is one where the Liouville momenta is shifted away from integers.

For large imaginary $|s|$, the integrand decays exponentially, so the integral is convergent in that region. Near the origin, the contour needs to be defined because the integrand has poles which lie on the origin. We do this by shifting the contour a little to the left of the imaginary axis.

Let us list the arguments of the functions S_1 for our case:

$$\begin{aligned} U_1 &= -1+\sigma_1+\sigma_2+k_1, & V_1 &= 2+\sigma_2-\sigma_3+k_1+k_3, \\ U_2 &= 1-\sigma_1+\sigma_2+k_1, & V_2 &= 2+\sigma_2-\sigma_3+k_1-k_3, \\ U_3 &= 1+\sigma_2-\sigma_3-k_2, & V_3 &= 2\sigma_2, \\ U_4 &= 1+\sigma_2-\sigma_3+k_2, & V_4 &= 2. \end{aligned} \tag{B.2}$$

The poles from the numerator and the denominator are at[8]

$$s+U_i=-n_i \quad \text{and} \quad s+V_i=2+m_i, \qquad (n_i,m_i=0,1,2,\cdots) \tag{B.3}$$

respectively. For Re $U_i>0$ and Re $V_i\leq 2$, the poles arising from the numerator are all in the left half-plane and those from the denominator are in the right half-plane[9]. The imaginary axis is therefore a well-defined contour and thanks to the asymptotic behaviour, the integral has a finite value.

For general values of k_i, the integral is defined by analytic continuation of the above prescription. Specifically, this means that as we vary k_i (or equivalently, U_i, V_i) smoothly, some of the poles from the LHP cross the imaginary axis and enter the RHP, and vice-versa. In such a case, one deforms the contour such that the poles from the numerator and the denominator are always separated by the contour. Alternatively, this could be done by an equivalent deformation as follows. Suppose, a pole of the numerator migrates to the LHP. The new (deformed) contour now consists of two parts, one is the old one and another a small circle around the 'migrating' pole. The latter will pick up the residue of the integrand around that pole. However, this also gives a finite contribution and will not be of our final interest.

The integral diverges if two poles, one originating in numerator and another in denominator, approach towards each other to coincide. In this case, the contour is 'pinched between' the two poles. Alternatively, the migrating pole hits another pole. This divergence dominates over the finite piece and it is this which is of interest to us. In order to extract the leading divergence in such cases, let us deform b away from the value $b=1$ by

[8] Here we have already plugged in $b=1$, the general formula has simple poles at $s+U_i=-nb-mb^{-1}$. At $b=1$, these simple poles coalesce to a pole of high order.

[9] The V_4 factor has a pole at the origin, but we have shifted the contour a little to the left as indicated in (4.2). With this understanding, we shall continue to call it the imaginary axis.

an amount ϵ and make the circle around a migrating pole very small. As it hits a would-be singularity at $b = 1$, we determine the divergent residue as a power of ϵ.

The condition for collision between the poles (B.3) is $s = -U_i - n_i = 2 - V_j + m_j$, $(n_i, m_j = 0, 1, 2, \cdots)$, *i.e.*,

$$V_j - U_i = 2 + m, \ m = 0, 1, 2, \cdots . \tag{B.4}$$

For generic σ_i, this can only happen when V_1 collides with U_3 or U_4. Moreover, $V_1 - U_3 = 1 + k_1 + k_2 + k_3 = V_1 - U_4 + 2k_2$, so the divergence from the collision of V_1 and U_3 dominates and it is sufficient to consider only that. Let

$$s + (\sigma_2 - \sigma_3) = n \in \mathbb{Z}. \tag{B.5}$$

Then, a collision between the poles (B.3) happens when

$$1 + n - k_2 = -n_3, \qquad n + k_1 + k_3 = m_1, \qquad (n_3, m_1 = 0, 1, 2, \cdots). \tag{B.6}$$

This happens when $-k_1 - k_3 \leq n \leq k_2 - 1$. This set is non-empty for $(k_1, k_2) \neq (0, 0)$.

The divergence of the integrand for a particular value of n, as defined in Eq.(B.5) above, contributes an amount to the integral $\mathcal{B}_4$ that we denote $\mathcal{B}_4^{(n)}$. Hence,

$$\mathcal{B}_4 = \sum_n \mathcal{B}_4^{(n)} \tag{B.7}$$

The range of values of n over which the sum is to be performed will be determined below.

The net order of divergence of the integrand at a given value of n comes from counting the poles/zeroes in U_3, U_4, V_1 and V_2 (keeping σ_i are generic), and (using Eq.(B.5) and the formula for the divergence of the S_1-function given in Appendix A) is equal to:

$$-(n - k_2) - (n + k_2) + (1 + n + k_1 + k_3) + (1 + n + k_1 - k_3) = 2 + 2k_1. \tag{B.8}$$

One of these poles is the migrant one with a circular contour around it, so the divergent part of the residue[10] is $1/(2\pi\epsilon)^{2k_1+1}$.

[10] Let us see how the same result is obtained with the equivalent regulator in which $b = 1$ but k is shifted away from an integer. The contour integral is about a pole of higher order, say $M \equiv n + k_1 + k_3$, if the migrant pole is from V_1. The residue is then the $(M-1)$th derivative of the other factor which has a pole of order $2k_1 + 2 - M$. The dominant singularity comes from differentiating this singular part, leading to the same final answer.

The finite piece of the residue is due to the other four S_1-functions. Once again, using (B.5), we can write the contribution $\mathcal{B}_4^{(n)}$ as:

$$\mathcal{B}_4^{(n)} = \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \frac{S_1(-1+n+k_1+\sigma_1+\sigma_3)\, S_1(1+n+k_1-\sigma_1+\sigma_3)}{S_1(n+\sigma_2+\sigma_3)\, S_1(2+n-\sigma_2+\sigma_3)}. \tag{B.9}$$

Combining $\mathcal{B}_3$ and $\mathcal{B}_4^{(n)}$ and using some inversions of S_1 along the way,

$$\begin{aligned}
\mathcal{B}_3\, \mathcal{B}_4^{(n)} =& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \frac{S_1(-1+n+k_1+\sigma_1+\sigma_3)}{S_1(-1-k_3+\sigma_1+\sigma_3)} \frac{S_1(-1+k_2+\sigma_2+\sigma_3)}{S_1(n+\sigma_2+\sigma_3)} \\
& \times \frac{S_1(1+k_3+\sigma_1-\sigma_3)}{S_1(1-n-k_1+\sigma_1-\sigma_3)} \frac{S_1(-n+\sigma_2-\sigma_3)}{S_1(1-k_2+\sigma_2-\sigma_3)} \\
=& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \, (2\sin\pi(\sigma_1+\sigma_3))^{k_1+k_3+n} (2\sin\pi(\sigma_2+\sigma_3))^{k_2-n-1} \\
& \times (2\sin\pi(\sigma_1-\sigma_3))^{k_1+k_3+n} (2\sin\pi(\sigma_2-\sigma_3))^{k_2-n-1} \\
=& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(2\, \frac{\mu_{1B}-\mu_{3B}}{\sqrt{\mu}}\right)^{k_1+k_3+n} \left(2\, \frac{\mu_{2B}-\mu_{3B}}{\sqrt{\mu}}\right)^{k_2-n-1}.
\end{aligned} \tag{B.10}$$

Finally we have to sum over all these residues, since the contour is a disjoint sum of all these circles at various values of s labelled by an integer n, which ranges from $-k_1-k_3$ to k_2-1. This is a geometric series. Evaluating the sum, we get:

$$\begin{aligned}
\mathcal{B}_3\, \mathcal{B}_4 &= \mathcal{B}_3 \sum_{n=-k_1-k_3}^{k_2-1} \mathcal{B}_4^{(n)} \\
&= \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(\frac{2\mu_{21}}{\sqrt{\mu}}\right)^{-1} \left\{ \left(\frac{2\mu_{23}}{\sqrt{\mu}}\right)^{\sum_i k_i} - \left(\frac{2\mu_{13}}{\sqrt{\mu}}\right)^{\sum_i k_i} \right\},
\end{aligned} \tag{B.11}$$

where we have defined:

$$\mu_{ij} \equiv \mu_{iB} - \mu_{jB}. \tag{B.12}$$

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