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$c = 1$ Matrix Models: Equivalences and Open-Closed String Duality

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Abstract: We give an explicit demonstration of the equivalence between the Normal Matrix Model (NMM) of $c = 1$ string theory at selfdual radius and the Kontsevich-Penner (KP) model for the same string theory. We relate macroscopic loop expectation values in the NMM to condensates of the closed string tachyon, and discuss the implications for open-closed duality. As in $c < 1$, the Kontsevich-Miwa transform between the parameters of the two theories appears to encode open-closed string duality, though our results also exhibit some interesting differences with the $c < 1$ case. We also briefly comment on two different ways in which the Kontsevich model originates.

Keywords: String theory, Random matrices.

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1. Introduction

In the last few years, enormous progress has been made in understanding noncritical string theory. One line of development started with the work of Refs.[1, 2, 3], in the context of D-branes of Liouville theory. These and subsequent works were inspired by the beautiful CFT computations that gave convincing evidence for the consistency of these branes[4, 5, 6], as well as Sen's picture of the decay of unstable D-branes via tachyon condensation[7]. Another independent line of development that has proved important was the attempt to formulate new matrix models to describe noncritical string theories and their deformations, including black hole deformations[8, 9, 10, 11].

Some of the important new results are related to nonperturbatively stable type 0 fermionic strings[12, 13], but even in the bosonic context, many old and new puzzles concerning matrix models as well as Liouville theory have been resolved. For $c < 1$ matter coupled to Liouville theory, a beautiful picture emerged of a Riemann surface governing the semiclassical dynamics of the model. Both ZZ and FZZT branes were identified as properties of this surface: the former are located at singularities while the latter arise as line integrals. This picture was obtained in Ref.[14] within the continuum Liouville approach and subsequently re-derived in the matrix model formalism in Ref.[15] using earlier results of Ref.[16]. However, later it was realised[17] that the exact, as opposed to semiclassical, picture is considerably simpler: the Riemann surface disappears as a result of Stokes' phenomenon and is replaced by a single sheet. In the exact (quantum) case, correlation functions of macroscopic loop operators go from multiple-valued functions to the Baker-Akhiezer functions of the KP hierarchy, which are analytic functions of the boundary cosmological constant. Thus, for these models (and also their type 0 extensions) a rather complete picture now exists.

Another remarkable development in this context is an explicit proposal to understand open-closed string duality starting from open string field theory. This was presented in Ref.[18] and implemented there for the $(2, q)$ series of minimal models coupled to gravity (which can be thought of as perturbations of the "topological point" or $(2, 1)$ minimal model). The basic idea of Ref.[18] was to evaluate open string field theory on a collection of N FZZT branes in the $(2, 1)$ closed string background. This leads to the Kontsevich matrix model[19], which depends on a constant matrix A whose eigenvalues are the N independent boundary cosmological constants for this collection of branes. Now the Kontsevich model computes the correlators of *closed-string* observables in the same $(2, 1)$ background. So this relationship was interpreted as open-closed duality, following earlier ideas of Sen[20].

A different way of understanding what appears to be the same open-closed duality emerged in Ref.[17] for the $(2, 1)$ case. Extending some older observations in Ref.[21], the authors showed that if one inserts macroscopic loop operators $\det(x_i - \Phi)$, representing FZZT branes (each with its own boundary cosmological constant x_i) in the Gaussian matrix model, and takes a double-scaling limit, one obtains the Kontsevich matrix model. The constant matrix A in this model again arises as the boundary cosmological constants of the FZZT branes[1].

The situation is more complicated and less well-understood for $c = 1$ matter coupled to Liouville theory, namely the $c = 1$ string. The results of FZZT were derived for generic Liouville central charge c_L, but become singular as $c_L \to 25$, the limit that should give the $c = 1$ string. Attempts to understand FZZT branes at $c = 1$ (Refs.[25, 26]) rely on this limit from the $c < 1$ case which brings in divergences and can therefore be problematic. In particular, there is as yet no definite computation exhibiting open-closed duality at $c = 1$ starting from open string field theory in the $c = 1$ Liouville background. One should expect such a computation to give rise to the $c = 1$ analogue of the Kontsevich matrix model, namely the Kontsevich-Penner model[2] of Ref.[27].

In the present work we take a different approach to understand D-branes and open-closed duality in the $c = 1$ string, more closely tied to the approach of Refs.[17,

[1]This has been generalised[22] by starting with macroscopic loops in the double-scaled 2-matrix models that describe $(p, 1)$ minimal model strings. After double-scaling, one obtains the generalised Kontsevich models of Refs.[23, 24].

[2]This model is valid only at the selfdual radius $R = 1$.

22]. The obvious point of departure at $c = 1$ would be to consider macroscopic loops in the Matrix Quantum Mechanics (MQM) and take a double-scaling limit. Indeed, FZZT branes at $c = 1$ have been investigated from this point of view, for example in Refs.[28, 29]. However, we will take an alternative route that makes use of the existence of the Normal Matrix Model (NMM)[11] for $c = 1$ string theory (in principle, at arbitrary radius R). This model is dual in a certain precise sense to the more familiar MQM, namely, the grand canonical partition function of MQM is the partition function of NMM in the large-N limit. Geometrically, the two theories correspond to different real sections of a single complex curve. More details about the interrelationship between MQM and NMM can be found in Ref.[11].

One good reason to start from the NMM is that it is a simpler model than MQM and does not require a double-scaling limit. Also, it has been a longstanding question whether the KP model and NMM are equivalent, given their structural similarities, and if so, what is the precise map between them. It is tempting to believe that open-closed duality underlies their mutual relationship. Indeed, the NMM does not have a parameter suggestive of a set of boundary cosmological constants, while the KP model has a Kontsevich-type constant matrix A. So another natural question is whether the eigenvalues of A are boundary cosmological constants for a set of FZZT branes/macroscopic loop operators of NMM.

In what follows we examine these questions and obtain the following results. First of all we find a precise map from the NMM (with arbitrary tachyon perturbations) to the KP model, thereby demonstrating their equivalence. While the former model depends on a non-Hermitian matrix Z constrained to obey $[Z, Z^\dagger] = 1$, the latter is defined in terms of a positive definite Hermitian matrix M. We find that the eigenvalues z_i and m_i are related by $m_i = z_i\bar{z}_i$. The role of the large-N limit in the two models is slightly different: in the KP model not only the random matrix but also the number of parameters (closed string couplings) is reduced at finite N. On the contrary, in the NMM the number of parameters is always infinite for any N, but one is required to take $N \to \infty$ to obtain the right theory (this was called "Model I" in Ref.[11]). The two models are therefore equivalent only on a subspace of the parameter space at finite N, with the limit $N \to \infty$ being required to obtain full equivalence. This is an important point to which we will return.

Next in § 5 we consider macroscopic loop operators of the form $\det(\xi - Z)$ in the NMM, and show that these operators when inserted into the NMM, *decrease* the value of the closed-string tachyon couplings in a precise way dictated by the Kontsevich-Miwa transform. On the contrary, operators of the form $1/\det(\xi - Z)$ play the role of increasing, or turning on, the closed-string tachyon couplings. In particular, insertion of these *inverse determinant* operators in the (partially unperturbed) NMM leads to the Kontsevich-Penner model. (By partially unperturbed, we mean the couplings of the positive-momentum tachyons are switched off, while those of the negative-momentum tachyons are turned on at arbitrary values.) Calculationally, this result is a corollary of our derivation of the KP model from the perturbed NMM in § 4.

These results bear a rather strong analogy to the emergence of the Kontsevich model from the insertion of determinant operators at $c < 1$[17]. In both cases, the parameters of macroscopic loop operators turn into eigenvalues of a Kontsevich matrix. Recall that in Ref.[17], one inserts n determinant operators into the $N \times N$ Gaussian matrix model and then integrates out the Gaussian matrix. Taking $N \to \infty$ (as a double-scaling limit) we are then left with the Kontsevich model of rank n. In the $c = 1$ case, we insert n inverse determinant operators in the NMM. As we will see, $N - n$ of the normal matrix eigenvalues then decouple, and we are left with a Kontsevich-Penner model of rank n (here one does not have to take $N \to \infty$). We see that the two cases are rather closely analogous.

The main difference between our case at $c = 1$ and the $c < 1$ case of Ref.[17] is that we work with inverse determinant rather than determinant operators. However at infinite n we can remove even this difference: it is possible to replace the inverse determinant by the determinant of a different matrix, defining a natural pair of mutually "dual" Kontsevich matrices[3]. In terms of the dual matrix, one then recovers a relation between correlators of determinants (rather than inverse determinants) and the KP model.

In the concluding section we examine a peculiar property of the NMM, namely that it describes the $c = 1$ string even at finite N, if we set $N = \nu$, where ν is the analytically continued cosmological constant $\nu = -i\mu$. This was noted in Ref.[11], where this variant of the NMM was called "Model II". Now it was already observed in Ref.[27] that setting $N = \nu$ in the KP model (and giving a nonzero value to one of the deformation parameters) reduces the KP model to the original Kontsevich model that describes $(2, q)$ minimal strings. Thus we have a (two-step) process leading from the NMM to the Kontsevich model. However, we also know from Ref.[17] that the Kontsevich model arises from insertion of macroscopic loops in the double-scaled Gaussian matrix model. We will attempt to examine to what extent these two facts are related.

2. Normal Matrix Model

We start by describing the Normal Matrix Model (NMM) of $c = 1$ string theory[11] and making a number of observations about it. The model originates from some well-known considerations in the Matrix Quantum Mechanics (MQM) description of the Euclidean $c = 1$ string at radius R. Here, $R = 1$ is the selfdual radius, to which we will specialise later. The MQM theory has discrete "tachyons" T_k, of momentum $\frac{k}{R}$, where $k \in Z$. Let us divide this set into "positive tachyons" $T_k, k > 0$ and "negative

[3] This dual pair is apparently unrelated to the dual pair of boundary cosmological constants at $c < 1$.

tachyons" $T_k, k < 0$. (The zero-momentum tachyon is the cosmological operator and is treated separately). We now perturb the MQM by these tachyons, using coupling constants $t_k, k > 0$ for the positive tachyons and $\bar{t}_k, k > 0$ for the negative ones.

The grand canonical partition function of MQM is denoted $\mathcal{Z}(\mu, t_k, \bar{t}_k)$. At $t_k = \bar{t}_k = 0$, it can easily be shown to be:

$$\mathcal{Z}(\mu, t_k = 0, \bar{t}_k = 0) = \prod_{n\geq 0}^{\infty} \Gamma\left(-\frac{n+\frac{1}{2}}{R} - i\mu + \frac{1}{2}\right) \tag{2.1}$$

But this is also the partitition function of the matrix integral:

$$\begin{aligned}\mathcal{Z}_{NMM} &= \int [dZdZ^\dagger]\, e^{-\mathrm{tr}W(Z,Z^\dagger)} \\ &= \int [dZdZ^\dagger]\, e^{\mathrm{tr}\left(-\nu(ZZ^\dagger)^R + \left[\frac{1}{2}(R-1)+(R\nu-N)\right]\log ZZ^\dagger\right)}\end{aligned} \tag{2.2}$$

where $\nu = -i\mu$ and $N \to \infty$. Here Z, $Z^\dagger$ are $N \times N$ matrices satisfying:

$$[Z, Z^\dagger] = 0 \tag{2.3}$$

Since the matrix Z commutes with its adjoint, the model defined by Eq. (2.2) is called the Normal Matrix Model (NMM)[4].

The equality above says that the unperturbed MQM and NMM theories are equivalent. The final step is to note that the tachyon perturbations correspond to infinitely many Toda "times" in the MQM partition function, which becomes a τ-function of the Toda integrable hierarchy. The same perturbations on the NMM side are obtained by adding to the matrix action the terms:

$$W(Z, Z^\dagger) \to W(Z, Z^\dagger) + \nu \sum_{k=1}^{\infty} \left(t_k Z^k + \bar{t}_k {Z^\dagger}^k\right) \tag{2.4}$$

It follows that the Normal Matrix Model, even after perturbations, is equivalent to MQM.

The equivalence of the full perturbed MQM and NMM gives an interesting interpretation of the perturbations in NMM in terms of the Fermi surface of the MQM. The unperturbed MQM Hamiltonian is given by:

$$H_0 = \frac{1}{2}\mathrm{tr}(-\hbar^2 \frac{\partial^2}{\partial X^2} - X^2) \tag{2.5}$$

[4]For the most part we follow the conventions of Ref.[11]. However we use the transcription $(1/i\hbar)_{\rm them} \to \nu_{\rm us}$ and $\mu_{\rm them} \to 1_{\rm us}$. The partition function depends on the ratio $(\mu/i\hbar)_{them} \to \nu_{us} = -i\mu_{\rm us}$. Our conventions for the NMM will be seen to match with the conventions of Ref.[27] for the KP model. Note that the integral is well-defined for all complex ν with a sufficiently large real part. It can then be extended by analytic continuation to all complex values of the parameter ν, other than those for which the argument of the Γ function is a negative integer. This is sufficient, since everything is ultimately evaluated at purely imaginary values of ν.

where X is an $N \times N$ Hermitian matrix (here the compactification radius is R). In the $SU(N)$-singlet sector this system is described by N non-relativistic fermions moving in an inverted harmonic oscillator potential. The eigenvalues of X describe the positions of these fermions. In terms of eigenvalues the Hamiltonian can be written as:

$$H_0 = \frac{1}{2} \sum_{i=1}^{N} (\hat{p}_i^2 - \hat{x}_i^2), \tag{2.6}$$

p_i being the momenta conjugate to x_i. We now want to consider perturbations of Eq. (2.6) by tachyon operators. For this it is convenient to change variables from $\hat{p}$, $\hat{x}$ to the "light cone" variables $\hat{x}_\pm$:

$$\hat{x}_\pm = \frac{\hat{x} \pm \hat{p}}{\sqrt{2}} \tag{2.7}$$

Since $[\hat{p}, \hat{x}] = -i\hbar$ it follows that $[\hat{x}_+, \hat{x}_-] = -i\hbar$ also. The MQM Hamiltonian in terms of the new variables is:

$$H_0 = -\sum_{i=1}^{N} \hat{x}_{+i}\hat{x}_{-i} - \frac{i\hbar N}{2} \tag{2.8}$$

In the phase space (x_+, x_-) the equation of the Fermi surface for the unperturbed MQM is given by:

$$x_+ x_- = \mu \tag{2.9}$$

The tachyon perturbations to the MQM Hamiltonian H_0 are given in terms of the new variables by:

$$H = H_0 - \sum_{k \geq 1} \sum_{i=1}^{N} \left(k\, t_{\pm k}\, x_{\pm i}^{\frac{k}{R}} + v_{\pm k}\, x_{\pm i}^{-\frac{k}{R}} \right) \tag{2.10}$$

In the above equation the v's are determined in terms of the t's from the orthonormality of the Fermion wavefunctions. The conventions chosen above simplifies the connection with NMM perturbations. The Fermi surface of the perturbed MQM is given by:

$$x_+ x_- = \mu + \sum_{k \geq 1} \left(k\, t_{\pm k}\, x_\pm^{\frac{k}{R}} + v_{\pm k}\, x_\pm^{-\frac{k}{R}} \right) \tag{2.11}$$

The equivalence between NMM and MQM relates the tachyon perturbations in Eq. (2.4) and Eq. (2.10) with the following identification between the tachyon operators of the two models:

$$\mathrm{tr} X_+^{\frac{n}{R}} = \mathrm{tr} Z^n$$
$$\mathrm{tr} X_-^{\frac{n}{R}} = \mathrm{tr} Z^{\dagger n}$$

The coefficients $t_\pm$ are the same as $t, \bar{t}$ in the NMM. This means that any tachyon perturbation in the NMM is mapped directly to a deformation of the Fermi surface of MQM by Eq. (2.11).

At the selfdual radius R=1, the NMM simplifies and the full perturbed partition function can be written as:

$$\mathcal{Z}_{NMM}(t,\bar{t}) = \int [dZdZ^\dagger]\, e^{\mathrm{tr}\left(-\nu ZZ^\dagger + (\nu - N)\log ZZ^\dagger - \nu \sum_{k=1}^{\infty}\left(t_k Z^k + \bar{t}_k {Z^\dagger}^k\right)\right)} \tag{2.12}$$

We note several properties of this model.

(i) The unperturbed part depends only on the combination $ZZ^\dagger$ and not on $Z, Z^\dagger$ separately.

(ii) The model can be reduced to eigenvalues, leading to the partition function:

$$\mathcal{Z}_{NMM}(t,\bar{t}) = \int \prod_{i=1}^{N} dz_i d\bar{z}_i\ \Delta(z)\Delta(\bar{z})\, e^{\sum_{i=1}^{N}\left(-\nu z_i\bar{z}_i + (\nu-N)\log z_i\bar{z}_i - \nu\sum_{k=1}^{\infty}\left(t_k z_i^k + \bar{t}_k \bar{z}_i^k\right)\right)} \tag{2.13}$$

(iii) The model is symmetric under the interchange $t_k \leftrightarrow \bar{t}_k$, as can be seen by interchanging Z and $Z^\dagger$. In spacetime language this symmetry amounts to the transformation $X \to -X$ where X is the Euclidean time coordinate, which interchanges positive and negative momentum tachyons.

(iv) The correlator:

$$\langle \mathrm{tr}Z^{k_1}\mathrm{tr}Z^{k_2}\cdots\mathrm{tr}Z^{k_m}\mathrm{tr}{Z^\dagger}^{\ell_1}\mathrm{tr}{Z^\dagger}^{\ell_2}\cdots\mathrm{tr}{Z^\dagger}^{\ell_n}\rangle_{t_k=\bar{t}_k=0} \tag{2.14}$$

vanishes unless

$$\sum_m k_m = \sum_n \ell_n \tag{2.15}$$

This correlator is computed in the unperturbed theory. The above result follows by performing the transformation:

$$Z \to e^{i\theta} Z \tag{2.16}$$

for some arbitrary angle θ. The unperturbed theory is invariant under this transformation, therefore correlators that are not invariant must vanish. In spacetime language this amounts to the fact that tachyon momentum in the X direction is conserved.

(v) As a corollary, we see that if we set all $t_k = 0$, the partition function becomes independent of $\bar{t}_k$:

$$\mathcal{Z}_{NMM}(0,\bar{t}_k) = \mathcal{Z}_{NMM}(0,0) \tag{2.17}$$

(vi) For computing correlators of a finite number of tachyons, it is enough to turn on a *finite* number of $t_k, \bar{t}_k$, i.e. we can always assume for such purposes that $t_k, \bar{t}_k = 0, k > k_{max}$ for some finite integer k_{max}. In that case, apart from the log term we have a polynomial matrix model.

(vii) We can tune away the log term by choosing $\nu = N$. This choice has been called Model II in Ref.[11]. In this case the model reduces to a Gaussian model (but of a normal, rather than Hermitian, matrix) with perturbations that are holomorphic + antiholomorphic in the matrix Z (i.e., in the eigenvalues z_i). If we assume that the couplings $t_k, \bar{t}_k$ vanish for $k > k_{max}$, as in the previous comment, then the perturbations are also polynomial. We will return to this case in a subsequent section.

3. The Kontsevich-Penner or W_∞ model

The Kontsevich-Penner or W_∞ model[27] (for a more detailed review, see Ref.[30]) is a model of a single positive-definite hermitian matrix, whose partition function is given by:

$$\mathcal{Z}_{KP}(A, \bar{t}) = (\det A)^\nu \int [dM]\, e^{\mathrm{tr}\left(-\nu MA + (\nu - N)\log M - \nu \sum_{k=1}^\infty \bar{t}_k M^k\right)} \tag{3.1}$$

where $\bar{t}_k$ are the couplings to negative-momentum tachyons, N is the dimensionality of the matrix M and A is a constant matrix. The eigenvalues of this matrix determine the couplings t_k to positive-momentum tachyons via the Kontsevich-Miwa (KM) transform:

$$t_k = -\frac{1}{\nu k}\mathrm{tr}(A^{-k}) \tag{3.2}$$

This model is derived by integrating the W_∞ equations found in Ref.[31]. The parameter ν appearing in the action above is related to the cosmological constant μ of the string theory by $\nu = -i\mu$. The model can also be obtained from the Penner matrix model[32, 33] after making a suitable change of variables (as explained in detail in Ref.[30]) and adding perturbations.

We now note some properties that are analogous to those of the NMM, as well as others that are quite different.

(i) By redefining $MA \to M$ we can rewrite the partition function without any factor in front, as:

$$\mathcal{Z}_{KP}(A, \bar{t}) = \int [dM]\, e^{\mathrm{tr}\left(-\nu M + (\nu - N)\log M - \nu \sum_{k=1}^\infty \bar{t}_k (MA^{-1})^k\right)} \tag{3.3}$$

(ii) This model has no radius deformation, and describes the $c = 1$ string theory directly at selfdual radius $R = 1$.

(iii) In view of the logarithmic term, the model is well-defined only if the integral over the eigenvalues m_i of the matrix M is restricted to the region $m_i > 0$.

(iv) The model can be reduced to eigenvalues, leading to the partition function:

$$\mathcal{Z}_{KP}(A, \bar{t}) = \left(\prod_{i=1}^N a_i\right)^\nu \int \prod_{i=1}^N dm_i\, \frac{\Delta(m)}{\Delta(a)}\, e^{\sum_{i=1}^N \left(-\nu m_i a_i + (\nu - N)\log m_i - \nu \sum_{k=1}^\infty \bar{t}_k m_i^k\right)} \tag{3.4}$$

(vi) In the representation Eq. (3.1), the operators $\mathrm{tr}M^k$ describe the negative-momentum tachyons. But there are no simple operators that directly correspond to positive-momentum tachyons. Nevertheless this model generates tachyon correlators of the $c = 1$ string as follows:

$$\langle \mathcal{T}_{k_1}\mathcal{T}_{k_2}\cdots\mathcal{T}_{k_m}T_{-\ell_1}T_{-\ell_2}\cdots T_{-\ell_n}\rangle = \frac{\partial}{\partial t_{k_1}}\frac{\partial}{\partial t_{k_2}}\cdots\frac{\partial}{\partial t_{k_m}}\frac{\partial}{\partial \bar{t}_{\ell_1}}\frac{\partial}{\partial \bar{t}_{\ell_2}}\cdots\frac{\partial}{\partial \bar{t}_{\ell_n}}\log\mathcal{Z}_{KP} \quad (3.5)$$

where derivatives in t_k are computed using Eq. (3.2) and the chain rule.

(v) The symmetry of the partition function under the interchange of $t_k, \bar{t}_k$ is not manifest, since one set of parameters is encoded through the matrix A while the other appears explicitly.

(vi) The transformation

$$A \to \alpha A, \qquad \bar{t}_k \to \alpha^k\, \bar{t}_k \quad (3.6)$$

for arbitrary α, is a symmetry of the model (most obvious in the representation Eq. (3.3)). As a consequence, the tachyon correlators satisfy momentum conservation.

(vi) The partition function satisfies the "puncture equation":

$$\mathcal{Z}_{KP}(A-\epsilon, \bar{t}_k + \delta_{k,1}\,\epsilon) = e^{\epsilon\nu^2 t_1}\,\mathcal{Z}_{KP}(A, \bar{t}_k) \quad (3.7)$$

as can immediately be seen from Eq. (3.1).

4. Equivalence of the matrix models

4.1 $N = 1$ case

We start by choosing the selfdual radius $R = 1$, and will later comment on what happens at other values of R. As we have seen, in the perturbed NMM there are two (infinite) sets of parameters $t_k, \bar{t}_k$, all of which can be chosen independently. This is the case even at finite N, though the model describes $c = 1$ string theory only at infinite N (or at the special value $N = \nu$, as noted in Ref.[11], a point to which we will return later). In contrast, the Kontsevich-Penner model has one infinite set of parameters $\bar{t}_k$, as well as N additional parameters from the eigenvalues of the matrix A. The latter encode the t_k, as seen from Eq. (3.2) above. From this it is clear that at finite N, there can only be N independent parameters t_k ($k = 1, 2, \ldots N$) while the remaining ones ($t_k, k > N$) are dependent on these.

This makes the possible equivalence of the two models somewhat subtle. To understand the situation better, let us compare both models in the limit that is farthest away from $N \to \infty$, namely $N = 1$. While this is a "toy" example, we will see that it provides some useful lessons.

In this case the NMM partition function is:

$$\mathcal{Z}_{NMM,N=1}(t_k,\bar{t}_k) = \int dz\, d\bar{z}\ e^{-\nu z\bar{z}+(\nu-1)\log z\bar{z}-\nu\sum_{k=1}^{\infty}(t_k z^k+\bar{t}_k\bar{z}^k)} \tag{4.1}$$

while the Kontsevich-Penner partition function is:

$$\mathcal{Z}_{KP,N=1}(a,\bar{t}_k) = a^\nu \int dm\ e^{-\nu ma+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k m^k} \tag{4.2}$$

We will now show that the two integrals above are equivalent if we assume that t_k in the NMM is given by:

$$t_k = -\frac{1}{\nu k}\, a^{-k} \tag{4.3}$$

which is the KM transform Eq. (3.2) in the special case where A is a 1×1 matrix, denoted by the single real number a. Note that this determines all the infinitely many t_k in terms of a.

To obtain the equivalence, insert the above relation and also perform the change of integration variable:

$$z = \sqrt{m}\, e^{i\theta} \tag{4.4}$$

in the NMM integral. Then we find that (up to a numerical constant):

$$\begin{aligned}\mathcal{Z}_{NMM,N=1}(a,\bar{t}_k) &= \int dm\, d\theta\ e^{-\nu m+(\nu-1)\log m+\sum_{k=1}^{\infty}\frac{1}{k}(\frac{\sqrt{m}}{a})^k e^{ik\theta}-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}}\\ &= \int dm\, d\theta\ \frac{1}{1-\frac{\sqrt{m}\, e^{i\theta}}{a}}\ e^{-\nu m+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}}\end{aligned} \tag{4.5}$$

Strictly speaking the last step is only valid for $\sqrt{m}/a < 1$, since otherwise the infinite sum fails to converge. Hence we fix m and a to satisfy this requirement and continue by evaluating the θ-integral. This can be evaluated by defining $e^{-i\theta} = w$ and treating it as a contour integral in w. We have

$$d\theta\, \frac{1}{1-\frac{\sqrt{m}\, e^{i\theta}}{a}} \to dw\, \frac{1}{w-\frac{\sqrt{m}}{a}} \tag{4.6}$$

Since the rest of the integrand is well-defined and analytic near $w=0$, we capture the simple pole at $w=\sqrt{m}/a$. That brings the integrand to the desired form. Now we can lift the restriction $\sqrt{m}/a < 1$, and treat the result as valid for all m by analytic continuation. Therefore we find:

$$\begin{aligned}\mathcal{Z}_{NMM,N=1}(a,\bar{t}_k) &= \int dm\ e^{-\nu m+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k(ma^{-1})^k}\\ &= a^\nu \int dm\ e^{-\nu ma+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k m^k}\\ &= \mathcal{Z}_{KP,N=1}(a,\bar{t}_k)\end{aligned} \tag{4.7}$$

Thus we have shown that the perturbed 1×1 Normal Matrix Model at $R = 1$ is equivalent to the 1×1 Kontsevich-Penner model. However, this equivalence only holds when we perform the 1×1 KM transform, which fixes all the perturbations t_k in terms of a single independent parameter a (while the $\bar{t}_k$ are left arbitrary).

An important point to note here is the sign chosen in Eq. (4.3). Changing the sign (independently of k) amounts to the transformation $t_k \to -t_k$. This is apparently harmless, leading to some sign changes in the correlation functions, but there is no way at $N = 1$ (or more generally at any finite N) to change a (or the corresponding matrix A) to compensate for this transformation. The sign we have chosen, given the signs in the original NMM action, is therefore the only one that gives the KP model. This point will become important later on.

Returning to the NMM-KP equivalence at $N = 1$, it is interesting to generalise it by starting with the NMM at an arbitrary radius R instead of $R = 1$ as was the case above. As seen from Eq. (2.2), the coupling of the log term is modified in this case as:

$$(\nu - 1) \to \frac{1}{2}(R - 1) + (R\nu - 1) \tag{4.8}$$

and also the bilinear term $z\bar{z}$ is modified to $(z\bar{z})^R$. The above derivation goes through with only minor changes, and we end up with:

$$\mathcal{Z}_{NMM,N=1}(a, \bar{t}_k) = a^{\frac{1}{2}(R-1)+\nu} \int dm\; e^{-\nu(ma)^R + \left[\frac{1}{2}(R-1)+(R\nu-1)\right]\log m - \nu\sum_{k=1}^{\infty}\bar{t}_k m^k} \tag{4.9}$$

This appears to suggest that there is a variant of the Kontsevich-Penner model valid at arbitrary radius (or at least arbitrary integer radius, since otherwise it may become hard to define the integral). This would be somewhat surprising as such a model has not been found in the past. As we will see in the following subsection, the above result holds only for the $N = 1$ case. Once we go to $N \times N$ matrices, we will see that NMM leads to a KP matrix model only at $R = 1$, consistent with expectations.

Another generalisation of the above equivalence seems more interesting. In principle, even for the 1×1 matrix model, we can carry out a KM transform using an $n \times n$ matrix A where n is an arbitrary integer. Indeed, there is no logical reason why the dimension of the constant matrix A must be the same as that of the random matrices occurring in the integral. The most general example of this is to take $N \times N$ random matrices $Z, Z^\dagger$ in the NMM and then carry out a KM transform with A being an $n \times n$ matrix. The "usual" transform then emerges as the special case $n = N$. Of course all this makes sense only within the NMM and not in the KP model. If $n \neq N$ then the KP model, which has a $\mathrm{tr}MA$ term in its action, cannot even be defined. So we should not expect to find the KP model starting with the NMM unless $n = N$, but it is still interesting to see what we will find.

Here we will see what happens if we take $N = 1$ and $n > 1$. The full story will appear in a later subsection. Clearly the KM transform Eq. (3.2) permits more independent parameters t_k as n gets larger. Let us take the eigenvalues of A to be $a_1, a_2, \ldots, a_n$. Then it is easy to see that:

$$\begin{aligned}\mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) &= \int dm\, d\theta\; e^{-\nu m+(\nu-1)\log m+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k}(\frac{\sqrt{m}}{a_i})^k e^{ik\theta}-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}} \\ &= \int dm\, d\theta\; \frac{1}{\prod_{i=1}^{n}(1-\frac{\sqrt{m}\,e^{i\theta}}{a_i})}\; e^{-\nu m+(\nu-1)\log m-\nu\sum_{k=1}^{\infty}\bar{t}_k(\sqrt{m})^k e^{-ik\theta}}\end{aligned} \tag{4.10}$$

Converting to the w variable as before, we now encounter n poles. Picking up the residues, we get:

$$\mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) = \int dm\; e^{-\nu m+(\nu-1)\log m}\sum_{l=1}^{n}\left(\frac{1}{\prod_{i\neq l}(1-\frac{a_l}{a_i})}e^{-\nu\sum_{k=1}^{\infty}\bar{t}_k(\frac{m}{a_l})^k}\right) \tag{4.11}$$

This in turn can be expressed as a sum over n 1×1 Kontsevich-Penner models:

$$\mathcal{Z}_{NMM,N=1}(a_i, \bar{t}_k) = \sum_{l=1}^{n}\frac{1}{\prod_{i\neq l}(1-\frac{a_l}{a_i})}\mathcal{Z}_{KP,N=1}(a_l, \bar{t}_k) \tag{4.12}$$

Note that if in this expression we take $a_n \to \infty$, one of the terms in the above equation (corresponding to $l = n$) decouples, and a_n also drops out from the remaining terms. Therefore we recover the same equation with $n \to n-1$. In this way we can successively decouple all but one of the a_i's.

To summarise, at the level of the 1×1 NMM, we have learned some interesting things: this model is equivalent to the 1×1 KP model if we specialise the parameters t_k to a 1-parameter family via the KM transform, while it is equivalent to a sum over n different 1×1 KP models if we specialise the parameters t_k to an n-parameter family. We also saw a 1×1 KP model arise when we are at a finite radius $R \neq 1$. In the next section we will see to what extent these lessons hold once we work with $N\times N$ random matrices.

4.2 General case

In this section we return to the $N\times N$ Normal Matrix Model. With the substitution Eq. (3.2) (where A is also an $N\times N$ matrix), its partition function becomes:

$$\mathcal{Z}_{NMM} = \int [dZ\, dZ^\dagger]\, e^{\mathrm{tr}\left(-\nu ZZ^\dagger+(\nu-N)\log ZZ^\dagger+\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}(A^{-k})Z^k-\nu\sum_{k=1}^{\infty}\bar{t}_k Z^{\dagger k}\right)} \tag{4.13}$$

or, in terms of eigenvalues:

$$\begin{aligned}\mathcal{Z}_{NMM} &= \int\prod_{i=1}^{N}d^2z_i\;\Delta(z)\Delta(\bar{z})\,e^{-\nu\sum_{i=1}^{N}z_i\bar{z}_i+(\nu-N)\sum_{i=1}^{N}\log z_i\bar{z}_i} \\ &\quad\times e^{\sum_{i,j=1}^{N}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_i}{a_j}\right)^k}\, e^{-\nu\sum_{i=1}^{N}\sum_{k=1}^{\infty}\bar{t}_k\bar{z}_i^k}\end{aligned} \tag{4.14}$$

where $\Delta(z)$ is the Vandermonde determinant. Because of the normality constraint $[Z, Z^\dagger] = 0$ there is only one Vandermonde for z_i and one for $\bar{z}_i$.

The sum over k in the second line of Eq. (4.14) converges if $\frac{z_i}{a_j} < 1$ for all i, j, in which case it can be evaluated immediately giving:

$$\mathcal{Z}_{NMM} = \int \prod_{i=1}^{N} d^2 z_i \; |\Delta(z)|^2 \prod_{i,j=1}^{N} \frac{1}{1 - \frac{z_i}{a_j}} \; e^{\sum_{i=1}^{N}\left[-\nu z_i \bar{z}_i + (\nu - N)\log z_i \bar{z}_i - \nu \sum_{k=1}^{\infty} \bar{t}_k \bar{z}_i^k\right]} \tag{4.15}$$

To make contact with the Penner model, first change variables $z_i \to \sqrt{m_i}\, e^{i\theta_i}$ and then replace $e^{-i\theta_i}$ by w_i as before. Then we get $d^2 z_i \to dm_i \frac{dw_i}{w_i}$ and:

$$\begin{aligned}\mathcal{Z}_{NMM} = &\int \prod_{i=1}^{N} dm_i \oint \prod_{i=1}^{N} \frac{dw_i}{w_i} \prod_{i<j} \left(\frac{\sqrt{m_i}}{w_i} - \frac{\sqrt{m_j}}{w_j} \right) \left(\sqrt{m_i} w_i - \sqrt{m_j} w_j \right) \\ &\times \prod_{i,j=1}^{N} \frac{1}{1 - \frac{\sqrt{m_i}}{w_i a_j}} \; e^{\sum_{i=1}^{N}\left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k (\sqrt{m_i} w_i)^k\right]}\end{aligned} \tag{4.16}$$

The contour integrals can be evaluated once this is rewritten in the more convenient form:

$$\begin{aligned}\mathcal{Z}_{NMM} = &\int \prod_{i=1}^{N} dm_i \oint \prod_{i=1}^{N} dw_i \prod_{i<j} \left(\sqrt{m_i} w_j - \sqrt{m_j} w_i \right) \left(\sqrt{m_i} w_i - \sqrt{m_j} w_j \right) \\ &\times \prod_{i,j=1}^{N} \frac{1}{w_i - \frac{\sqrt{m_i}}{a_j}} \; e^{\sum_{i=1}^{N}\left[-\nu m_i + (\nu - N)\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k (\sqrt{m_i} w_i)^k\right]}\end{aligned} \tag{4.17}$$

Next we pick up the residues at the poles. During the intermediate steps, we will assume that the eigenvalues a_i of the matrix A are non-degenerate. From the above expression, each integration variable w_i has a pole at each of the values:

$$w_i = \frac{\sqrt{m_i}}{a_j} \tag{4.18}$$

for all j. Thus the contributions can be classified by the set of poles:

$$(w_1, w_2, \ldots, w_N) = \left(\frac{\sqrt{m_1}}{a_{j_1}}, \frac{\sqrt{m_2}}{a_{j_2}}, \ldots, \frac{\sqrt{m_N}}{a_{j_N}} \right) \tag{4.19}$$

We now notice that the set $(j_1, j_2, \ldots, j_N)$ must consist of distinct elements, in other words it forms a permutation of $(1, 2, \ldots, N)$. This is because if two values of j_i coincide, one of the Vandermonde factors of the type $(\sqrt{m_i} w_j - \sqrt{m_j} w_i)$ vanishes and there is no contribution.

We start by considering the simplest permutation, the identity, namely:

$$(j_i, j_2, \ldots, j_N) = (1, 2, \ldots, N) \tag{4.20}$$

In this case the residues from the denominator and Vandermonde factors become:

$$\prod_{i<j}^{N}\left(\frac{\sqrt{m_im_j}}{a_j}-\frac{\sqrt{m_im_j}}{a_i}\right)\left(\frac{m_i}{a_i}-\frac{m_j}{a_j}\right)\prod_{j\neq i}^{N}\frac{1}{\frac{\sqrt{m_i}}{a_i}-\frac{\sqrt{m_i}}{a_j}}=\frac{\prod_{i<j}^{N}(m_ia_j-m_ja_i)}{\Delta(a)} \tag{4.21}$$

while the exponential measure factor becomes:

$$e^{\sum_{i=1}^{N}\left[-\nu m_i+(\nu-N)\log m_i-\nu\sum_{k=1}^{\infty}\bar{t}_k\left(\frac{m_i}{a_i}\right)^k\right]} \tag{4.22}$$

It is easy to check that for all the other possible permutations of $(j_1, j_2, \ldots, j_N)$ besides the identity permutation, a corresponding permutation of the integration variables m_i brings the above answer (exponential measure as well as prefactors) back to the same form as for the identity permutation. This means that (dropping a factor of $\frac{1}{N!}$) we have proved:

$$\begin{aligned}\mathcal{Z}_{NMM} &= \int\prod_{i=1}^{N}dm_i\,\frac{\prod_{i<j}^{N}(m_ia_j-m_ja_i)}{\Delta(a)}e^{\sum_{i=1}^{N}\left[-\nu m_i+(\nu-N)\log m_i-\nu\sum_{k=1}^{\infty}\bar{t}_k\left(\frac{m_i}{a_i}\right)^k\right]}\\ &= \Big(\prod_{i=1}^{N}a_i\Big)^{\nu}\int\prod_{i=1}^{N}dm_i\,\frac{\Delta(m)}{\Delta(a)}\,e^{\sum_{i=1}^{N}\left[-\nu m_ia_i+(\nu-N)\log m_i-\nu\sum_{k=1}^{\infty}\bar{t}_km_i^k\right]}\end{aligned} \tag{4.23}$$

where in the last step we have replaced $m_i \to m_i a_i$.

This is precisely the eigenvalue representation Eq. (3.4) of the KP matrix model Eq. (3.1). Thus we have provided a direct proof of equivalence of the perturbed Normal Matrix Model and the Kontsevich-Penner model. Notice that in performing the KM transform we reduced the independent t_k of the NMM to a finite number, namely N, so that eventually the $N \to \infty$ limit is required in order to encode all the independent parameters.

In the previous subsection we considered taking different ranks for the constant matrix A arising in the KM transform and the random matrix Z. The most general case is to take Z to be $N \times N$ and A to be $n \times n$. The computation is a simple extension of the one done above. We find the following results. When $n > N$ we again get a sum over Kontsevich-Penner models. The number of terms in the sum is the binomial coefficient ${}^nC_N$. This is a generalisation of the result given in Eq. (4.12) for $N = 1$, where we found n terms. In the general case let us denote by $a_{\{i,l\}}$ the i^{th} element of the set formed by one possible choice of N a_i's from a total of n, the index l labeling the particular choice. The complementary set, formed by the rest of the a_i's is denoted by $a_{\{\tilde{i},l\}}$, the index $\tilde{i}$ taking $n - N$ values. We then have:

$$\mathcal{Z}_{NMM}(a_i,\bar{t}_k)=\sum_{l=1}^{{}^nC_N}\prod_{i=1}^{N}\prod_{\tilde{i}=1}^{N-n}\frac{1}{\left(1-\frac{a_{\{i,l\}}}{a_{\{\tilde{i},l\}}}\right)}\mathcal{Z}_{KP}(a_{\{l\}},\bar{t}_k) \tag{4.24}$$

so that the NMM is again expressed as a sum over KP models.

The other case, $n < N$, can be obtained by starting with $n = N$ and successively decoupling $N - n$ eigenvalues a_i by taking them to infinity. This is similar to what we observed in the $N = 1$ case following Eq. (4.12). In the present case one can easily show that $N - n$ matrix eigenvalues m_i also decouple in this limit (apart from a normalisation). In fact, it is straightforward to derive the formula:

$$\lim_{a_N \to \infty} Z_{KP}^{(N,\nu)}(A^{(N)}, \bar{t}_k) = \frac{\Gamma(\nu - N + 1)}{\nu^{\nu - N + 1}} Z_{KP}^{(N-1,\nu)}(A^{(N-1)}, \bar{t}_k) \tag{4.25}$$

which can then be iterated. Thus after $N - n$ eigenvalues a_i are decoupled, we find up to normalisation the KP model of rank n. As we remarked in the introduction, this exhibits a strong analogy to the insertion of n determinant operators in the Gaussian model, as described in Ref.[17], where the result is the $n \times n$ Kontsevich model[5].

4.3 Radius dependence

Finally, we can ask what happens to the radius-dependent NMM under the above procedure. Again the steps are quite straightforward and one arrives at the following generalisation of Eq. (4.23):

$$\mathcal{Z}_{NMM,R} = \Big(\prod_{i=1}^{N} a_i\Big)^{\frac{1}{2}(R-1)+\nu} \int \prod_{i=1}^{N} dm_i \; \frac{\Delta(m)}{\Delta(a)} \times e^{\sum_{i=1}^{N}\left[-\nu(m_i a_i)^R + \left[\frac{1}{2}(R-1)+(\nu R - N)\right]\log m_i - \nu \sum_{k=1}^{\infty} \bar{t}_k m_i^k\right]} \tag{4.26}$$

The problem is that the above eigenvalue model cannot (as far as we can see) be converted back to a matrix model. The key to doing so in the $R = 1$ case was the linear term $\sum_i m_i a_i$ in the action, which (after absorbing the Vandermondes and using the inverse of the famous Harish Chandra formula) can be summed back into $\mathrm{tr}MA$. The quantity $\sum_i (m_i a_i)^R$ cannot be converted back into a matrix trace unless $R = 1$.

This clarifies a longstanding puzzle: while a KP model could only be found at $R = 1$, the NMM exists and describes the $c = 1$ string for any R. We see now that the correct extension of the KP model to $R \neq 1$ is the eigenvalue model given by Eq. (4.26) above, but unfortunately this does not correspond to a matrix model.

5. Loop operators in the NMM

In this section we will examine loop operators in the NMM. Our goal here is to understand whether correlation functions of these operators can be related to the

[5]This paragraph corrects an error in a previous version of this paper. As a result, the analogy with $c < 1$ is now *stronger* than we had previously claimed. We are grateful to the referee for helpful suggestions in this regard.

Kontsevich-Penner model of Ref.[27], thereby providing the $c = 1$ analogue of the corresponding observations in Refs.[17, 22]. Though there are some similarities, we will also find some striking differences between this and the $c < 1$ case.

Macroscopic loops in a model of random matrices Φ are described by insertions of the operator:

$$W(x) = \mathrm{tr}\log(x - \Phi) \tag{5.1}$$

which creates a boundary in the world sheet. Here x is the boundary cosmological constant. The corresponding generating function for multiple boundaries is[34, 35, 36, 37, 17]:

$$e^{W(x)} = \det(x - \Phi) \tag{5.2}$$

Such operators have been studied extensively in $c < 1$ matrix models, describing (p, q) minimal models coupled to 2d gravity.

We will consider expectation values of operators of the form $\det(a - Z)$ in the NMM, where a is a real parameter. These operators create a hole in the dual graph in the Feynman diagram expansion of the matrix model. Since the NMM has vertices that are holomorphic/antiholomorphic in Z, the dual graph will have faces that are dual to Z or $Z^\dagger$. The loop operator $\det(a - Z)$ creates a hole in a Z-face, while its complex conjugate creates a hole in a $Z^\dagger$-face.

As we would expect, this means that the correlators are complex, but we have the identity[6]:

$$\Big\langle \prod_i \det(a_i - Z) \Big\rangle_{t_k, \bar{t}_k} = \Big\langle \prod_i \det(a_i - Z^\dagger) \Big\rangle_{\bar{t}_k, t_k} \tag{5.3}$$

where on the RHS the role of the deformations $t_k, \bar{t}_k$ has been interchanged. Therefore as long as we consider correlators only of $\det(a_i - Z)$ or $\det(a_i - Z^\dagger)$ the result is effectively the same. As we will see in a moment, a stronger statement is true: on the subspace of parameter space dictated by the KM transform, the unmixed correlators are individually real. Later we will also consider mixed correlators.

As a start, notice that in the 1×1 case,

$$\begin{aligned}
\mathcal{Z}_{NMM,N=1}(t_k = 0, \bar{t}_k) &= \int d^2z \; e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu\sum_{k=1}^{\infty}\bar{t}_k\bar{z}^k} \\
&= \int d^2z \; (a - z)\frac{1}{(a - z)} \; e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu\sum_{k=1}^{\infty}\bar{t}_k\bar{z}^k} \\
&= \frac{1}{a}\int d^2z \; (a - z)\, e^{-\nu z\bar{z} + (\nu-1)\log z\bar{z} - \nu\sum_{k=1}^{\infty}(t_k^0 z^k + \bar{t}_k\bar{z}^k)} \\
&= \frac{1}{a}\Big\langle (a - z) \Big\rangle_{t_k^0, \bar{t}_k} \; Z_{NMM,N=1}(t_k^0, \bar{t}_k)
\end{aligned} \tag{5.4}$$

[6]Here and in the rest of this section, all correlators are understood to be normalised correlators in the NMM.

where the expectation value in the last line is evaluated in the NMM with

$$t_k^0 = -\frac{1}{\nu k}\, a^{-k} \tag{5.5}$$

We see that the t_k^0 dependence drops out in the RHS because insertion of the loop operator cancels the dependence in the partition function. In fact, more is true: even the $\bar{t}_k$ dependence cancels out between the different factors on the RHS. This is a consequence of the property exhibited in Eq. (2.17).

A more general statement in the 1×1 case is:

$$\mathcal{Z}_{NMM,N=1}(t_k - t_k^0, \bar{t}_k) = \frac{1}{a}\Big\langle (a - z) \Big\rangle_{t_k,\bar{t}_k} Z_{NMM,N=1}(t_k, \bar{t}_k) \tag{5.6}$$

In other words, insertion of the macroscopic loop operator has the effect of decreasing the value of t_k, leaving $\bar{t}_k$ unchanged.

In the more general case of $N \times N$ random matrices, the corresponding result is as follows. The expectation value of a single exponentiated loop operator $\det(a - Z)$ is:

$$\Big\langle \det(a - Z) \Big\rangle_{t_k,\bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)}\, a^N \tag{5.7}$$

with t_k^0 again given by Eq. (5.5). Now we would like to consider multiple loop operators. Therefore consider the expectation value:

$$\Big\langle \prod_{i=1}^{n} \det(a_i - Z) \Big\rangle_{t_k,\bar{t}_k} \tag{5.8}$$

As noted in Ref.[17], this can be thought of as a single determinant in a larger space. Define the $n \times n$ matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ and extend it to an $(n+N) \times (n+N)$ matrix $A \otimes 1\!\!1_{N\times N}$. Similarly, extend the $N \times N$ matrix Z to an $(n+N) \times (n+N)$ matrix $1\!\!1_{n\times n} \otimes Z$. Now we can write

$$\prod_{i=1}^{n} \det(a_i - Z) = \det(A \otimes 1\!\!1 - 1\!\!1 \otimes Z) = \prod_{i=1}^{n}\prod_{j=1}^{N}(a_i - z_j) \tag{5.9}$$

Rewriting this as:

$$\prod_{i=1}^{n} \det(a_i - Z) = (\det A)^N \det(1\!\!1 \otimes 1\!\!1 - A^{-1} \otimes Z) \tag{5.10}$$

and expanding the second factor, we find:

$$\Big\langle \prod_{i=1}^{n} \det(a_i - Z) \Big\rangle_{t_k,\bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)}\, (\det A)^N \tag{5.11}$$

where now:

$$t_k^0 = -\frac{1}{\nu k}\,\mathrm{tr}A^{-k} \tag{5.12}$$

Thus we see that macroscopic loop correlators in this model are obtained by simply shifting the parameters t_k in the partition function, the shift being given by the KM transform.

The above considerations can be extended to mixed correlators as follows. Consider correlation functions of the form:

$$\Big\langle \prod_{i=1}^{n} \det(a_i - Z) \prod_{j=1}^{m} \det(b_j - Z^\dagger) \Big\rangle \tag{5.13}$$

Then, defining the $m \times m$ matrix $B = \mathrm{diag}(b_1, b_2, \ldots, b_m)$, the parameters t_k^0 as in Eq. (5.12), and the parameters $\bar{t}_k^0$ by:

$$\bar{t}_k^0 = -\frac{1}{\nu k}\,\mathrm{tr}B^{-k} \tag{5.14}$$

we find

$$\Big\langle \prod_{i=1}^{n} \det(a_i - Z) \prod_{j=1}^{m} \det(b_j - Z^\dagger) \Big\rangle_{t_k, \bar{t}_k} = \frac{\mathcal{Z}_{NMM}(t_k - t_k^0, \bar{t}_k - \bar{t}_k^0)}{\mathcal{Z}_{NMM}(t_k, \bar{t}_k)} \,(\det A \det B)^N \tag{5.15}$$

In the above we have seen how to re-express correlations of loop operators in terms of shifted closed-string parameters. This in itself is quite reminiscent of an open-closed duality. However we did not yet encounter the KP model. To do so, we note that besides the exponentiated loop operator $\det(a - Z)$, we can consider its inverse: $1/\det(a - Z)$. Just as insertion of $\det(a - Z)$ has the effect of decreasing each t_k by t_k^0 given by Eq. (5.5), insertion of the inverse operator *increases* t_k by the same amount.

Thus we may consider correlators like:

$$\Big\langle \prod_{i=1}^{n} \frac{1}{\det(a_i - Z)} \Big\rangle = \frac{1}{(\det A)^N} \Big\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \Big\rangle \tag{5.16}$$

As before, the two factors of the direct product in the above equation refer to $n \times n$ and $N \times N$ matrices. It is easy to see that the correlation function on the RHS has the effect of increasing the t_k by t_k^0 as given in Eq. (5.12).

Although in principle n and N are independent, here we will consider the case $n = N$. Now the inverse operator

$$\Big\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \Big\rangle \tag{5.17}$$

has already made an appearance in § 4, where one finds it in the eigenvalue basis (see for example Eq. (4.15)):

$$\prod_{i,j=1}^{N} \frac{1}{1 - \frac{z_i}{a_j}} \tag{5.18}$$

The interesting property of the inverse determinant operators is that they can be used to create the KP model starting from the *partially unperturbed* NMM (where $t_k = 0$ but $\bar{t}_k$ are arbitrary). Computationally this is similar to the derivation in § 4 of the KP model from the perturbed NMM. Thus we have:

$$\left\langle \frac{1}{\det(\mathbb{1} \otimes \mathbb{1} - A^{-1} \otimes Z)} \right\rangle_{0,\bar{t}_k} \mathcal{Z}_{NMM}(0, \bar{t}_k) = Z_{KP}(A, \bar{t}_k) \tag{5.19}$$

Here $\mathcal{Z}_{NMM}(0, \bar{t}_k)$ can be replaced by $\mathcal{Z}_{NMM}(0, 0)$ as we have noted previously. This equation then is the precise statement of one of our main observations, that inverse determinant expectation values in the (partially unperturbed) NMM give rise to the KP partition function.

It is clearly desirable to have a target space interpretation for these loop operators. Since the NMM is derived from correlators computed from matrix quantum mechanics, in principle one should be able to understand the loop operators of NMM starting from loop operators (or some other operators) in MQM. While that is beyond the scope of the present work, we will instead exhibit some suggestive properties of our loop operators and leave their precise interpretation for future work.

In matrix models for the $c < 1$ string, which are described by constant random matrices, exponentiated loop operators are determinants just like the ones discussed here for the NMM. In those models it has been argued that the loop operators represent FZZT branes. One striking observation is that in the Kontsevich/generalised Kontsevich description of $c < 1$ strings, the eigenvalues of the constant matrix A come from the boundary cosmological constants appearing in the loop operators. Moreover, Eq. (5.2) has been interpreted as evidence that the FZZT-ZZ open strings there are fermionic[37, 17].

In the present case, we see that the parameters a_i in the loop operators turn precisely into the eigenvalues of the constant matrix A of the Kontsevich-Penner model. We take this as evidence that our loop operators are likewise related in some way to FZZT branes. Indeed, one is tempted to call them FZZT branes of the NMM. Pursuing this analogy further, the role played by inverse determinants in the present discussion appears to suggest that the corresponding strings in the NMM are *bosonic* rather than fermionic. But the relationship of these operators to the "true" FZZT branes of matrix quantum mechanics remains to be understood, as we have noted above[7].

[7]In light of the discussions about the MQM Fermi surface in § 2 we can give an interpretation

In the limit of infinite N, the inverse loop operators depending on a matrix A can be thought of as loop operators for a different matrix $\tilde{A}$. Thus, only in this limit, the inverse determinant operators can be replaced by more conventional determinants. This proceeds as follows. We have already seen that the KM transformation Eq. (3.2) encodes infinitely many parameters t_k via a constant $N \times N$ matrix A, in the limit $N \to \infty$. Now for fixed t_k, suppose we considered the (very similar) transform:

$$t_k = \frac{1}{\nu k} \mathrm{tr} \tilde{A}^{-k} \tag{5.20}$$

that differs only by a change of sign. The point is that this apparently harmless reversal of the t_k brings about a significant change in the matrix A. Moreover this is possible only in the infinite N limit, since we are trying to satisfy:

$$\mathrm{tr} A^{-k} = -\mathrm{tr} \tilde{A}^{-k} \tag{5.21}$$

for all k. Now it is easy to see that the matrices A and $\tilde{A}$ satisfy the following identity:

$$\det(1\!\!1 \otimes 1\!\!1 - A^{-1} \otimes Z) = \frac{1}{\det(1\!\!1 \otimes 1\!\!1 - \tilde{A}^{-1} \otimes Z)} \tag{5.22}$$

Therefore a correlator of inverse loop operators can be rewritten in terms of usual loop operators using:

$$\left\langle \frac{1}{\det(A \otimes 1\!\!1 - 1\!\!1 \otimes Z)} \right\rangle = \frac{1}{(\det A\tilde{A})^N} \left\langle \det(\tilde{A} \otimes 1\!\!1 - 1\!\!1 \otimes Z) \right\rangle \tag{5.23}$$

In the light of our previous observation that inverse determinant operators might indicate the bosonic nature of FZZT-ZZ strings at $c = 1$, it is tempting to think of Eq. (5.23) as a statement of fermi-bose equivalence!

In terms of the operator $\det(\tilde{A} \otimes 1\!\!1 - 1\!\!1 \otimes Z)$, we can make the statement that its insertion into the partially unperturbed NMM gives rise to the KP model depending on the "dual" Kontsevich matrix A.

6. Normal matrix model at finite N

The correspondence between NMM and KP model demonstrated in § 4 is valid for any N, as long as the parameters of the former are restricted to a subspace. The NMM itself is supposed to work at $N \to \infty$, in which case this restriction goes away. However, as noted in Ref.[11], there is another way to implement the NMM:

to both determinant and inverse determinant operators in the NMM. Since their insertions lead to opposite shifts in the t_k's, by virtue of the equivalence between MQM and NMM discussed above we can map each one directly to a corresponding deformation to the Fermi surface, which can be read off from Eq. (2.11). This fact should facilitate direct comparison with the MQM.

by setting $N = \nu R$ (which amounts to $N = \nu$ for $R = 1$), which they labelled as "Model II". In other words, these authors argue that:

$$\lim_{N\to\infty} Z_{NMM}(N, t, \nu) = Z_{NMM}(N = \nu R, t, \nu) \tag{6.1}$$

Thus the NMM describes the $c = 1$ theory at this finite value of N, after analytically continuing the cosmological constant $\mu = i\nu$ to an imaginary value[8].

The key property of this choice is that the logarithmic term in the matrix potential of the NMM gets tuned away. Let us take $R = 1$ from now on. Suppose we evaluate the expectation value of the inverse determinant operator at this N (for the moment we assume that this special value is integral). For N insertions of the inverse determinant, it gives the KP model with $N = \nu$. Thus, as one would expect, the log term of the KP model is also tuned away. Now if we choose $\bar{t}_k = c\,\delta_{k3}$, with c some constant, then the KP model reduces to the Kontsevich model, as observed in Ref.[27]. This shows that the Kontsevich model is a particular deformation of the $c = 1$ string theory after analytic continuation to imaginary cosmological constant and condensation of a particular tachyon (T_3). Note that at the end of this procedure, the rank of the Kontsevich matrix is the same as that of the NMM matrix.

As mentioned earlier, there is a different route to the Kontsevich model starting from the Gaussian Matrix Model (GMM)[17]. Here one starts with a Gaussian matrix model of rank $\hat{N}$, with N insertions of the determinant operator, and takes $\hat{N} \to \infty$ as a double-scaling limit by focussing on the edge of the eigenvalue distribution. The result is the Kontsevich matrix model. This time the rank $\hat{N}$ of the original matrix has disappeared from the picture (it was sent to infinity) while the Kontsevich matrix inherits its rank from the number of determinant insertions N.

A diagram of the situation is given in Fig.1. ¿From the figure one sees that the diagram can be closed if we find a suitable relation of the NMM to the Gaussian matrix model. This is not hard to find at a qualitative level. In fact with $\nu = N$ and $t_k, \bar{t}_k = 0$ the NMM *is* a Gaussian matrix model. We choose the rank to be $\hat{N}$. The NMM eigenvalue distribution $\rho(z, \bar{z})$ is constant inside a disc in the z-plane (for $R = 1$)[11]. If we look at a contour along the real axis in the z-plane, then the effective eigenvalue distribution

$$\rho(x) = \int dy\ \rho(x, y) \tag{6.2}$$

is a semi-circle law, and we find the Gaussian matrix model. However, this picture of eigenvalue distributions is valid only at large $\hat{N}$. Inserting N determinant operators and taking $\hat{N} \to \infty$ as a double-scaling limit, one recovers the Kontsevich model. In

[8] Whereas the authors of Ref.[11] presented this as the analytic continuation of N to the imaginary value $-i\mu$, we prefer to think of it as continuing the cosmological constant μ to the imaginary value iN.

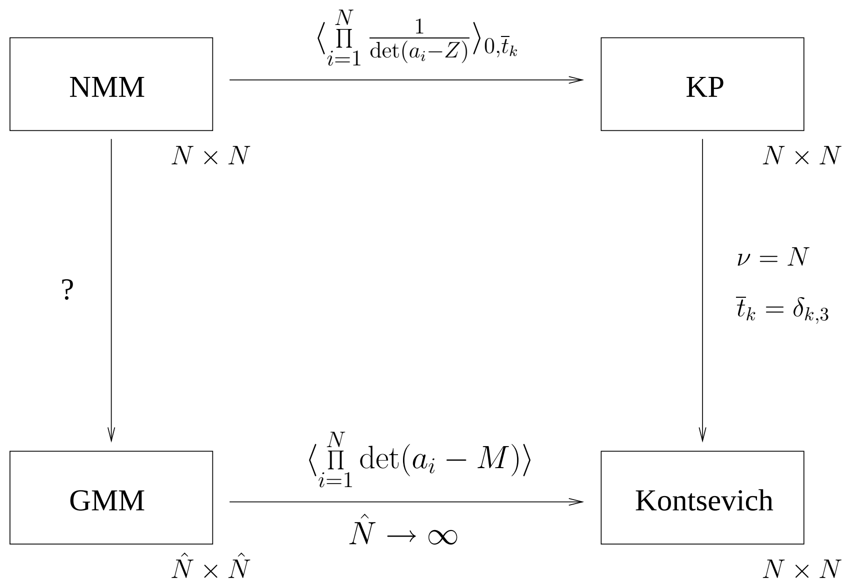

Figure 1: Two routes from NMM to Kontsevich

this way of proceeding, the cubic coupling of the Kontsevich model is switched on automatically during the double-scaling limit. In the alternative route through the KP model, one has to switch on the coupling $\bar{t}_3$ by hand. A more detailed understanding of these two routes and their relationship should illuminate the question of how minimal model strings are embedded in $c = 1$. We leave this for future work.

7. Conclusions

We have established the equivalence between two matrix models of the $c = 1$ string (at selfdual radius): the Normal Matrix Model of Ref.[11] and the Kontsevich-Penner model of Ref.[27]. Both matrix models were initially found as solutions of a Toda hierarchy, so this equivalence is not very surprising. However, it is still helpful to have an explicit derivation, which also uncovered a few subtleties. Also we ended up showing why the KP matrix model does not exist at radius $R \neq 1$.

The more interesting aspect of this equivalence is that correlation functions of inverse determinant operators in the partially unperturbed NMM give rise to the KP model. This is analogous to corresponding results in Refs.[17, 22], with two important differences. In those cases, one considered determinants rather than inverse determinants, and their correlators were computed in a double-scaled matrix model. In the NMM there is no double-scaling as it already describes the grand canonical partition function of the double-scaled Matrix Quantum Mechanics. Another difference is that the N of the final (KP) model is equal to that of the NMM, and part of the matrix variables in NMM survive as the matrices of the KP model. All this suggests that, if one makes an analogy with the topological minimal models, the

NMM occupies a position half-way between the original matrix model arising from dynamical triangulation of random surfaces (which requires a double-scaling limit to describe continuum surfaces) and the final "topological" model. If this is true, we may have only described half the story of open-closed duality at $c = 1$ while the correspondence between MQM and NMM constitutes the previous half. Further work may lead to a more coherent picture of the steps involved and thereby a deeper understanding of open-closed string duality at $c = 1$.

As we commented earlier, the inverse determinant operators seem to suggest bosonic statistics for FZZT-ZZ branes (at least in the NMM context) in contrast to fermionic statistics for $c < 1$. Another way to think of this is that both determinant and inverse determinant operators expand out to give the same set of macroscopic loops, the only difference being a minus sign for an odd number of loops in the latter case. Alternatively one can think of the basic loop operator as being changed by a sign to $-\mathrm{tr}\log(a - Z)$. Either way, the role of inverse determinant operators clearly calls for further investigation.

We commented earlier that trying to take the $c = 1$ limit of $c < 1$ FZZT correlators is problematic and therefore a derivation of the KP model from open-string field theory analogous to Ref.[18] has not been forthcoming. While this may yet be achieved, the situation recalls a historical parallel. In the 1990's, attempts to derive $c = 1$ closed string theory as a limit of the $c < 1$ theories were not very successful. Eventually it was found that at least at selfdual radius, the $c = 1$ string is a nonstandard case – rather than a limit – of the $c < 1$ models. This was understood by going over to the topological[38, 39, 40] rather than conventional, formulation of these string theories. It emerged that while the (p, q) minimal models for varying q were described by topological models labelled by an integer $k = p - 2 \geq 0$ (for example, $SU(2)_k/U(1)$ twisted Kazama-Suzuki models or twisted $\mathcal{N} = 2$ Landau-Ginzburg theories with superpotential X^{k+2}), the $c = 1$ string at selfdual radius was instead described by "continuations" of these models to $k = -3$[41, 40, 42, 43], rather than the more naive guess one might have made, namely $k \to \infty$. Therefore progress on FZZT branes at $c = 1$ in the continuum formulation might most naturally emerge in the context of topological D-branes in the twisted $SU(2)_{-3}/U(1)$ Kazama-Suzuki model or X^{-1} Landau-Ginzburg theory. Indeed, Ref.[44] represents important progress in this direction, and the Kontsevich model has been obtained there in the topological setup, predating the more recent derivations of Refs.[18, 17]. In fact, the KP model of Ref.[27] was also obtained in Ref.[44].

Extension of the NMM/KP models to include winding modes of the $c = 1$ string, as well as a better understanding of 2d black holes from matrix models[8, 9, 10, 11], remain open problems and perhaps the open-closed duality studied here will be helpful in this regard.

We have not pursued here an observation made in Ref.[30] that the KP model simplifies when we exponentiate the matrix variable via $M = e^{\Phi}$. The resulting model, which was named the "Liouville matrix model" there, is suggestive of N D-instantons moving in a Liouville plus linear potential. A similar exponentiation can be carried out in the NMM. In either case this is an almost trivial change of variables, therefore it does not seem important for the considerations in the present paper. However, in the light of the present work, these changes of variables might lead to new and more satisfying interpretations of the matrix models themselves.

As a final comment, we note that open-closed duality has in recent times been given a more fundamental basis in the Gopakumar programme[45, 46, 47] where closed string theory is proposed to be derived from quite general large-N field theories. Now this programme is expected to apply not just to noncritical strings but to all string theories. We know that the Kontsevich and Penner models compute topological invariants of the moduli space of Riemann surfaces, but the above works seem to suggest that these models play a role in more complicated string theories too. If so, equivalences and open-closed dualities such as we have discussed here may have more far-reaching implications than just providing examples in simplified string backgrounds.

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