Preprint typeset in JHEP style - HYPER VERSION

hep-th/0612054
TIFR/TH/06-38

Noncritical-Topological Correspondence: Disc Amplitudes and Noncompact Branes

Anindya Mukherjee *, **Sunil Mukhi** †, **and Rahul Nigam** ‡

Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400 005, India

Abstract: We examine the duality between type 0 noncritical strings and topological B-model strings, with special emphasis on the flux dependence. The former theory is known to exhibit holomorphic factorisation upto a subtle flux-dependent disc term. We give a precise definition of the B-model dual and propose that it includes both compact and noncompact B-branes. The former give the factorised part of the free energy, while the latter violate holomorphic factorisation and contribute the desired disc term. These observations are generalised to rational radii, for which we derive a nonperturbatively exact result. We also show that our picture extends to a proposed alternative topological-anti-topological picture of the correspondence for type 0 strings.

Keywords: String theory.

*Email: anindya_m@theory.tifr.res.in
†Email: mukhi@tifr.res.in
‡Email: rahulnig@theory.tifr.res.in

Contents

1. Introduction

It has been known for some time[1, 2, 3, 4, 5] that noncritical string theories in two spacetime dimensions have a topological description. Subsequently the actual correspondence between them and their topological duals on a Calabi-Yau manifold was found by Ghoshal and Vafa[6]. Noncritical $c = 1$ string theory at the self-dual radius is perturbatively equivalent to topological string theory on a deformed conifold. For integer multiples of the self-dual radius, the corresponding topological theory lives on a $\mathbb{Z}_n$ orbifold of the conifold geometry. The noncritical-topological equivalence was shown both via Landau-Ginsburg models and using the ground ring[7] construction.

The above correspondence has led to considerable illumination of the existence and properties of noncritical strings. But because the bosonic $c = 1$ string is non-perturbatively unstable, it has not been possible to extend the equivalence to the nonperturbative level. Such an extension can be explored in the case of the nonperturbatively stable Type 0 string theories in two dimensions. These too have a description in terms of topological string theory, first proposed in Ref.[8] and further explored in Refs.[9, 10, 11]. The Calabi-Yau dual to noncritical type 0 strings at a special radius is a $\mathbb{Z}_2$ orbifold of the conifold, while for integer multiples of this radius it is a $\mathbb{Z}_{2n}$ orbifold.

One of the most interesting aspects of noncritical type-0 strings is the possibility of turning on background RR fluxes: type 0A theory has two RR gauge fields and type 0B theory has an RR scalar whose equations of motion admit linear growth in space and time[12]. Thus in both cases there is a pair of independent RR fluxes q and $\tilde{q}$. At the level of the closed string perturbation expansion the theory depends only on $|q|+|\tilde{q}|$, but at the disc level there is a subtle and important additional term in the free energy which depends on $|q|-|\tilde{q}|$, as found by Maldacena and Seiberg in [13]. Only after this extra term is included in the free energy, one finds a satisfactory physical interpretation wherein one of the two fluxes is sourced by ZZ 0-branes while the other has no sources.

However, the dual topological B-model of Refs.[8] and [10] depends on the complex-structure moduli of the orbifolded conifold, which in turn depend on the fluxes only through the combination $|q|+|\tilde{q}|$. Thus it is not obvious how the correspondence can be extended to incorporate this effect. Our aim here will be to re-examine the duality with particular reference to flux dependence. We will make the existing proposal more precise and will then argue that the topological side must be extended to incorporate a new feature, namely noncompact branes[1]. When placed at appropriate locations, they contribute precisely the desired disc term in the free energy. Thereafter we generalise these observations to integer multiples of the special radius and to infinite radius.

One key difference between our analysis and previous ones is that we use the result of Ref.[13] which we consider to be rigorously true as a convergent integral representation of the full (nonperturbative) free energy of type 0 strings.

[1] A tentative indication of a role for noncompact branes in this duality was found in Ref.[8].

2. Noncritical-topological duality

2.1 Bosonic case

In this section we briefly review some relevant aspects of topological string theory on noncompact Calabi-Yau spaces. The simplest example is the conifold, described by the equation

$$zw - px = 0, \tag{2.1}$$

where z, w, p, x are complex coordinates of $\mathbb{C}^4$. This is therefore a three complex dimensional non-compact manifold. It has a singularity at the origin. The singularity can be removed by blowing up an S^3 cycle at the origin, after which the equation becomes:

$$zw - px = \mu, \tag{2.2}$$

where μ is in general complex and its modulus determines the size of the S^3. Eq. (2.2) is known as the deformed conifold (DC) and μ is its complex structure parameter. The singularity in Eq. (2.1) can alternatively be removed by blowing up an S^2 at the origin. The resulting manifold is the resolved conifold (RC).

The topological A model on any given Calabi-Yau is a theory of quantised deformations of the Calabi-Yau, sensitive only to the Kähler moduli. The B model is similar but depends only on the complex structure moduli. The noncritical-topological duality proposed by Ghoshal and Vafa[6] stems from the observation[7] that the ground ring of $c = 1$ string theory at the self-dual radius has four generators z, w, p, x that are worldsheet operators of conformal dimension 0 in the BRST cohomology, satisfying the conifold relation Eq. (2.2) where

$$\mu = i g_s \mu_M \tag{2.3}$$

and μ_M is the cosmological constant on the worldsheet in the noncritical string theory[2]. Based on this and other evidence, they argued that the $c = 1$ string with cosmological

[2]The factor of i exhibited here is often dropped in the literature, though it has been correctly placed in Refs.[9, 11]. It is important because the genus expansion of the topological string free energy is $F^{top} = \sum_g \chi_g \mu^{2-2g}$, with coefficients that alternate in sign (given by the virtual Euler characteristic of the moduli space of genus g Riemann surfaces), while that of the $c = 1$ string is $F_{c=1} = \sum_g |\chi_g| (g_s \mu_M)^{2-2g}$ and is therefore positive in every genus as befits a unitary theory. A discussion of this point may be found in Ref.[14].

constant μ_M at self-dual radius is equivalent to the topological B-model on the deformed conifold with deformation parameter μ. In particular their argument requires the genus-g partition functions of the two theories to coincide. Writing the genus expansions of the free energies of the $c = 1$ theory and the topological theory on the deformed conifold as:

$$\begin{aligned}\mathcal{F}^{c=1}(\mu_M) &= \sum_{g=0}^{\infty} \mathcal{F}_g^{c=1} \mu_M^{2-2g} \\ \mathcal{F}^{top,DC}(\mu) &= \sum_{g=0}^{\infty} \mathcal{F}_g^{top,DC} \mu^{2-2g}\end{aligned} \tag{2.4}$$

the claim then amounts to:

$$\mathcal{F}_g^{c=1} = (ig_s)^{2-2g} \mathcal{F}_g^{top,DC}, \quad \text{all } g \tag{2.5}$$

for which ample evidence has been found[15, 16]. There is also expected to be a 1-1 correspondence between the physical observables (tachyons in the $c = 1$ case and deformations of S^3 in the B-model case) and their correlators (for a recent discussion, see Ref.[17]).

Going beyond the self-dual radius, it has long been known[18] that the ground ring of the $c = 1$ string at integer multiples of the self-dual radius, $R = p$, is a $\mathbb{Z}_p$ orbifold of the conifold. This space has p singularities connected by (complex) lines. The deformed version of this space is described by the equation:

$$zw - \prod_{k=1}^{p} (px - \mu_k) = 0 \tag{2.6}$$

which has n homology 3-spheres of size $\mu_1, \mu_2, \ldots, \mu_p$, each concealing one of the singularities. The geometry develops a conifold singularity if any of the μ_i's become zero, and a line singularity if $\mu_i = \mu_j$ for $i \neq j$. If the μ_i's are all distinct and nonzero, the manifold is non-singular.

We expect the n deformation parameters to be in correspondence with p distinct (non-normalisable) deformations of the noncritical string theory[19]. If we only choose to perform the cosmological constant deformation μ_M then these p deformation parameters must be determined in terms of μ_M. It has been shown via a Schwinger computation[20] that for an integer radius the parameters μ_k are given by $ig_s\frac{\mu_M+ik}{p}$, and $k=-\frac{p-1}{2},-\frac{p-1}{2}+1,\cdots,\frac{p-1}{2}$. Moreover, the free energy factorises[3] into a sum of contributions as follows:

$$\mathcal{F}^{R=p}_{c=1}(\mu)=\mathcal{F}^{top,DOC_p}(\{\mu_k\})=\sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}}\mathcal{F}^{top,DC}(\mu_k) \tag{2.7}$$

This factorisation can be understood in the Riemann surface formulation of [21]. In this approach one thinks of the following class of noncompact Calabi-Yaus:

$$zw-H(p,x)=0 \tag{2.8}$$

as a fibration described by the pair of equations:

$$zw=H,\quad H(p,x)=H \tag{2.9}$$

The fibre is $zw=H$, a complex hyperbola, and the base is the complexified p,x plane. Above points in the base satisfying $H(p,x)=0$, the fibre degenerates to $zw=0$, a pair of complex planes intersecting at the origin. Such points in the base form a Riemann surface, and it is this surface that governs the physics of the topological string theory. Moreover the function $H(p,x)$ plays the role of a Hamiltonian and lends an integrable structure to the system.

In the present case of the orbifolded conifold of Eq. (2.6), the Hamiltonian is:

$$H(p,x)=\prod_{k=1}^{p}(px-\mu_k) \tag{2.10}$$

and hence the Riemann surface $H(p,x)=0$ factorises into disjoint Riemann surfaces[10]. This is the physical reason for the factorisation of the free energy into a sum of contributions, one for each branch of the Riemann surface.

The above statements are meaningful only at the level of string perturbation theory, since the bosonic $c=1$ is not well-defined nonperturbatively. Moreover, the computation of Ref.[20] is performed by manipulating a divergent series. Later we will discuss the analogous relation for the type 0A string, and will demonstrate factorisation of the free energy without ever using perturbation expansions or divergent series. In this way we will reliably show that it is nonperturbatively exact.

[3]We use the word "factorise" even though the free energy splits into a sum, rather than a product, of terms. What factorises is of course the partition function.

2.2 Type 0 case, $R = 1$

In Ref.[8] and subsequently Ref.[9, 10, 11], the above ideas were applied to the case of the type 0A string. Here it is convenient to choose units in which $\alpha' = 2$. A new feature of the type 0A string relative to the bosonic case (for more details, see Refs.[12, 22, 13] and references therein) is that it has two distinct quantised parameters q and $\tilde{q}$. In the Liouville description these arise as the fluxes of two distinct Ramond-Ramond 2-form field strengths, $F_{t\phi}, \tilde{F}_{t\phi}$. The theory has a symmetry, labelled S-duality, under which the cosmological constant μ_M changes sign and at the same time, $F \leftrightarrow \tilde{F}$. In the more powerful matrix quantum mechanics (MQM) description of the same string theory, the fluxes have quite an asymmetric origin. For $\mu_M < 0$, q is the difference in the number of $D0$ and $\bar{D}0$ branes, or the net number of $D0$ branes, in the MQM. On the other hand, $\tilde{q}$ is the coefficient of a Chern-Simons term involving gauge fields on the branes and anti-branes. For $\mu_M > 0$ the roles of $q, \tilde{q}$ are reversed. On reducing to eigenvalues, each of the integers q and $\tilde{q}$ can be interpreted as the quantised angular momentum of fermions moving in the complex plane. Moreover, if both are turned on there is an additional coupling term arising from projection to nonsinglet sectors, such that the Hamiltonian eventually depends only on $(q + \tilde{q})$.

The Euclidean type 0A theory has a special value of the radius, $R = 1$ in these units, at which the correspondence with the topological string is simplest. This radius is the analogue of the self-dual radius for the bosonic $c = 1$ string. For type 0A noncritical strings at the special radius, the corresponding dual geometry in the topological string has been proposed[8] to be a deformed $\mathbb{Z}_2$ orbifold of the conifold (DOC). The identification is again based on the analysis of the ground ring of the noncritical theory. The DOC dual to the type 0A string has two S^3's whose complex structure parameters are identified[4] with the type 0A parameters $\mu_M, \hat{q} = q + \tilde{q}$ as:

$$\begin{aligned} \mu &= \ \ i g_s(\mu_M - \tfrac{i\hat{q}}{2}) \ = \frac{g_s}{2} y \\ \mu' &= -i g_s(\mu_M + \tfrac{i\hat{q}}{2}) = \frac{g_s}{2} \bar{y} \end{aligned} \tag{2.11}$$

with:

$$y = \hat{q} + 2i\mu_M \tag{2.12}$$

[4]Again, this identification substantially agrees with that in Refs.[9, 11] but differs from that in Refs.[8, 10] by factors of i.

Thus the equation of the DOC is:

$$zw + (px - \mu)(px - \mu') = 0 \tag{2.13}$$

Notice that complex conjugation exchanges the moduli of the two S^3's and acts as S-duality of the noncritical string. This is because both conjugation and S-duality act as $\hat{q} \to \hat{q}, \mu_M \to -\mu_M$. As a result the S-duality of type 0A noncritical strings is explicitly geometrised in the topological B-model dual.

We note at this point that a different point of view about noncritical-topological duality for type 0 strings is espoused in Ref.[11], according to which the topological string is defined on the "holomorphic square root" of the space we have been discussing, which is an ordinary conifold rather than an orbifolded one. The noncritical-topological correpondence then has to be reformulated by saying that we have to add topological and anti-topological free energies. While this seems to fit in with the picture of topological strings emerging from black hole studies[23, 24], it is not clear that in practical terms it differs from the older proposal of Ref.[8]. However we will see later that our proposal for a precise topological dual involving noncompact branes can also be phrased in topological-anti-topological language.

The manifold Eq. (2.13) exists and is nonsingular for all nonzero $\mu \neq \mu'$. However, the topological B-model on it is dual to type 0A noncritical string theory only in the special case $\mu' = \bar{\mu}$. With this restriction, the space is nonsingular as long as μ has an imaginary part. From Eq. (2.11), this will in turn be the case as long as the cosmological constant μ_M of the noncritical theory is nonzero, which is natural since μ_M cuts off the strong coupling end of the Liouville direction. Of course from the matrix model point of view there is still a sensible string theory when $\mu_M = 0$, but one in which the standard genus expansion of the continuum theory does not hold, and where the role of the string coupling is played by the inverse RR flux. The region where the RR flux is of the same order as, or larger than, the cosmological constant has received some discussion in the literature[25, 26].

The above identification leads to the following proposed equality between type 0A string and topological B model free energies:

$$\mathcal{F}_{0A}(\mu_M, q, \tilde{q}, R=1) = \mathcal{F}^{top,DOC}\left(\mu = \frac{g_s}{2}y, \mu' = \frac{g_s}{2}\bar{y}\right) \tag{2.14}$$

Using arguments analogous to those described above for the bosonic string, we also find that the RHS perturbatively factorises:

$$\mathcal{F}^{top,DOC}\left(\mu=\frac{g_s}{2}y,\mu'=\frac{g_s}{2}\bar{y}\right)=\mathcal{F}^{top,DC}\left(\frac{g_s}{2}y\right)+\mathcal{F}^{top,DC}\left(\frac{g_s}{2}\bar{y}\right) \tag{2.15}$$

In principle we can now investigate the validity of the above correspondence beyond perturbation theory. This point was considered in Refs.[9, 11]. However, the methods used there involve manipulation of divergent series, and we will be able to derive all our correspondences using convergent integral representations of the relevant special functions.

Let us see how this works in some detail. As in Ref.[10], we consider the open-string dual of the DOC obtained from the Gopakumar-Vafa correspondence[27]. This theory lives on a resolved orbifolded conifold (ROC) with two P^1's whose (complex) size parameter is irrelevant in the B-model but which have respectively N_1, N_2 2-dimensional B-branes wrapped over them, where:

$$\begin{aligned}N_1&=\frac{y}{2}=\frac{\hat{q}}{2}+i\mu_M\\ N_2&=\frac{\bar{y}}{2}=\frac{\hat{q}}{2}-i\mu_M\end{aligned} \tag{2.16}$$

The number of branes in this correspondence is inevitably complex, and therefore a prescription is required to complexify starting from real integer values[5].

In the open string description, the partition function arises as follows. Using Eqs.(2.11) and (2.12), we find:

$$\begin{aligned}\mathcal{F}^{top,DOC}\left(\mu=\frac{g_s}{2}y,\mu'=\frac{g_s}{2}\bar{y}\right)&=\mathcal{F}^{top,ROC}\left(N_1=\frac{y}{2},N_2=\frac{\bar{y}}{2}\right)\\ &=\mathcal{F}^{top,RC}\left(N=\frac{y}{2}\right)+\mathcal{F}^{top,RC}\left(N=\frac{\bar{y}}{2}\right)\end{aligned} \tag{2.17}$$

where in the last step, factorisation of the Hamiltonian $H(p,x)$ has been used.

On an ordinary resolved conifold, the free energy of N D-branes is given by the log of the matrix integral:

$$e^{-\mathcal{F}^{top,RC}(N)}=\frac{1}{\mathrm{vol}(U(N))}\int dM e^{-\frac{1}{2}\mathrm{tr}M^2}=\frac{(2\pi)^{\frac{N^2}{2}}}{\mathrm{vol}(U(N))} \tag{2.18}$$

[5] However, we see that the *total* number of branes in the background $N_1+N_2=\hat{q}$ is real and integer. This is striking, and somewhat reminiscent of fractional branes, though we do not have an explanation of this fact.

Now we use[28]

$$\mathrm{vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}(N^2+N)}}{G_2(N+1)} \tag{2.19}$$

where $G_2(x)$ is the Barnes double-Γ function[29] defined by:

$$G_2(z+1) = \Gamma(z) G_2(z), \quad G_2(1) = 1 \tag{2.20}$$

Thus we find

$$-\mathcal{F}^{top,RC}\left(N = \frac{y}{2}\right) - \mathcal{F}^{top,RC}\left(N = \frac{\bar{y}}{2}\right) = \left(\log G_2\left(\frac{y}{2}+1\right) - \frac{y}{4}\log 2\pi\right) + c.c. \tag{2.21}$$

Let us compare the above with what we know about the noncritical string starting from the matrix model. In Ref.[13] the authors have given a complete nonperturbative solution for the free energy of Type 0 noncritical strings at arbitrary radius R. The free energy of type 0A theory is given by:

$$-\mathcal{F}_{0A}(\mu_M, q, \tilde{q}, R) = \Omega(y, R) + \Omega(\bar{y}, R) + \frac{\pi \mu_M R}{2}(|q| - |\tilde{q}|) \tag{2.22}$$

where the function Ω is defined by the convergent (for $\mathrm{Re}\, y > -\left(1 + \frac{1}{R}\right)$) integral:

$$\Omega(y, R) \equiv -\int_0^\infty \frac{dt}{t}\left[\frac{e^{-\frac{yt}{2}}}{4\sinh\frac{t}{2}\sinh\frac{t}{2R}} - \frac{R}{t^2} + \frac{Ry}{2t} + \left(\frac{1}{24}\left(R + \frac{1}{R}\right) - \frac{Ry^2}{8}\right)e^{-t}\right] \tag{2.23}$$

At the special radius $R = 1$ it is easily shown from the integral form that:

$$\Omega(y, R=1) = \log G_2\left(\frac{y}{2}+1\right) - \frac{y}{4}\log 2\pi \tag{2.24}$$

where G_2 is the Barnes function discussed above.

If we temporarily ignore the last term in Eq. (2.22), we see that the free energy is the sum of holomorphic and antiholomorphic contributions. Moreover, each of these is known to be the (complexified) free energy of the bosonic $c = 1$ string at radius R[30]. This is in agreement with Eqs.(2.14),(2.15).

However, the last term in Eq. (2.22) does not seem to come from the topological string. We will discuss this issue in the following section. First we will generalise the considerations of this subsection to the case where the radius of the time circle is different from $R = 1$, in particular to integer radii. We will also comment on the case of rational radii $R = \frac{p}{p'}$.

2.3 Integer radius

We have seen that the $c = 1$ bosonic string at $R = p$ (an integer multiple of the self-dual radius $R = 1$) is dual to a topological string living on a Z_n orbifold of the conifold. An analogous result has been proposed for the type 0A string[11]. We will provide a simple and general derivation of this result using only properties of convergent integral representations.

Inserting the value $R = p$ into the expression for Ω, Eq. (2.23), we rewrite the first term in the integrand:

$$\frac{e^{-\frac{yt}{2}}}{4\sinh\frac{t}{2}\sinh\frac{t}{2p}} \to \frac{e^{-\frac{yt}{2}}}{4(\sinh\frac{t}{2})^2}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} \tag{2.25}$$

Next, use:

$$\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} = \sum_{k=1}^{p} e^{\frac{t}{2p}(p-(2k-1))} \tag{2.26}$$

Now define:

$$y_k = y + \frac{-p + (2k-1)}{p}, \quad k = 1, 2, \ldots, p \tag{2.27}$$

Using the identities:

$$\begin{aligned} \sum_{k=1}^{p}\frac{1}{t^2} &= \frac{p}{t^2} \\ \sum_{k=1}^{p}\frac{y_k}{2t} &= \frac{py}{2t} \\ \sum_{k=1}^{p}\left(\frac{1}{12} - \frac{y_k^2}{8}\right) &= \frac{1}{24}\left(p + \frac{1}{p}\right) - \frac{py^2}{8} \end{aligned} \tag{2.28}$$

of which only the third one is not completely obvious, but nonetheless easy to prove. It follows that:

$$\Omega\Big(y, R = p\Big) = \sum_{k=1}^{p}\Omega(y_k, R = 1) \tag{2.29}$$

We see that the free energy at rational radius factorises into $2p$ distinct contributions, of which p are holomorphic in y and the remaining are anti-holomorphic. Each of the contributions corresponds to a theory at $R = 1$, or equivalently to the contribution of topological B-branes. The factorisation is exact.

Let us analyse this in some more detail. First, by definition $\mathrm{Re}\, y \geq 0$, which not only ensures convergence of the LHS of Eq. (2.29), but also ensures that the RHS is convergent since this implies that $\mathrm{Re}\, y_k > -\left(1 + \frac{1}{R}\right) = -2$ for all k. Therefore the equality is between convergent integral representations as promised.

From the above result we can conclude that for type 0 string theory at every integer radius $R = p$, there is an exact noncritical-topological correspondence where the corresponding topological string lives on a Z_{2p} orbifold[18] of the conifold, whose deformed version is:

$$zw - \prod_{k=1}^{k=p}(px - \mu_k) \prod_{k=1}^{k=p}(px - \bar{\mu}_k) \tag{2.30}$$

where

$$\mu_k = \frac{g_s}{2} y_k \tag{2.31}$$

and y_k are defined in Eq. (2.27). This manifold has $2p$ independent 3-cycles that occur in complex conjugate pairs. The factorisation into contributions from these cycles is nonperturbatively exact upto non-universal terms, and even those terms vanish identically at integer 0A radius.

The resolved version of this correspondence would involve the same Z_{2p} orbifold of the conifold but now with the $2p$ singularities blown up into P^1's with N_k B-branes wrapped over each of the first p cycles, and the complex conjugate number of branes on the remaining p cycles, where:

$$N_k = \frac{y_k}{2} \tag{2.32}$$

As before, the partition function in this picture arises from the vol$(U(N)$ factors associated to each set of N_k branes, giving the most direct derivation of the noncritical-topological correspondence.

This generalised correspondence too can be phrased in topological-anti-topological language. In this case the topological theory lives on a Z_p orbifold, with p cycles labelled by an integer k and N_k branes wrapped on each of them. The remaining contribution to the free energy arises on combining with the anti-topological version of this theory.

2.4 Rational radius

Let us now consider more general rational radii of the form $R = \frac{p}{p'}$, with p and p' co-prime. A similar derivation to the previous one goes through in this case, though the interpretation presents some subtleties that we will discuss.

Inserting the value of R into the expression for Ω, Eq. (2.23), we send $t \to \frac{t}{p'}$ and then rewrite the first term in the integrand:

$$\frac{e^{-\frac{yt}{2p'}}}{4\sinh\frac{t}{2p'}\sinh\frac{t}{2p}} \to \frac{e^{-\frac{yt}{2p'}}}{4(\sinh\frac{t}{2})^2}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p'}}\frac{\sinh\frac{t}{2}}{\sinh\frac{t}{2p}} \tag{2.33}$$

Using Eq. (2.26) and defining:

$$y_{k,k'} = \frac{y-p'+(2k'-1)}{p'} + \frac{-p+(2k-1)}{p}, \quad k=1,2,\ldots,p; \quad k'=1,2,\ldots,p' \tag{2.34}$$

we find the following identities, generalising Eq. (2.28):

$$\begin{aligned} \sum_{k=1}^{p}\sum_{k'=1}^{p'}\frac{1}{t^2} &= \frac{pp'}{t^2} \\ \sum_{k=1}^{p}\sum_{k'=1}^{p'}\frac{y_{k,k'}}{2t} &= \frac{py}{2t} \\ \sum_{k=1}^{p}\sum_{k'=1}^{p'}\left(\frac{1}{12}-\frac{y_{k,k'}^2}{8}\right) &= \frac{1}{24}\left(\frac{p}{p'}+\frac{p'}{p}\right)-\frac{py^2}{8p'} \end{aligned} \tag{2.35}$$

Thus we find:

$$\Omega\Big(y, R=\frac{p}{p'}\Big) = \sum_{k'=1}^{p'}\sum_{k=1}^{p}\Omega(y_{k,k'}, R=1) - \left(\frac{1}{24}\Big(\frac{p}{p'}+\frac{p'}{p}\Big)-\frac{py^2}{8p'}\right)\log p' \tag{2.36}$$

Thus, at rational radius the free energy factorises into $2pp'$ distinct contributions, of which pp' are holomorphic in y and the remaining are anti-holomorphic. However, in general the factorisation is exact only upto an analytic and therefore non-universal term. If we consider the special case of $p'=1$, corresponding to integer radius in the type 0A theory, then the non-universal term vanishes. On the other hand if we take $p=1$, corresponding to even integer radius in the type 0B theory, then the non-universal term is present. Subtracting the two expressions (after scaling $y \to ym$ in one of them) we find:

$$\Omega\left(ym, R=\frac{1}{m}\right) - \Omega\left(y, R=m\right) = -\left(\frac{1}{24}\Big(m+\frac{1}{m}\Big)-\frac{y^2}{8m}\right)\log m \tag{2.37}$$

which is precisely Eq.(A.39) of [13]. There, we see that the apparent violation of T-duality by the extra term is actually harmless and can be understand as due to the difference in natural cutoffs for type 0A and 0B. This explains the presence of the non-universal term, and confirms that its presence can be ignored.

We would now like to interpret the above factorisation property in terms of contributions from singularities. For the bosonic string, the original ground ring analysis of Ref.[18] tells us that the (singular) ring at $R = \frac{p}{p'}$ is a $Z_p \times Z_{p'}$ orbifold of the conifold. Assuming that in type 0 strings the parameters are complexified and occur in complex-conjugate pairs, we expect in this case to find a $Z_{2p} \times Z_{2p'}$ orbifold of the form:

$$\prod_{k'=1}^{p'}(zw - \alpha_{k'}) \prod_{k'=1}^{p'}(zw - \bar{\alpha}_{k'}) = \prod_{k=1}^{p}(px - \beta_k) \prod_{k=1}^{p}(px - \bar{\beta}_k) \tag{2.38}$$

for some set of $p + p'$ complex parameters $\alpha_{k'}, \beta_k$. Such a space no longer has an interpretation as a fibration over a Riemann surface and the analysis of its partition function is therefore more complicated. We expect that for some (not necessarily simple) choice of the parameters, the free energy on this space can be written as a sum of terms as in Eq. (2.36) but will not be able to show this here.

An alternate interpretation of the factorised free energy is that it corresponds to a $Z_{2pp'}$ orbifold of the conifold:

$$zw - \prod_{\substack{k=1 \\ k'=1}}^{\substack{k=p \\ k'=p'}} (px - \mu_{k,k'}) \prod_{\substack{k=1 \\ k'=1}}^{\substack{k=p \\ k'=p'}} (px - \bar{\mu}_{k,k'}) \tag{2.39}$$

where

$$\mu_{k,k'} = \frac{g_s}{2} y_{k,k'} \tag{2.40}$$

and $y_{k,k'}$ are defined in Eq. (2.34). This manifold has $2pp'$ independent 3-cycles that occur in complex conjugate pairs. The resolved version of this space has the $2pp'$ singularities blown up into P^1's with $N_{k,k'}$ B-branes wrapped over each of the first pp' cycles, and the complex conjugate number of branes on the remaining pp' cycles, where:

$$N_{k,k'} = \frac{y_{k,k'}}{2} \tag{2.41}$$

The advantage of this latter interpretation is that it preserves the fibred structure of the manifold with a Riemann surface as the base, and therefore all previous computations manifestly go through in the same way. Unfortunately this interpretation is at variance with the original proposal[6] that the variety occurring on the topological B-model side is in correspondence with the ground ring on the noncritical side.

3. Disc amplitudes and noncompact branes

3.1 $R = 1$

In the correspondence between noncritical type 0A strings and the B-model on the conifold Eq. (2.13) (and more generally Eq. (2.30)) that we have discussed above, there is right away a puzzle. The former depends on three parameters, $q, \tilde{q}, \mu_M$, which in the continuum Liouville description arise as the two independent RR fluxes and the cosmological constant (in the matrix model description these three parameters arise as a net D-brane number, a Chern-Simons term and the Fermi level respectively[12, 13]). However the topological dual only depends on the complex number $y = |q| + |\tilde{q}| + 2i\mu_M$, and therefore on only two of these three parameters. It reproduces most of the free energy, which indeed depends only on two parameters and is the sum of mutually complex conjugate terms. However, the extra term in the free energy:

$$\mathcal{F}^{disc,2} = -\frac{\pi R}{2}\mu_M(|q| - |\tilde{q}|) \tag{3.1}$$

is unaccounted for (the reason for the label on this contribution will become clear shortly).

This term is responsible for an important effect. From the factorised part of the free energy one extracts the following disc contribution in the limit of large μ and fixed $\hat{q}$[13]:

$$\mathcal{F}^{disc,1} = +\frac{\pi R}{2}|\mu_M|(|q| + |\tilde{q}|) \tag{3.2}$$

Hence the total disc amplitude is:

$$\mathcal{F}^{disc} = \mathcal{F}^{disc,1} + \mathcal{F}^{disc,2} = \frac{\pi R}{2}\Big[\big(|\mu_M| - \mu_M\big)|q| + \big(|\mu_M| + \mu_M\big)|\tilde{q}|\Big] \tag{3.3}$$

This can be written as:

$$\begin{aligned} \mathcal{F}^{disc} &= (2\pi R)\frac{\mu_M}{2}|\tilde{q}|, \quad \mu_M > 0 \\ &= (2\pi R)\frac{|\mu_M|}{2}|q|, \quad \mu_M < 0 \end{aligned} \tag{3.4}$$

The physical interpretation is that for $\mu_M > 0$ the RR flux of $\tilde{q}$ units associated to the gauge field $\tilde{A}$ is supported by $|\tilde{q}|$ ZZ branes in the vacuum, with the contribution per brane to the free energy being given by the product of its extent in Euclidean time ($2\pi R$) and its tension ($\frac{|\mu_M|}{2}$). The other flux of q units associated to the gauge field A has no source. Similarly for $\mu_M < 0$ the vacuum contains $|q|$ ZZ branes sourcing the first flux while the other flux of $\tilde{q}$ units is not supported by any source..

Note that in the absence of the term $\mathcal{F}^{disc,2}$ there is no satisfactory physical interpretation of the disc amplitude in terms of ZZ branes. This makes the term extremely important for a consistent noncritical string theory.

We now propose that the missing term is supplied, on the topological side, by noncompact B-branes wrapping a degenerate fibre of the Calabi-Yau over the Riemann surface $H(p,x) = 0$. Such branes have been extensively studied in Refs.[31, 21] where they have been shown to give rise to the Kontsevich parameters of topological matrix models. These branes are, in particular, fermionic. Since we are considering the free energy of the string theory, we work in the vacuum where such Kontsevich branes are absent. However, as we now explain, it is still possible to place noncompact branes at infinity on the Riemann surface and they can reproduce just the desired term in the free energy.

Consider the case $R = 1$. Suppose we place a single noncompact B-brane along one branch of the degenerate fibre over a point x on the Riemann surface. We would like to isolate its contribution to the free energy compared with that of a brane at a fixed reference position x_*, or in other words we assume that the brane is asymptotically at x_* but its interior region has been moved to x. The action of such a brane has been shown[31, 21] to be[6]:

$$S(x) = \frac{1}{g_s}\int_{x_*}^{x} p(z)\,dz \tag{3.5}$$

As we have seen, for the case of interest to us the Riemann surface consists of two disjoint factors:

$$xp = \frac{g_s}{2}y, \quad xp = \frac{g_s}{2}\bar{y} \tag{3.6}$$

[6] In the language of Ref.[21], we place the branes in the "x-patch" and never move them to the "p-patch".

Thus a brane on the first branch contributes:

$$S(x) = \frac{\mu}{g_s} \ln \frac{x}{x_*} \tag{3.7}$$

Let us now place one noncompact brane above each of the two branches, and take their asymptotic positions to be at x_*, x'_* which will both be sent to infinity. Then their total contribution to the free energy is:

$$S(x, x') = \frac{1}{2} \left(y \ln \frac{x}{x_*} + \bar{y} \ln \frac{x'}{x'_*} \right) \tag{3.8}$$

Now we will choose our branes such that x, x' are also at infinity, but rotated by angles θ, θ' respectively along the circle at infinity relative to the original points x_*, x'_*. Namely:

$$x = x_* \, e^{i\theta}, \quad x' = x'_* \, e^{i\theta'} \tag{3.9}$$

It follows that:

$$\begin{aligned} S(x_1, x_2) &= \frac{i}{2}(y\,\theta + \bar{y}\,\theta') \\ &= -\mu_M(\theta - \theta') + i\frac{\hat{q}}{2}\,(\theta + \theta') \end{aligned} \tag{3.10}$$

The factors of g_s have conveniently cancelled out, and the real part of the above contribution is proportional to μ_M. Now if we choose:

$$\theta = -\theta' = \frac{\pi}{4}(|q| - |\tilde{q}|) \tag{3.11}$$

we find that the noncompact branes give a contribution:

$$S = -\frac{\pi}{2}\mu_M(|q| - |\tilde{q}|) \tag{3.12}$$

to the free energy, precisely equal to that in Eq. (3.1) at $R = 1$.

To summarise, we have shown that if we place a noncompact B-brane at $x \to \infty$ on each branch of the Riemann surface

$$H(p, x) = (px - \mu)(px - \mu') = 0 \tag{3.13}$$

and moreover require that the branes wind at infinity by the angles in Eq. (3.11), we precisely reproduce the disc contribution to the free energy of Eq. (3.1). This situation is depicted in Fig.1.

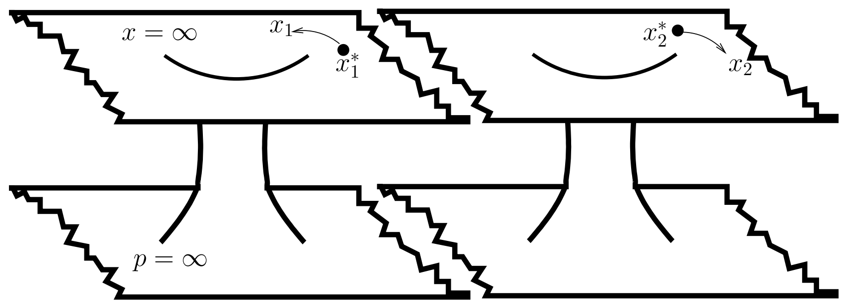

Figure 1: The Riemann surface with noncompact branes at infinity.

This then completes the definition of the topological dual to type 0A strings at the special radius.

The above system also has a description in topological-anti-topological language. As we have seen, the topological theory then lives on the pure conifold, having a Riemann surface with only one branch. Now we place a single noncompact brane on it with winding angle θ given by Eq. (3.11). Adding the anti-topological theory introduces the second noncompact brane with winding $-\theta$ and we recover the correct free energy.

3.2 Integer and rational radius

Let us now extend these considerations to other integer radii. At radius $R = p$, we have the possibility of placing noncompact branes at infinity on each of $2n$ branches of the Riemann surface $H(x, p) = 0$ obtained from Eq. (2.30). Parametrising the angles by which these branes wind as:

$$x_i = x_{*i}\, e^{i\,\theta_i}, \quad x'_i = x'_{*i}\, e^{i\,\theta'_i}, \tag{3.14}$$

the contribution of these branes to the free energy is:

$$\begin{aligned} S(x_i, x'_i) &= \frac{i}{2}\sum_{j=1}^{n}\left(y_j\,\theta_j + \bar{y}_j\,\theta'_j\right) \\ &= -\mu_M \sum_{j=1}^{n}(\theta_j - \theta'_j) + i\sum_{j=1}^{n}\frac{\hat{q}_j}{2}(\theta_j + \theta'_j) \end{aligned} \tag{3.15}$$

where

$$\hat{q}_j = \hat{q} - 1 + \frac{2j-1}{n}, \quad j = 1, 2, \ldots, n \tag{3.16}$$

It is natural to take

$$\theta_j = -\theta'_j = \frac{\pi}{4}\left(|q| - |\tilde{q}|\right), \quad \text{all } j = 1, 2, \ldots, n \tag{3.17}$$

which leads to a contribution to the free energy:

$$S = -\frac{\pi n}{2}\mu_M(|q| - |\tilde{q}|) \tag{3.18}$$

in precise agreement with Eq. (3.1) for $R = n$.

It appears as if in this case the noncompact brane configuration is not unique. However, note that choosing $\theta_j = -\theta'_j$ for all j is essential to make the free energy real. After this, the choice we have made is the most symmetric one which gives the correct disc amplitude.

In the topological-anti-topological approach, we would instead have p branches in the Riemann surface and therefore p noncompact branes with associated angles θ_k. The remaining noncompact branes with angles $-\theta_k$ then arise on the anti-topological side.

It is quite nontrivial that we were able to reproduce the subtle disc term by a simple configuration of noncompact branes in every case. The scaling with g_s of the holomorphic Chern-Simons action and of the complex-structure moduli $\mu_{k,k'}$ defined in Eq. (2.40) exactly cancel out. Moreover, $\mu_{k,k'}$ all have a common imaginary part proportional to μ_M. These facts were important in allowing us to obtain the desired contribution from noncompact branes.

Now let us briefly consider rational radius. If we accept the $Z_{2p} \times Z_{2p'}$ orbifold interpretation of Eq. (2.38) then it is not clear how to extend the above considerations to radius $R = \frac{p}{p'}$. This is because the manifold is no longer of the form $zw = H(p, x)$ and therefore the Riemann surface interpretation itself needs to be generalised, which lies beyond the scope of the present work.

4. Discussion

One of our main results has been that the noncritical-topological correspondence for type 0 noncritical strings has to include noncompact branes on the topological side. This introduces a dependence on a new parameter which we interpret as $|q| - |\tilde{q}|$ on the noncritical side, and renders the duality consistent with the dependence of the noncritical theory on three parameters: μ_M, q and $\tilde{q}$.

The identification between the phases of noncompact branes and the parameter $|q|-|\tilde{q}|$, via Eq. (3.11), appears rather ad hoc. From Eq. (3.11) it is tempting to imagine that there could be a missing normalisation factor of 8 which changes $\frac{\pi}{4}$ to 2π. In that case one could have postulated that the noncompact branes have an integer winding at infinity and this integer gets identified with the integer $|q|-|\tilde{q}|$. This would make the identification a little less ad hoc. However we did not find such a missing normalisation factor.

Given that the subtle disc term is required in the noncritical string by consistency, one may ask if the presence of noncompact branes in the topological theory is also a consistency requirement. However, this seems not to be the case. On the noncritical side there is the possibility of ZZ branes in the vacuum, and it is only after including the subtle term that the vacuum has a definite intepretation as containing or not containing such branes. However ZZ branes do not (so far) have a direct analogue on the topological side and so it is possible that the topological theory without the subtle disc term, and hence with an exactly holomorphically factorised free energy, is consistent by itself. The only thing that would fail is its correspondence to the noncritical theory. Nevertheless it would be interesting if there were a way to understand ZZ branes from the noncritical side. It would be equally interesting to understand the presence[13] of $q\tilde{q}$ fundamental strings in the vacuum, for which we have not found a direct topological explanation.

We also found that the free energy of the full type 0A theory has a nonperturbatively exact factorisation into contributions from compact and noncompact branes. Apparently there is no room for any interactions between these different branes, or in other words the open strings stretched among any two of these branes (both compact, or both noncompact, or one of each) seem to decouple completely. This is somewhat puzzling but must be related in some way to the topological nature of the theory as well as to having distinct branches of the Riemann surface $H(p,x)=0$.

It is amusing that compact and noncompact branes make use of different pieces of the holomorphic Chern-Simons theory restricted to a 2-cycle[32]:

$$S=\frac{1}{g_s}\int \mathrm{tr}(\Phi_1\bar{D}\Phi_0+W(\Phi_0)\omega) \tag{4.1}$$

For compact branes, the first term can be shown to be irrelevant while the second one gives a matrix-valued superpotential, which for our case is simply an independent quadratic for each branch of the Riemann surface. For noncompact branes it is the second term which is irrelevant (because the volume is infinite, we subtract the free energy of deformed compact branes from the undeformed ones[31]) while the first term leads to the expression $\int p dx$. In this case there is no matrix model, because we have placed only one brane on each branch.

It is clearly of interest to generalise our construction to include more noncompact branes that act as sources for incoming closed-string tachyons on the noncritical side[7], as well as non-normalisable deformations of the conifold which are associated to outgoing tachyons[21]. When carried out for the general orbifolded conifold Eq. (2.30), this will provide the analogue of the Normal Matrix Model[33, 34] for type 0 strings, valid for all rational radius. This is an important generalisation of the KP[35] model which has already been found using the topological string construction for both bosonic $c = 1$ strings[21] and type 0 strings[8].

As is well known, topological descriptions of noncritical strings are simplest at $R = 1$ in appropriate units, and can then be generalised to integer multiples of this radius, as done before for bosonic strings and in this paper for type 0 strings. In this way we can describe the Euclidean or finite-temperature version of the theory. To get to the zero-temperature case one then has to take the limit $R \to \infty$. This limit has been explored before, most recently in Ref.[36] where it was related to deconstruction. Our analysis in the present work can potentially add something to this story. Consider Eq. (2.34) at $p' = 1$ and take $p \to \infty$. In this limit we find that $y_{k,k'}$ varies continuously in the open interval $(y - 1, y + 1)$. From Eq. (2.12) this amounts to saying that the RR flux effectively varies continuously over the same interval. This suggests a higher-dimensional origin and may again link the topological theory to deconstruction in some way.

As we have seen, all our considerations extend to the topological-anti-topological picture of Ref.[11], which seems more natural in one sense. The Z_{2p} orbifolded conifold has in principle $2p$ independent complex structure parameters $\mu_k, \bar{\mu}_k$. The noncritical-topological correspondence requires half of them to be constrained to be complex conjugates of the other half, which is naturally achieved if we think of the system in the

[7]Our noncompact branes do not act as such sources precisely because they are located at $x \to \infty$.

topological-anti-topological way. In that case only the p parameters μ_k can be independent. However, as we have seen, the μ_k are all determined in terms of two parameters embodied in y, and the topological-anti-topological picture does not seem to help in explaining this fact. Therefore, if it is to be genuinely useful, perhaps it needs to be extended to a generalised principle where the holomorphic part of the free energy further factorises into contributions from p independent theories.

Acknowledgements

We would like to thank Debashis Ghoshal, Rajesh Gopakumar, K. Narayan and Ashoke Sen for very helpful discussions, and Sang-Heon Yi for a useful correspondence. Part of this work was carried out in the stimulating and pleasant environment of the Harish-Chandra Institute, whose hospitality we gratefully acknowledge. We thank the people of India for generously supporting our research. The work of A.M. was supported in part by CSIR Award No. 9/9/256(SPM-5)/2K2/EMR-I.

References

[1] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B371** (1992) 191–245.

[2] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of $c = 1$ string theory*, *Nucl. Phys.* **B407** (1993) 667–705, [hep-th/9301083].

[3] D. Ghoshal and S. Mukhi, *Topological Landau-Ginzburg model of two-dimensional string theory*, *Nucl. Phys.* **B425** (1994) 173–190, [hep-th/9312189].

[4] A. Hanany, Y. Oz, and M. Ronen Plesser, *Topological Landau-Ginzburg formulation and integrable structure of 2-D string theory*, *Nucl. Phys.* **B425** (1994) 150–172, [hep-th/9401030].

[5] D. Ghoshal, C. Imbimbo, and S. Mukhi, *Topological 2-D string theory: Higher genus amplitudes and W_∞ identities*, *Nucl. Phys.* **B440** (1995) 355–372, [hep-th/9410034].

[6] D. Ghoshal and C. Vafa, $c = 1$ *string as the topological theory of the conifold*, *Nucl. Phys.* **B453** (1995) 121–128, [hep-th/9506122].

[7] E. Witten, *Ground ring of two-dimensional string theory*, *Nucl. Phys.* **B373** (1992) 187–213, [hep-th/9108004].

[8] H. Ita, H. Nieder, Y. Oz, and T. Sakai, *Topological B-model, matrix models, $\hat{c}=1$ strings and quiver gauge theories*, *JHEP* **05** (2004) 058, [hep-th/0403256].

[9] U. H. Danielsson, M. E. Olsson, and M. Vonk, *Matrix models, 4d black holes and topological strings on non-compact calabi-yau manifolds*, *JHEP* **11** (2004) 007, [hep-th/0410141].

[10] S. Hyun, K. Oh, J.-D. Park, and S.-H. Yi, *Topological B-model and $\hat{c}=1$ string theory*, *Nucl. Phys.* **B729** (2005) 135–162, [hep-th/0502075].

[11] U. H. Danielsson, N. Johansson, M. Larfors, M. E. Olsson, and M. Vonk, *4d black holes and holomorphic factorization of the 0a matrix model*, *JHEP* **10** (2005) 046, [hep-th/0506219].

[12] M. R. Douglas *et al.*, *A new hat for the $c=1$ matrix model*, hep-th/0307195.

[13] J. M. Maldacena and N. Seiberg, *Flux-vacua in two dimensional string theory*, *JHEP* **09** (2005) 077, [hep-th/0506141].

[14] S. Mukhi, *Topological matrix models, Liouville matrix model and $c=1$ string theory*, hep-th/0310287.

[15] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, *$N=2$ type II heterotic duality and higher derivative F terms*, *Nucl. Phys.* **B455** (1995) 109–130, [hep-th/9507115].

[16] J. F. Morales and M. Serone, *BPS states and supersymmetric index in $N=2$ type I string vacua*, *Nucl. Phys.* **B501** (1997) 427–444, [hep-th/9703049].

[17] S. Gukov, K. Saraikin, and C. Vafa, *A stringy wave function for an S^3 cosmology*, *Phys. Rev.* **D73** (2006) 066009, [hep-th/0505204].

[18] D. Ghoshal, D. P. Jatkar, and S. Mukhi, *Kleinian singularities and the ground ring of $c=1$ string theory*, *Nucl. Phys.* **B395** (1993) 144–166, [hep-th/9206080].

[19] D. Ghoshal, P. Lakdawala, and S. Mukhi, *Perturbation of the ground varieties of $c=1$ string theory*, *Mod. Phys. Lett.* **A8** (1993) 3187–3200, [hep-th/9308062].

[20] R. Gopakumar and C. Vafa, *Topological gravity as large n topological gauge theory*, *Adv. Theor. Math. Phys.* **2** (1998) 413–442, [hep-th/9802016].

[21] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, *Topological strings and integrable hierarchies*, *Commun. Math. Phys.* **261** (2006) 451–516, [hep-th/0312085].

[22] S. Gukov, T. Takayanagi, and N. Toumbas, *Flux backgrounds in 2d string theory*, *JHEP* **03** (2004) 017, [hep-th/0312208].

[23] H. Ooguri, A. Strominger, and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146].

[24] H. Ooguri, C. Vafa, and E. P. Verlinde, *Hartle-hawking wave-function for flux compactifications*, *Lett. Math. Phys.* **74** (2005) 311–342, [hep-th/0502211].

[25] A. Kapustin, *Noncritical superstrings in a Ramond-Ramond background*, *JHEP* **06** (2004) 024, [hep-th/0308119].

[26] L. A. Pando Zayas and D. Vaman, *Condensing momentum modes in* 2d *0A string theory with flux*, hep-th/0507061.

[27] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415–1443, [hep-th/9811131].

[28] H. Ooguri and C. Vafa, *Worldsheet derivation of a large-N duality*, *Nucl. Phys.* **B641** (2002) 3–34, [hep-th/0205297].

[29] V. Adamchik, *Contributions to the theory of the Barnes function*, math.CA/0308086.

[30] I. R. Klebanov, *String theory in two dimensions*, hep-th/9108019.

[31] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, hep-th/0012041.

[32] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories*, *Nucl. Phys.* **B644** (2002) 3–20, [hep-th/0206255].

[33] S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, *2d string theory as normal matrix model*, *Nucl. Phys.* **B667** (2003) 90–110, [hep-th/0302106].

[34] A. Mukherjee and S. Mukhi, $c = 1$ *matrix models: Equivalences and open-closed string duality*, *JHEP* **10** (2005) 099, [hep-th/0505180].

[35] C. Imbimbo and S. Mukhi, *The topological matrix model of* $c = 1$ *string*, *Nucl. Phys.* **B449** (1995) 553–568, [hep-th/9505127].

[36] R. Dijkgraaf and C. Vafa, *N = 1 supersymmetry, deconstruction, and bosonic gauge theories*, hep-th/0302011.